PDE I – HOMEWORK ASSIGNMENT 9

Due Monday, November 8, 2010. **Please write clearly, and staple your work !**

1. Problem

Consider the Besov space $H^{s,q}$ which is the closure of $C_0^\infty(\mathbb{R}^n)$ relative to the norm

$$\|f\|_{H^{s,q}} := \Big(\sum_{k\in\mathbb{Z}} (1+2^k)^{sq} \|P_k f\|_{L^2(\mathbb{R}^n)}^q \Big)^{\frac{1}{q}} ,$$

for $1 \leq q < \infty$. Prove that $H^{s,1} \subset H^s$, and that $H^s = H^{s,2}$. Moreover, prove that

$$\|f\|_{L^\infty(\mathbb{R}^n)} \leq C \|f\|_{H^{n/2,1}} .$$

Hint: For the last part, use the Bernstein inequality in Paley-Littlewood theory.

2. Problem

Let $\underline{Sf} := (P_k f)_{k\in\mathbb{Z}}$ where $f = \sum_{k\in\mathbb{Z}} P_k f$ is the Paley-Littlewood decomposition of f, and let

$$Sf(x) = \|\underline{Sf}(x)\|_{\ell^2(\mathbb{Z})} = \Big(\sum_{k\in\mathbb{Z}} |P_k f(x)|^2 \Big)^{\frac{1}{2}}$$

denote the Hardy-Littlewood square function.

(a) Prove that

$$\|Sf\|_p \approx \|f\|_p$$

does *not* hold if $p = 1$ or $p = \infty$, but that S is a bounded map from L^1 to weak-L^1.

(b) Assume that instead of using smooth Fourier multipliers, P_k were defined via sharp multipliers, $\widehat{P_k f}(\xi) = \chi_{[2^k,2^{k+1})}(\xi)\widehat{f}(\xi)$. Explain why the proof given in class of $\|Sf\|_p \approx \|f\|_p$ for $1 < p < \infty$, and $f \in \mathcal{S}(\mathbb{R}^n)$, holds only for $p = 2$, but fails if $p \neq 2$.