Exponential functions (integer exponents)

$a^1 = a$ $\qquad a > 0$

$a^2 = a \cdot a$

$a^3 = a \cdot a \cdot a$

$\vdots$

$a^n = \underbrace{a \cdot a \cdots a}_{n \text{ times}}$ "a to the power n"

a base, n exponent.

$$\boxed{a^m \cdot a^n = \underbrace{a \cdots a}_{m} \underbrace{a \cdot a \cdots a}_{n} = a^{m+n}}$$

If $n > m$: $\dfrac{a^n}{a^m} = \dfrac{\cancel{\underbrace{a \cdots a}_{m}} \underbrace{a \cdots a}_{n-m}}{\cancel{\underbrace{a \cdots a}_{m}}} = a^{n-m}$

$$\Rightarrow \boxed{a^{-m} = \frac{1}{a^m}}$$

$$a^0 = a^{n-n} = \frac{a^n}{a^n} = 1 \qquad \text{for } a > 0 .$$

$$0^n = 0 , \quad \text{for } n > 0 .$$

$$0^0 = \text{not defined.}$$

$$\boxed{(a^m)^n} = \underbrace{a^m \cdot a^m \cdots a^m}_{n \text{ times}} = a^{m \cdot n} = \boxed{(a^n)^m}$$

Exponentials with fractional powers.

$a > 0$

$$\left(a^{5/2}\right)^2 = a^{\frac{5}{2}\cdot 2} = a^5 \implies a^{5/2} = \sqrt{a^5}$$

$p, q > 0$ integer.

$$\left(a^{\frac{p}{q}}\right)^q = a^{\frac{p}{q}\cdot q} = a^p \implies \boxed{\sqrt[q]{a^p} = a^{\frac{p}{q}}}.$$

Ex: $2^{1.212} = 2^{\frac{1212}{1000}} = \sqrt[1000]{2^{1212}} = \sqrt[200]{2^{303}}$

$$\frac{1212}{1000} = \frac{606}{500} = \frac{303}{250}$$

What if exponent is not a fraction?

$\pi = 3.1415\ldots$

<u>Ex</u> $a^{\pi} = ?$

a^{π} close to a^{3}

closer to $a^{3.1415}$

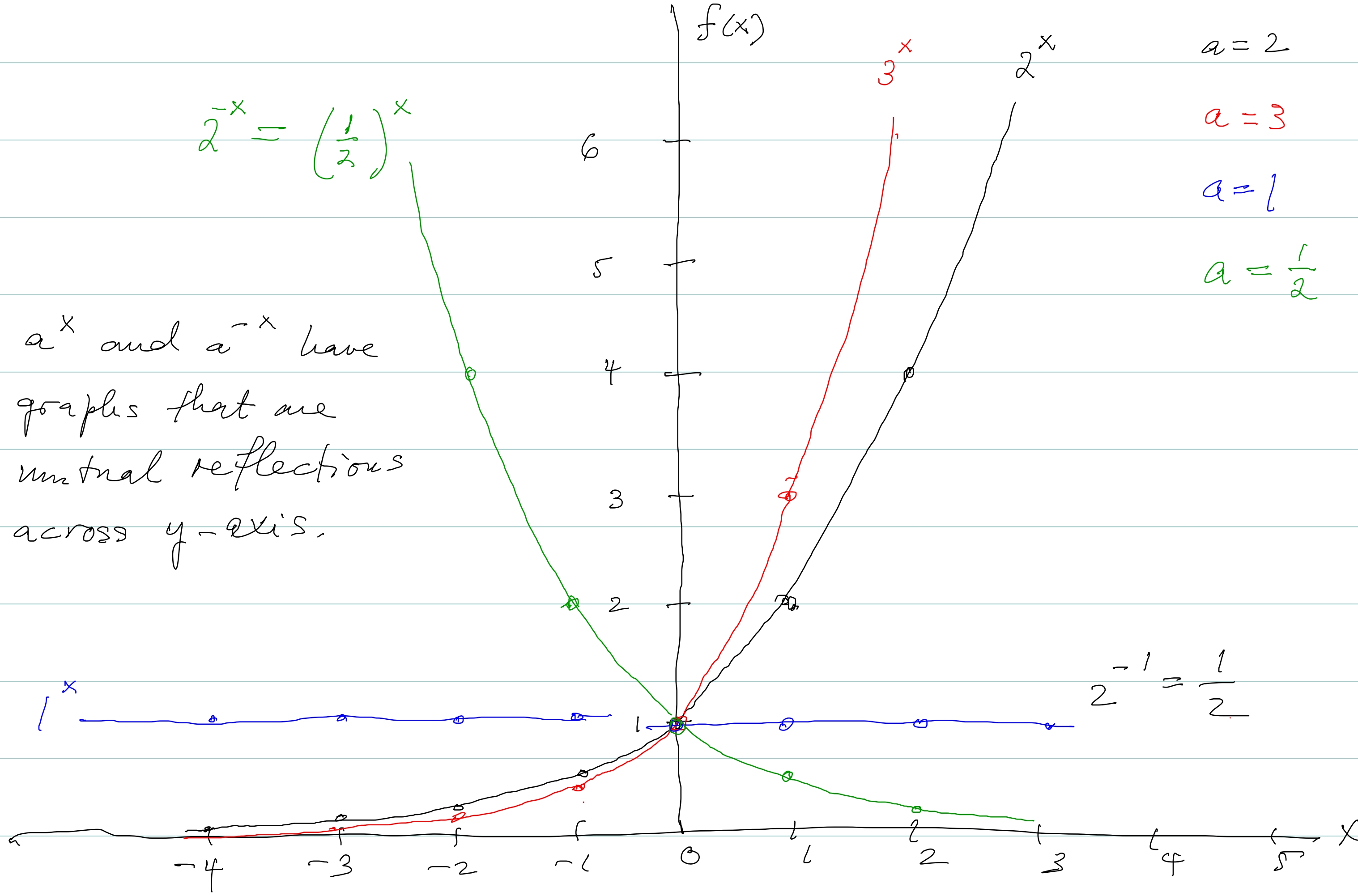
$\vdots$

more and more precise approximations.

$\Longrightarrow$ take a limit.

The function $f(x) = a^x$, $a > 0$

a^x and a^{-x} have graphs that are mutual reflections across y-axis.



Ex Growth of colonony of bacteria.

Assume that every second, each bacterium splits into 2

t time in seconds

$n(t)$ number of bacteria at time t.

$n(0)$

$n(1) = 2 \cdot n(0)$

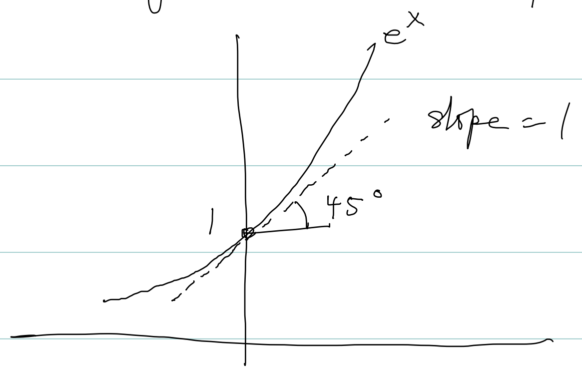
$n(2) = 2 \cdot n(1) = 2^2 n(0)$.

$\vdots$

$n(t) = 2^t n(0)$

There is a unique number for a such that

a^x crosses the y-axis with slope 1.

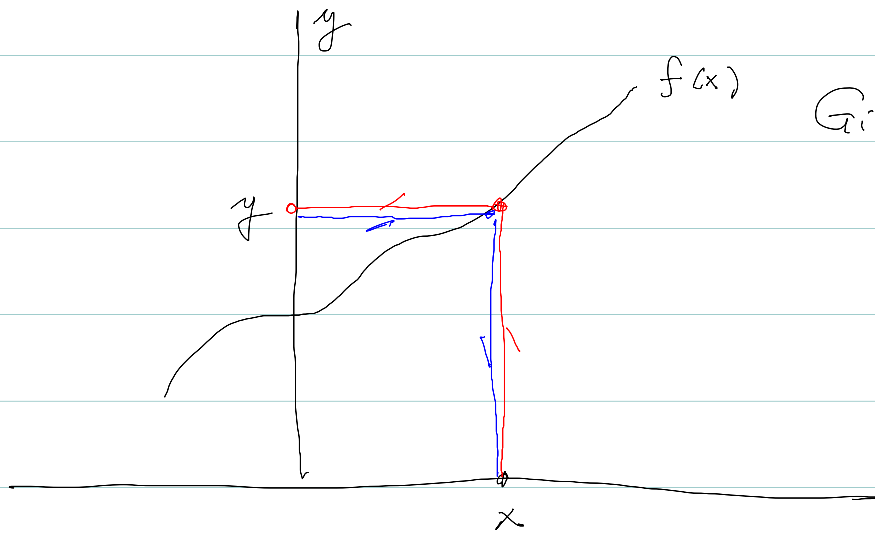


$e = 2.71828\ldots\ldots$

Euler number.

Inverse functions

Q: $y = f(x)$



Given $y = f(x)$

$$\Updownarrow$$

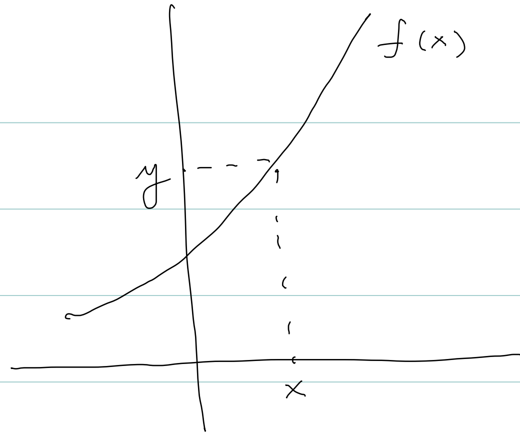
$x = f^{-1}(y)$

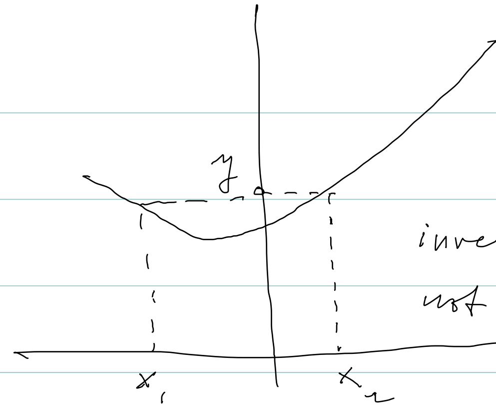
$f^{-1}(f(x)) = x$

$f(f^{-1}(y)) = y$

f and f^{-1} mutually undo the other one's effect

Ex

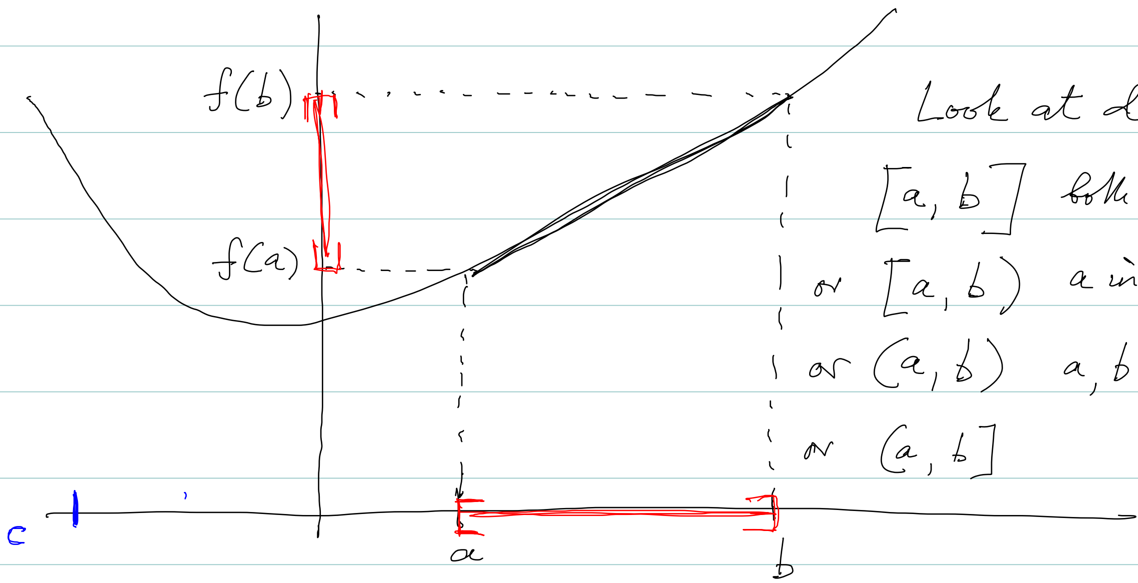




inverse fct not defined

Careful about domain and range of a fct.

To avoid the second case, we have to be more specific with questions concerning domain and range of f.



Look at domain

$[a,b]$ both a,b included

or $[a,b)$ a included, b not

or (a,b) a,b not included

or $(a,b]$

For domain $[a,b]$, the range of f is $[f(a), f(b)]$.

$(a,b]$ ——— " ——— $(f(a), f(b)]$

On $[a,b]$, f has precisely one y-value per x-value

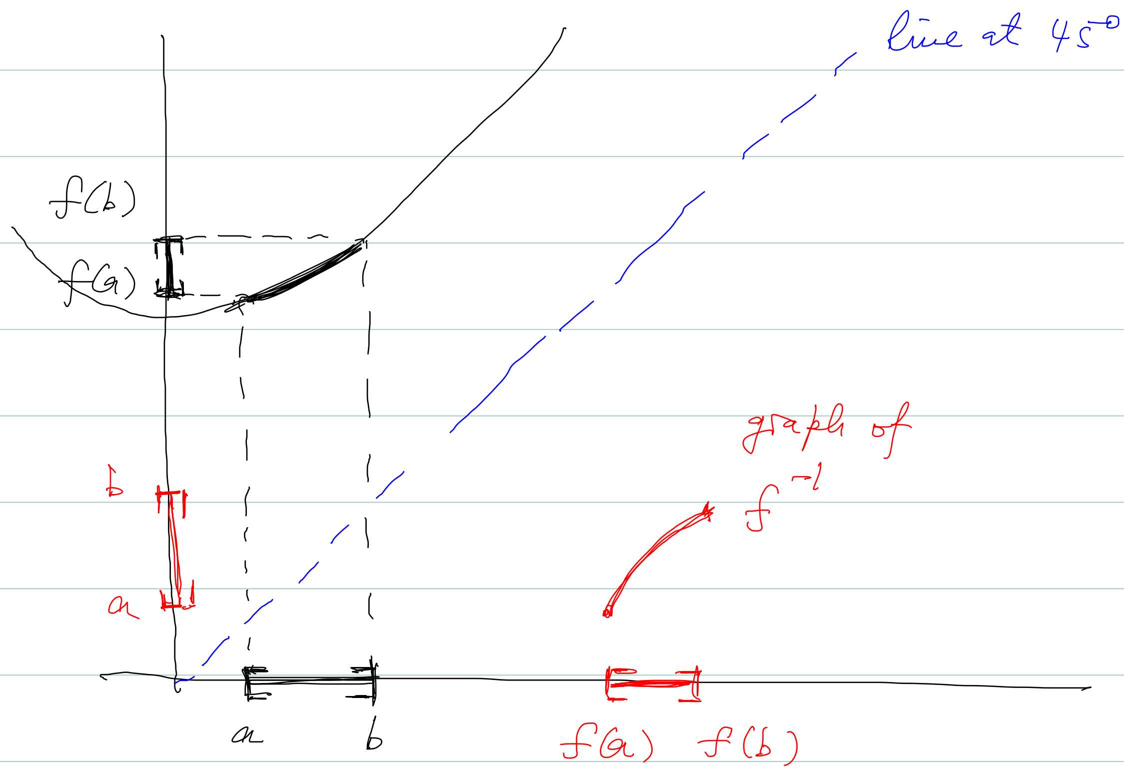
In this case, f is one-to-one (1-to-1), or injective.

Note: f not injective on $[c,b]$.

The inverse function for f exists when f is injective on its domain.

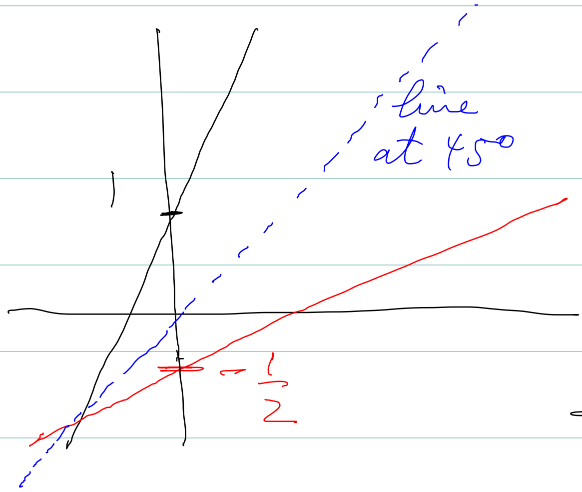
Then, the domain of f^{-1} is the range of f, and the range of f^{-1} is the domain of f.

Next draw the graph of f^{-1}.



Ex $f(x) = 2x + 1$, for $x \in (-\infty, \infty)$

"is an element of"



Find inverse function.

$y = f(x) = 2x + 1$

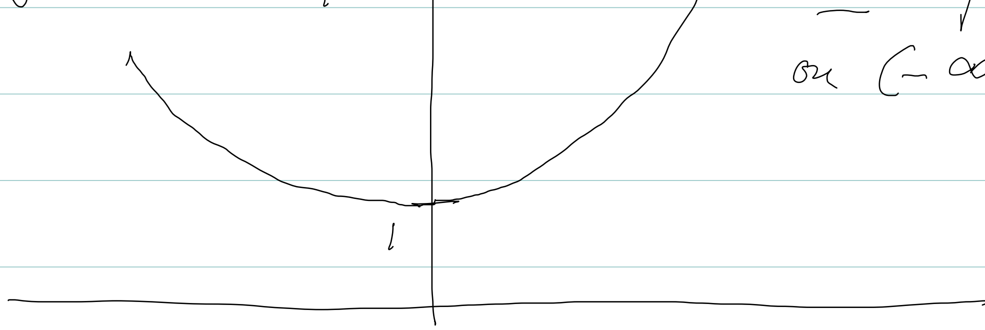
Solve for $x \Rightarrow 2x = y - 1$

$\Rightarrow x = \frac{y}{2} - \frac{1}{2} = f^{-1}(y)$

Switch $y \Leftrightarrow x$

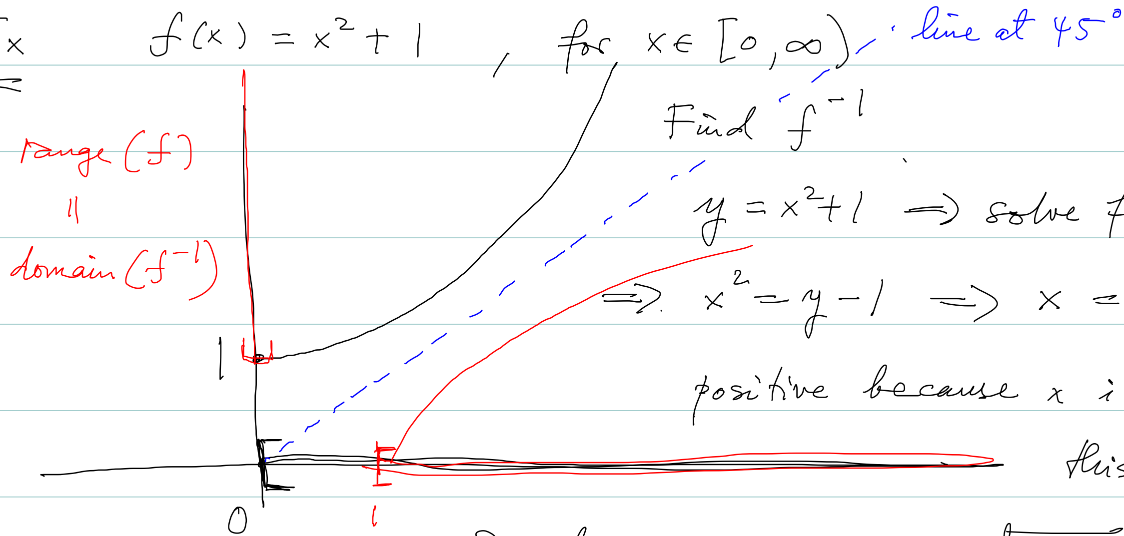
$y = \frac{x}{2} - \frac{1}{2} = f^{-1}(x)$

Ex $f(x) = x^2 + 1$



not injective on $(-\infty, \infty)$.

Ex $f(x) = x^2 + 1$, for $x \in [0, \infty)$



Find f^{-1}

$y = x^2 + 1 \Rightarrow$ solve for x.

$\Rightarrow x^2 = y - 1 \Rightarrow x = \sqrt{y-1} = f^{-1}(y)$

positive because x is positive on this domain.

Switch $y \Leftrightarrow x \Rightarrow y = \sqrt{x-1} = f^{-1}(x)$

$x \geq 1$ because domain$(f^{-1}) = [1, \infty)$

Ex $f(x) = x^2+1$, for $x \in (-\infty, 0]$

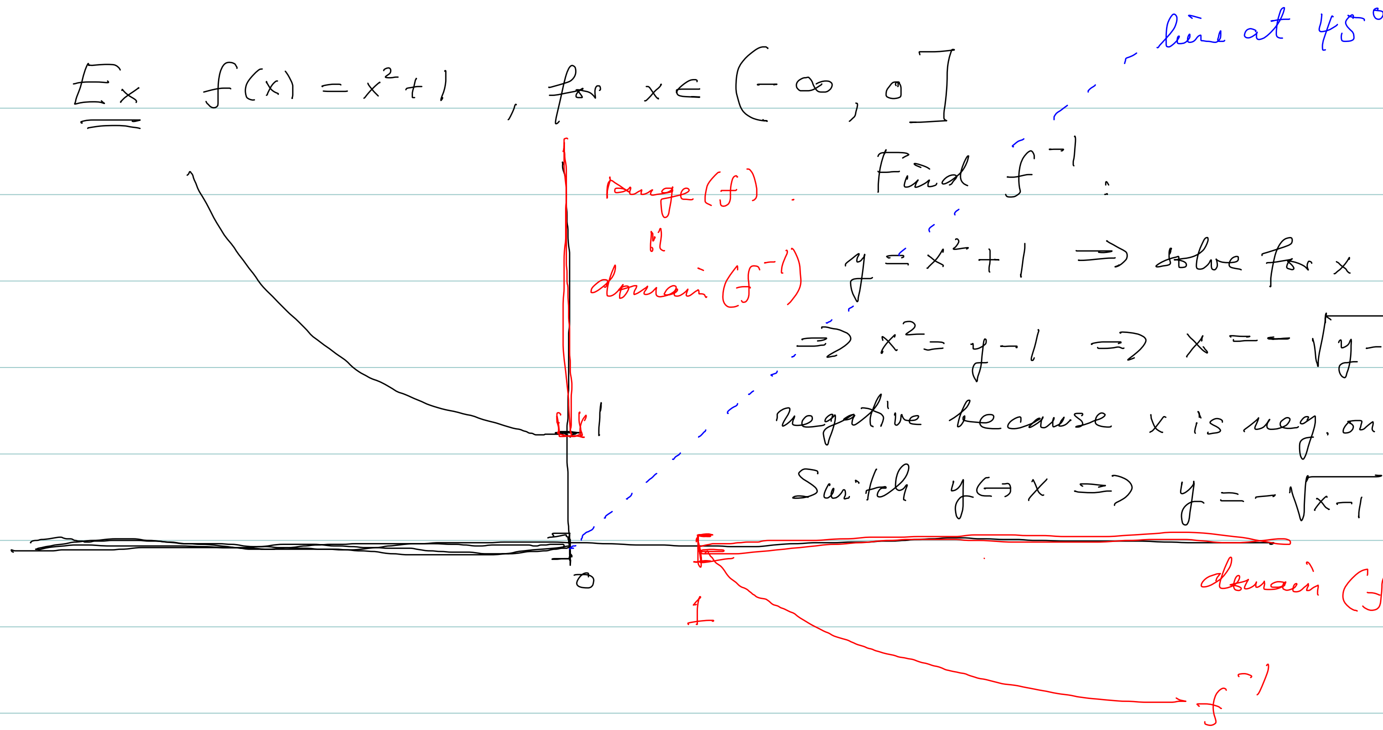
Find f^{-1}:

$y = x^2+1 \Rightarrow$ solve for x

$\Rightarrow x^2 = y-1 \Rightarrow x = -\sqrt{y-1} = f^{-1}(y)$

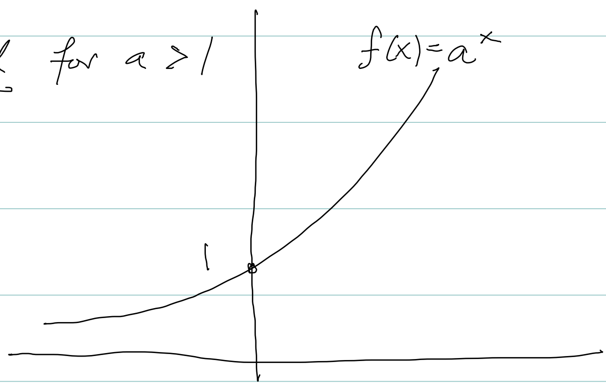
negative because x is neg. on this domain.

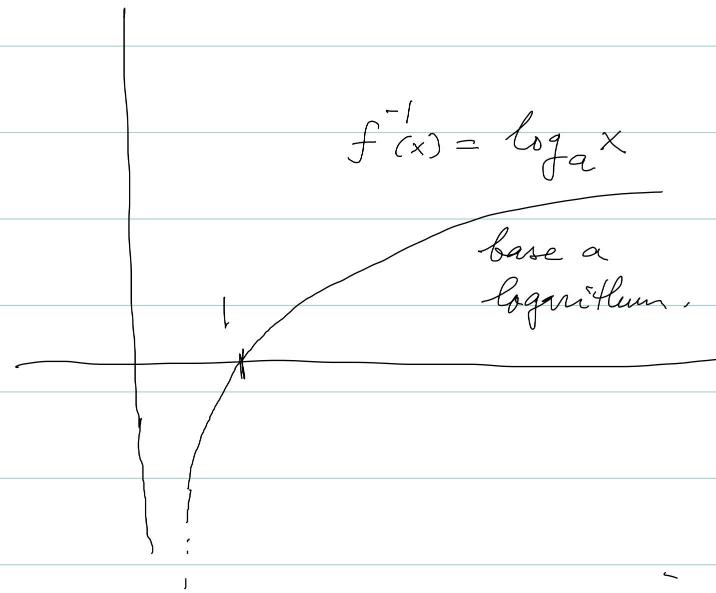
Switch $y \leftrightarrow x \Rightarrow y = -\sqrt{x-1} = f^{-1}(x)$



Logarithms as inverse functions of exponential functions.

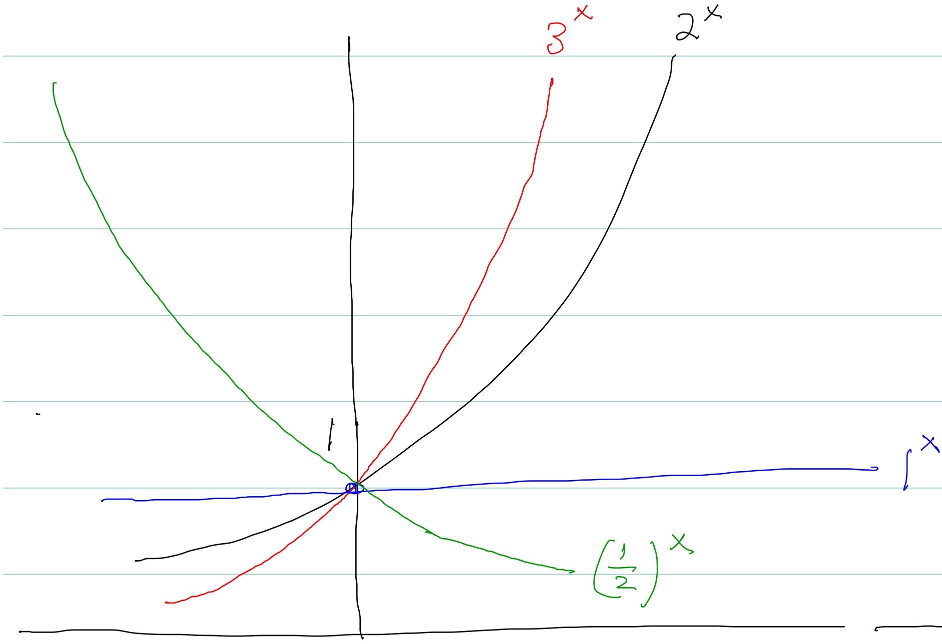
<u>Recall</u> for $a > 1$

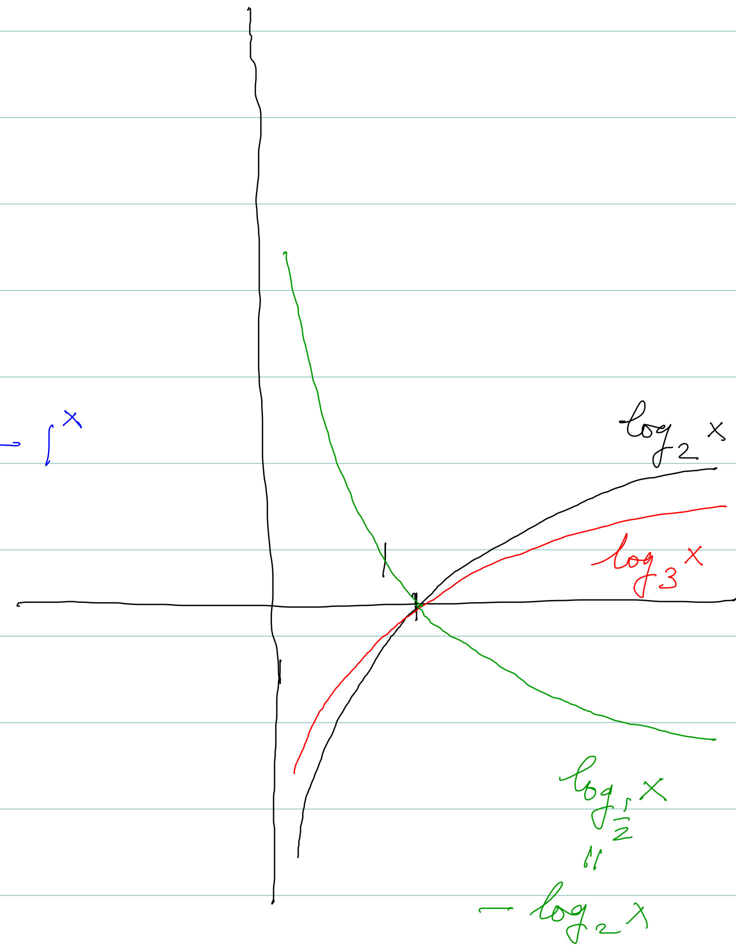




$$a^{\log_a x} = x$$

$$\log_a(a^x) = x$$





$\log_1 x$ is not defined.

$$\log_a x = -\log_{\frac{1}{a}} x$$

$$x \cdot y = \underbrace{a^{\log_a x}}_{x} \cdot \underbrace{a^{\log_a y}}_{y} = a^{\log_a x + \log_a y}$$

$a > 0$

$$= a^{\log_a (x \cdot y)} \qquad \text{for all } x, y > 0$$

$$\Rightarrow \boxed{\log_a (x \cdot y) = \log_a x + \log_a y}$$

$$\log_a \left(a^{\log_a (x \cdot y)} \right) = \log_a \left(a^{\log_a x + \log_a y} \right)$$

$$\frac{x}{y} = \underbrace{a^{\log_a x}}_{x} \cdot a^{-\log_a y} = a^{\log_a x - \log_a y}$$

$$= a^{\log_a \frac{x}{y}}$$

$x, y > 0$

$a > 0$

$$\log_a \left(a^{\log_a \frac{x}{y}} \right) = \log_a \left(a^{\log_a x - \log_a y} \right)$$

$x = 1$

$$\boxed{\log_a \frac{x}{y} = \log_a x - \log_a y} \Rightarrow \boxed{\log_a \frac{1}{y} = -\log_a y}$$

Ex $\log_a x^{10} = \log_a \underbrace{x \cdot x \cdots x}_{10 \text{ times}} = \log_a x + \log_a x^9 = \dots$

$= 10 \log_a x$

More generally: $\log_a x^r = r \log_a x$, for any $r \in (-\infty, \infty)$ and $a > 0$, $x > 0$

Ex: $\log_a \sqrt{x} = \log_a x^{\frac{1}{2}} = \frac{1}{2} \log_a x$

$\log_a \frac{1}{x} = \log_a x^{-1} = -\log_a x$

$\log_a 1 = \log_a x^0 = 0 \cdot \log_a x$
$= 0$

$\log_a 0$ not defined

$\log_0 x$ not defined.

$\log_1 x$ not defined.

$$\boxed{\log_b x = (\log_a x)\cdot(\log_b a)}$$

Ex: Check.

$$x = b^{\log_b x} \overset{?}{=} b^{(\log_b a)\cdot(\log_a x)}$$

$$= \left(\underbrace{b^{\log_b a}}\right)^{\log_a x}$$

$$= a^{\log_a x} = x$$

correct.

Ex: Find $\underbrace{\log_2 4^{17}}$ $-$ $\underbrace{3^{\log_9 81}}$ $= 34 - 9 = \underline{\underline{25}}$.

$$\log_2 (2^2)^{17} \qquad 3^{\log_9 9^2}$$

$$= \qquad\qquad =$$

$$17 \log_2 2^2 \qquad 3^{2 \overbrace{\log_9 9}^{1}}$$

$$= \qquad\qquad =$$

$$34 \underbrace{\log_2 2}_{1} \qquad 9$$

Notation: $\ln x = \log_e x$, $e = 2.718\ldots$ Euler number
natural log.

$\log x = \log_{10} x$.

Limits

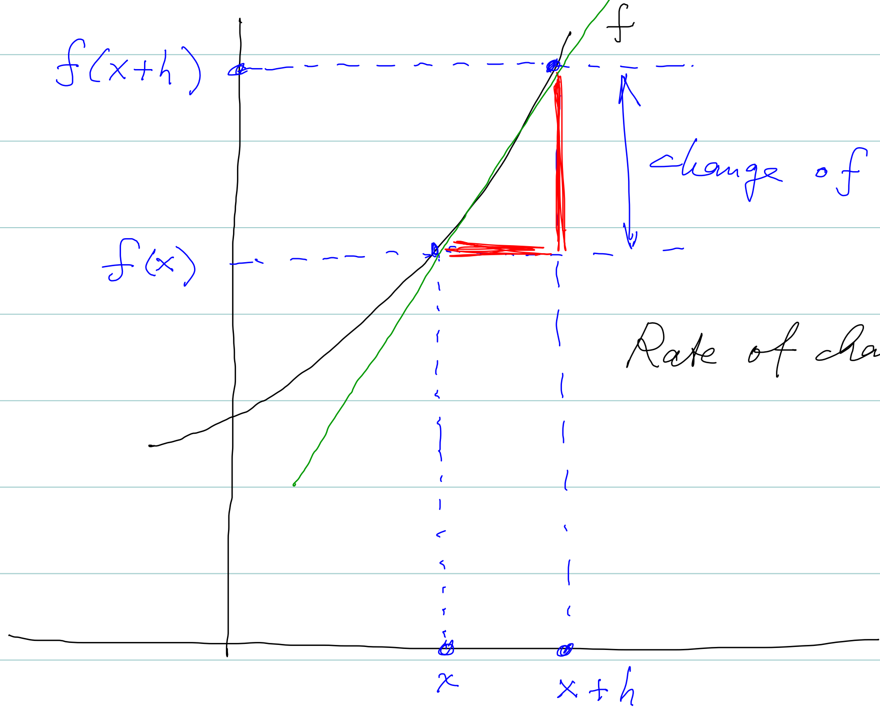
Calculus is very different from algebra in that we do not usually try to solve equations exactly in one step, but we will try to find better and better approximations.

control the error in the approximation.

Ex $1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\cdots$

Notation: $\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}\right)=2$ limit.

Rates of change of functions



$$\text{Rate of change} = \frac{\text{Change of } f}{\text{Change of } x}$$

$$= \frac{f(x+h) - f(x)}{\cancel{x} + h - \cancel{x}}$$

slope of the green line.

Def $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ derivative of f at x.

Slope of tangent to the graph of f at $f(x)$.

<u>Ex</u>

$$\lim_{h\to 0} \underbrace{\frac{h}{h}}_{1} = 1$$

$$\lim_{h\to 0} \frac{h^2}{h} = \lim_{h\to 0} h = 0$$

<u>$h>0$:</u>

$$\lim_{h\to 0} \frac{h}{h^2} = \lim_{h\to 0} \frac{1}{h} \longrightarrow \infty$$

Ex $f(x) = x^2$

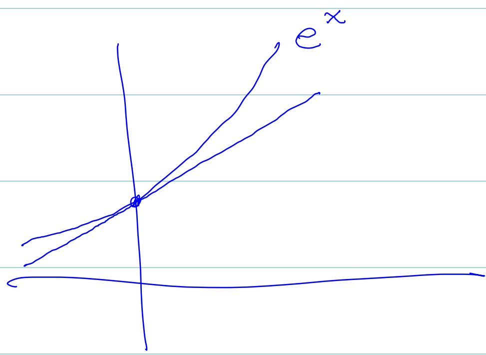
$$(x^2)' = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{\cancel{x^2} + 2x\cancel{h} + h^{\cancel{2}} - \cancel{x^2}}{\cancel{h}}$$

$$= \lim_{h\to 0} (2x + h) = 2x$$

$\underline{\text{Ex}}$ $f(x) = e^x$

$$(e^x)' = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x \cdot e^h - e^x}{h}$$

$$= e^x \cdot \underbrace{\lim_{h\to 0} \frac{e^h - 1}{h}}_{} = e^x$$



$$\lim_{h\to 0} \frac{e^{0+h} - e^0}{h} = \underline{1}$$

slope of tangent of e^x at $x=0$

<u>Ex</u>

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}} = \lim_{x \to 1} (x+1)$$

$$= 2$$

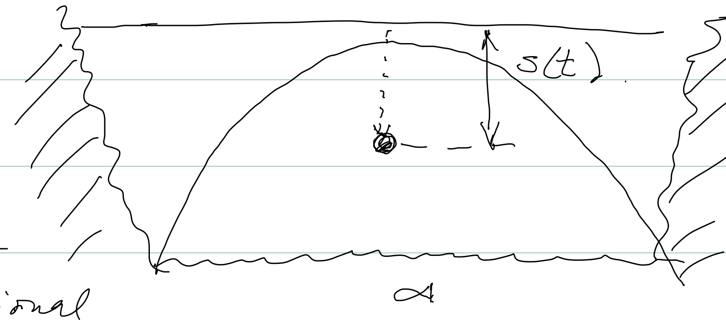
This means x gets closer and closer to 1, but is <u>never</u> exactly 1.

Ex (Average vs instantaneous velocity).

Drop a stone from a bridge.

$$s(t) = 4.9\, t^2 \ [m]$$

$\approx \frac{9.81}{2}$ ← $9.81\ m/sec^2$ gravitational constant.



How fast is it after 5 sec ?

Average velocity for 0.1 sec elapsing after 5 sec :

$$\overline{v} = \frac{s(5.1) - s(5)}{0.1} = \frac{4.9(5.1)^2 - 4.9 \cdot 5.1}{0.1} = 49.49\ m/sec.$$

Instantaneous velocity :

$$v = \lim_{h \to 0} \frac{s(5+h) - s(5)}{h} = s'(5) = 9.8t \Big|_{t=5} = 49\ m/sec$$

The limit of a function.

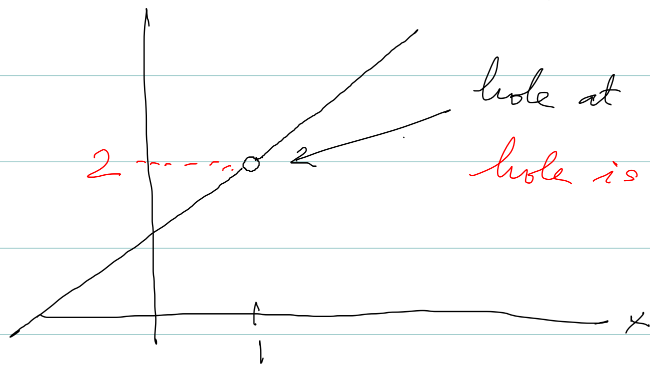
Recall the function $f(x) = \dfrac{x^2-1}{x-1}$.

Def We say that f has a function value at x if $f(x)$ can be properly calculated at x.

Ex $f(x) = \dfrac{x^2-1}{x-1}$ has a function value everywhere except at $x=1$.

For $x=1$, f has no function value.

When $x<1$ or $x>1$, then $f(x) = \dfrac{(x+1)\cancel{(x-1)}}{\cancel{x-1}} = x+1$



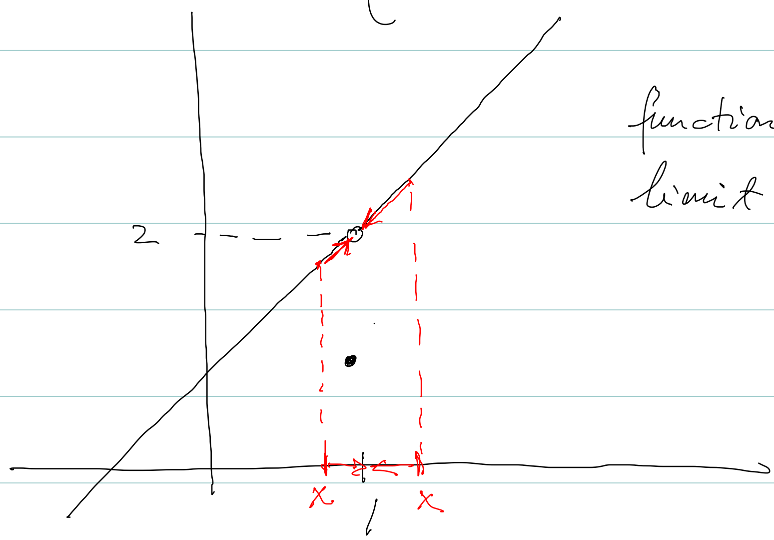
hole at $x=1$.

hole is located at 2 where $\lim_{x \to 1} f(x) = 2$

Observe: f has no fct value at $x=1$, but it has a limit at $x=1$.

Ex We could invent the following function:

$$g(x) = \begin{cases} \dfrac{x^2-1}{x-1} & \text{when } x \neq 1 \\ 1 & \text{when } x = 1 \end{cases}$$



function value at $x=1$: $g(1)=1$

limit at $x=1$: $\lim_{x\to 1} g(x) = 2$

im

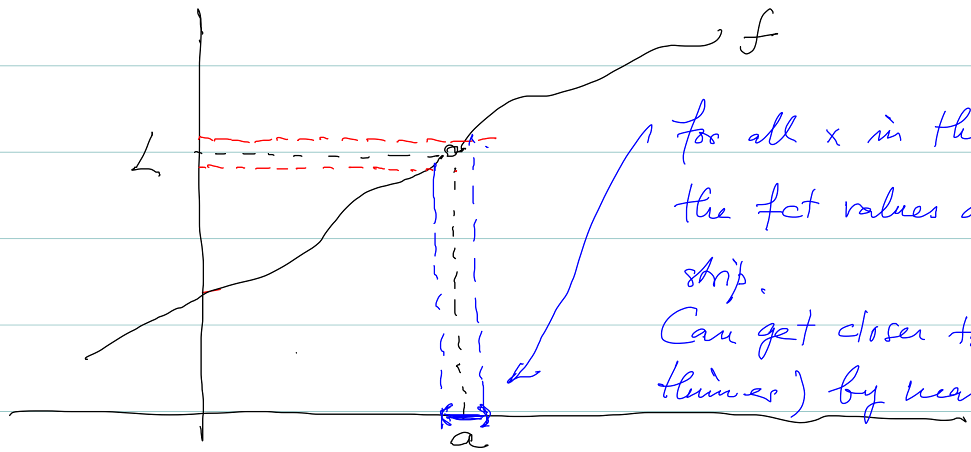
Def (Limit of a function).

Assume that near $x=a$, $f(x)$ has well-defined fct values.

Then, we write

$$\lim_{x\to a} f(x) = L$$

if we can bring $f(x)$ arbitrarily close to L by taking x sufficiently close to a.



for all x in the blue interval, the fct values are inside the red strip.
Can get closer to L (make red strip thinner) by making blue interval shorter.

Ex $\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2} = \lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2} \cdot \frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3}$

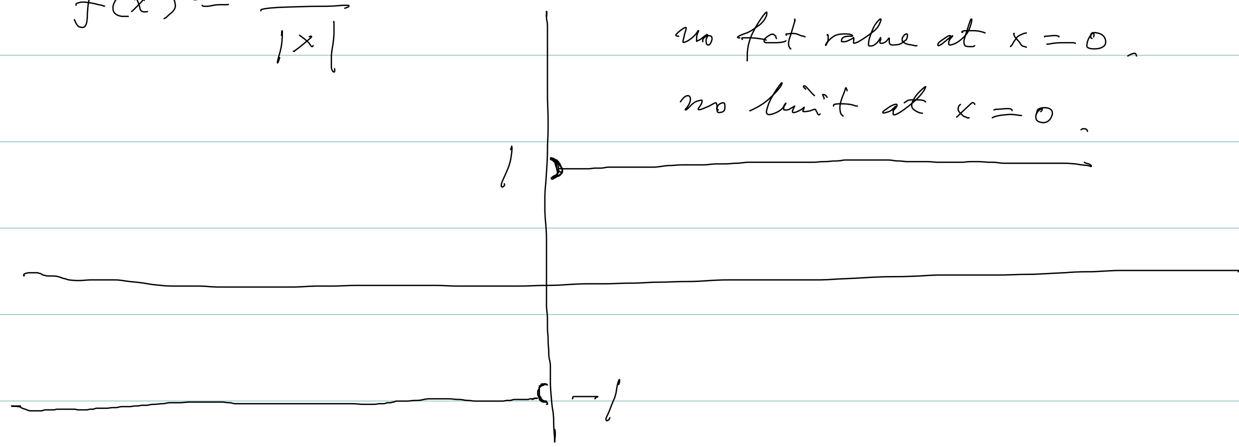
$= \lim_{t\to 0} \frac{\cancel{t^2}+\cancel{9}-\cancel{3^2}}{\cancel{t^2}\left(\sqrt{t^2+9}+3\right)} = \lim_{t\to 0} \frac{1}{\sqrt{t^2+9}+3}$

$= \frac{1}{6}$

Ex $f(x) = \frac{x}{|x|}$

no fct value at $x=0$.

no limit at $x=0$.



<u>Ex</u> $f(x) = \sin\left(\frac{1}{x}\right)$

no fct value at $x=0$.

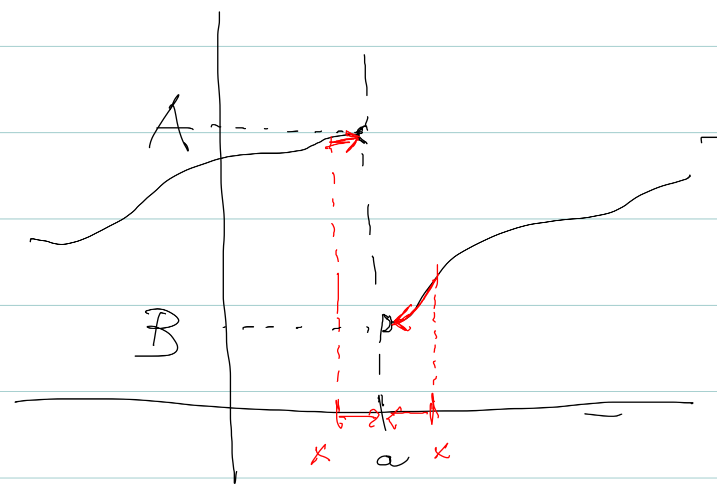
no limit at $x=0$.

The function oscillates crazier and crazier as x approaches 0.

Left- and right limit

Def Left limit $\lim_{x \to a^-} f(x) = A \iff f(x)$ approaches A as x approaches a from the left.

Right limit $\lim_{x \to a^+} f(x) = B \iff f(x)$ approaches B as x approaches a from the right.



$\underline{\underline{Ex}}$ $\lim_{x \to 0^-} \frac{x}{|x|} = -1$

$\lim_{x \to 0^+} \frac{x}{|x|} = 1$

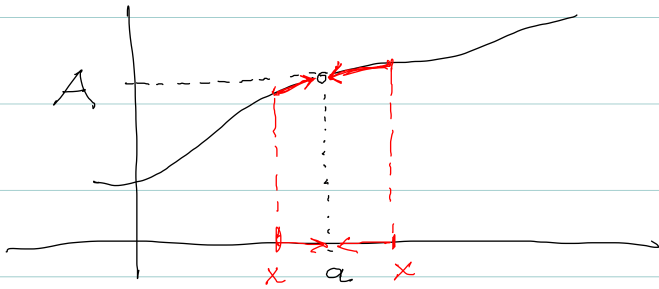
$\underline{\underline{Ex}}$

$$\left.\begin{array}{l} \lim_{x \to 1^-} \frac{x^2-1}{x-1} = 2 \\ \\ \lim_{x \to 1^+} \frac{x^2-1}{x-1} = 2 \end{array}\right\} \text{ limit exists at } x = 1.$$

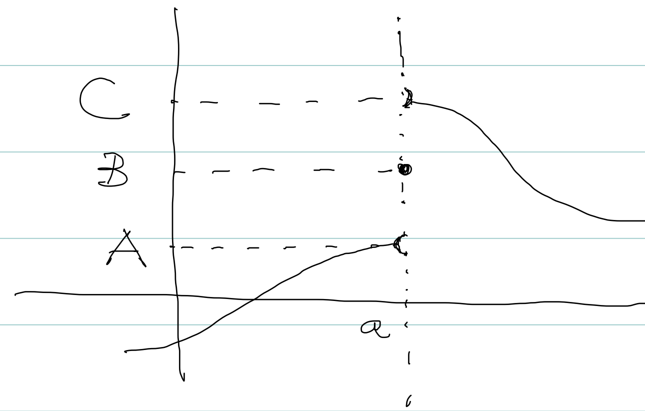
$\underline{Ex}$ $\lim_{x \to 0^+} \sin\left(\frac{1}{x}\right)$ does not exist.

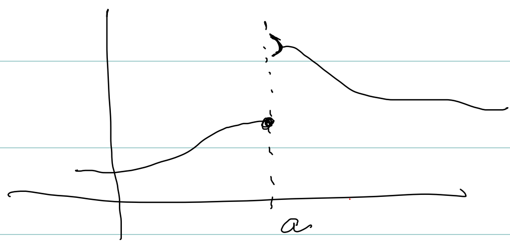
$\lim_{x \to 0^-} \sin\left(\frac{1}{x}\right)$ does not exist.

Theorem: The $\lim_{x \to a} f(x)$ exists if and only if both $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist, and have the same value.

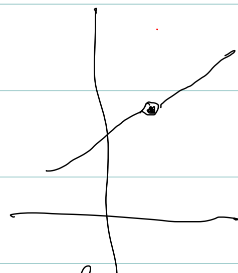


Limit exists.





No limit

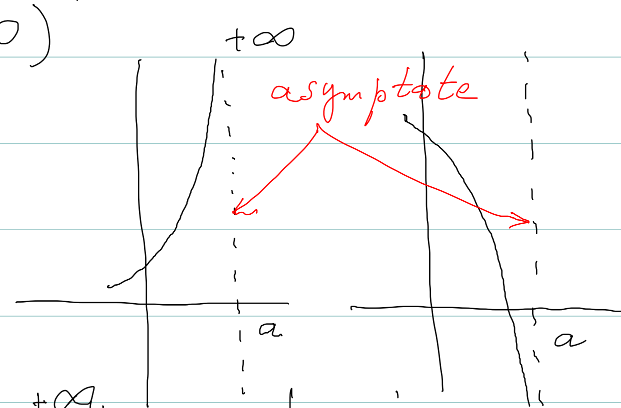


Continuous.

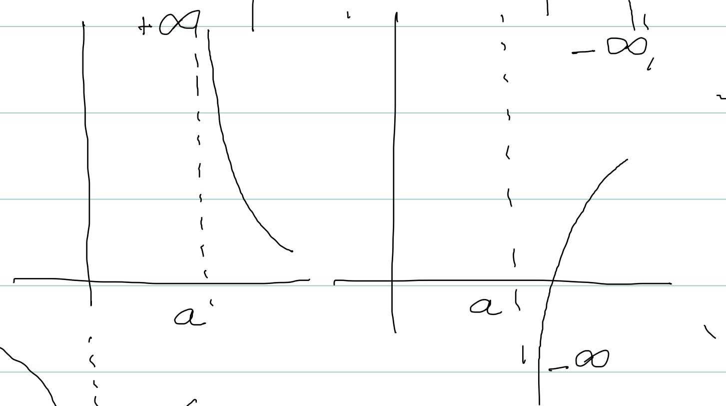
Left limit	Y
Right limit	Y
Fct value	Y
Limit	N

Def "$\lim\limits_{x \to a^-} f(x) = +\infty \ (\text{or } -\infty)$"

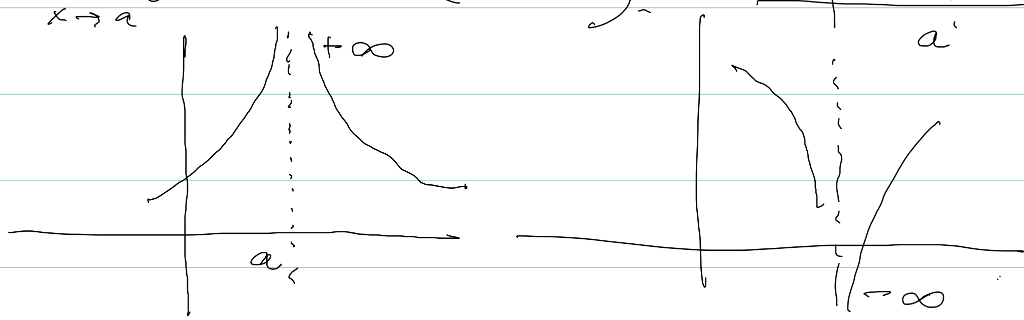
if $f(x)$ grows unboundedly to $+\infty$ (or decreases to $-\infty$) when $x \to a^-$



$$\lim_{x \to a^+} f(x) = +\infty \ (\text{or } -\infty)$$



$$\lim_{x \to a} f(x) = +\infty \ (\text{or } -\infty)$$



Limit laws

Theorem Assume that $\lim_{x\to a^{\pm}} f(x)$, $\lim_{x\to a^{\pm}} g(x)$ both exist and are finite.

Then: 1) $\displaystyle \lim_{x\to a^{\pm}} \big(\alpha f(x) \pm \beta g(x)\big) = \alpha \lim_{x\to a^{\pm}} f(x) \pm \beta \lim_{x\to a^{\pm}} g(x)$

for any numbers α, β.

2) $\displaystyle \lim_{x\to a^{\pm}} \big(f(x)\cdot g(x)\big) = \Big(\lim_{x\to a^{\pm}} f(x)\Big)\cdot\Big(\lim_{x\to a^{\pm}} g(x)\Big)$.

3) $\displaystyle \lim_{x\to a^{\pm}} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a^{\pm}} f(x)}{\lim_{x\to a^{\pm}} g(x)}$ if $\lim_{x\to a^{\pm}} g(x) \neq 0$.

$\underline{\underline{Ex}}$ $$\lim_{x\to 0} (2x+3)\cdot \sin x = \Big(\underbrace{\lim_{x\to 0} (2x+3)}_{3}\Big)\cdot\Big(\underbrace{\lim_{x\to 0} \sin x}_{0}\Big) = 0$$

$\underline{\underline{Ex}}$ $$\lim_{x\to 0^+} \big(x\cdot\cos(2+x)\big)\cdot\frac{1}{|x|}.$$

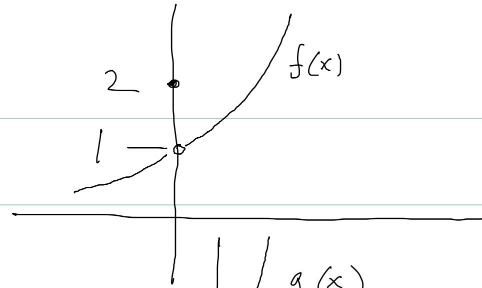
$$= \lim_{x\to 0^+} \cos(2+x)\cdot\frac{x}{|x|}$$

$$= \Big(\underbrace{\lim_{x\to 0^+} \cos(2+x)}_{\cos 2}\Big)\cdot\Big(\underbrace{\lim_{x\to 0^+} \frac{x}{|x|}}_{1}\Big) = \cos 2.$$

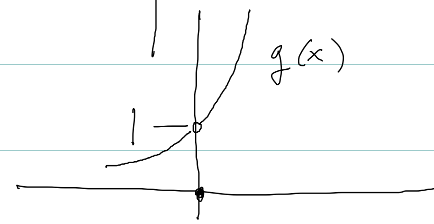
$$\lim_{x\to 0^-} \big(x\cdot\cos(2+x)\big)\cdot\frac{1}{|x|} = \Big(\underbrace{\lim_{x\to 0^-} \cos(2+x)}_{\cos 2}\Big)\Big(\underbrace{\lim_{x\to 0^-} \frac{x}{|x|}}_{-1}\Big) = -\cos 2$$

No limit (left & right limits don't agree)

Ex $\quad f(x) = \begin{cases} 2^x & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$



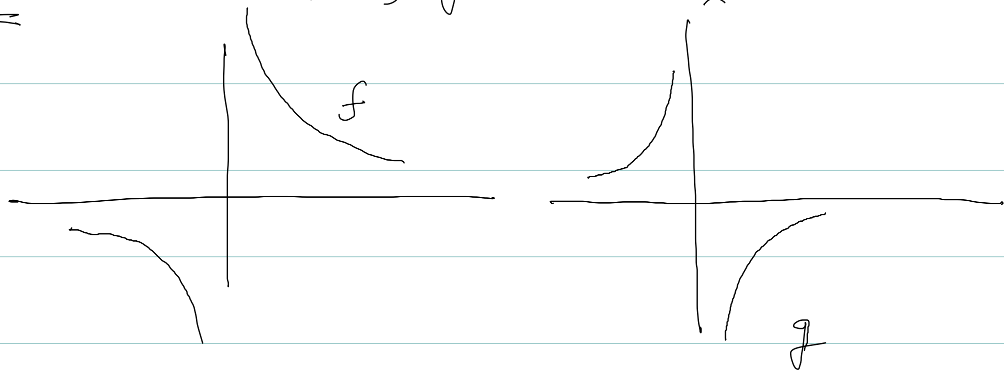
$$g(x) = \begin{cases} 3^x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$



$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{\lim_{x\to 0} f(x)}{\lim_{x\to 0} g(x)} = \frac{1}{1} = 1$$

But the function value of $\frac{f(x)}{g(x)}$ does not exist at $x=0$!

Ex $f(x) = \frac{1}{x}$, $g(x) = -\frac{2}{x}$.



$$\lim_{x \to 0^+} \left(f(x) + g(x) \right) = \lim_{x \to 0^+} \left(\frac{1}{x} - \frac{2}{x} \right) = \lim_{x \to 0^+} \left(-\frac{1}{x} \right) = -\infty$$

Compare to

$$\underbrace{\lim_{x \to 0^+} f(x)}_{\infty} + \underbrace{\lim_{x \to 0^+} g(x)}_{-\infty} = \text{undefined.}$$

Ex Find $\lim_{x \to 0^+} x^a$, depending on $a \in (-\infty, \infty)$

$$\lim_{x \to 0^+} x^a = \begin{cases} 0 & \text{if } a > 0 \\ 1 & \text{if } a = 0 \\ \infty & \text{if } a < 0 \end{cases}$$

because if $a < 0$, we have

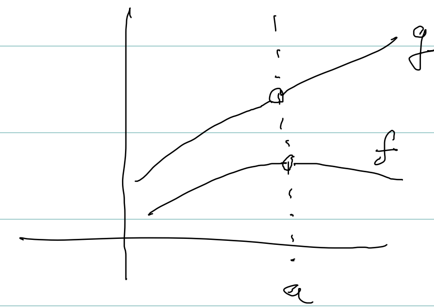
$$\lim_{x \to 0^+} x^a = \lim_{x \to 0^+} x^{-|a|} = \lim_{x \to 0^+} \frac{1}{x^{|a|}} = \infty .$$

Comparison methods.

Thm: If $f(x) \le g(x)$ for all x near a (except possibly at a) and if $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist (not necessarily finite)

Then:

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x).$$



This is also true for $\lim_{x\to a^-}$, or $\lim_{x\to a^+}$, instead of $\lim_{x\to a}$.

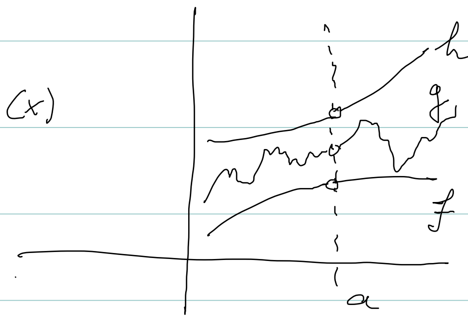
Ex $$\lim_{x\to 0^+} \frac{1+x^4}{x^3} \ge \lim_{x\to 0^+} \frac{1}{x^3} = +\infty$$

$\Rightarrow 1+x^4 \ge 1$ and $a, b, c > 0$ with $a \ge b \Rightarrow \frac{a}{c} \ge \frac{b}{c}$

$$\Rightarrow \lim_{x\to 0^+} \frac{1+x^4}{x^3} = \infty$$

Thm (squeeze). If $f(x) \le g(x) \le h(x)$ for x near a (except possibly at a), and all have well-defined limits at a (not necessarily finite)

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x) \le \lim_{x\to a} h(x)$$

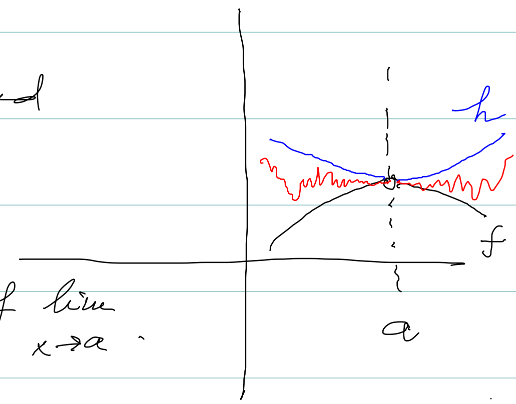


In particular, if

$$\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$$

then the limit of g also exists at a, and

$$\lim_{x\to a} g(x) = L.$$



Remark: Also true with $\lim_{x\to a^-}$ or $\lim_{x\to a^+}$ instead of $\lim_{x\to a}$.

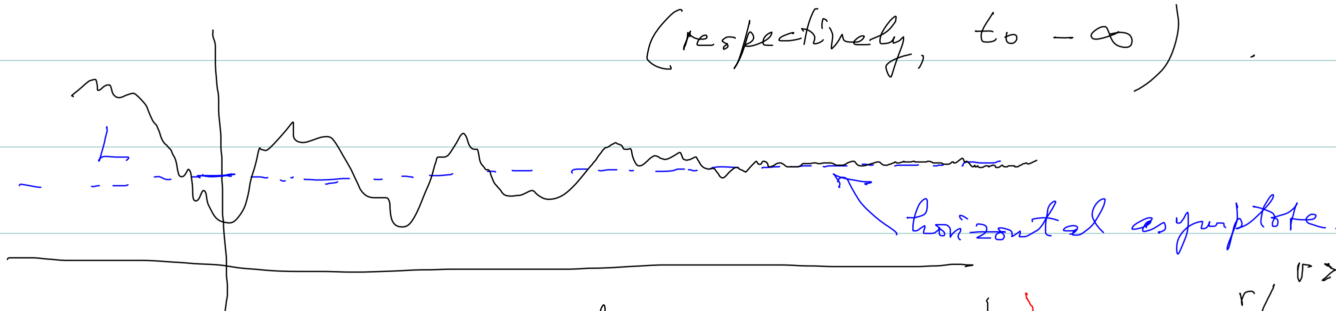
Ex $f(x) = x^2 \sin\frac{1}{x} \Rightarrow$ Find $\lim_{x \to 0} f(x)$ if it exists.

$$\underbrace{\lim_{x\to 0} x^2 \cdot (-1)}_{=0} \le \lim_{x\to 0} x^2 \underbrace{\sin\frac{1}{x}}_{-1 \le \sin\frac{1}{x} \le 1} \le \underbrace{\lim_{x\to 0} x^2 \cdot 1}_{=0}$$

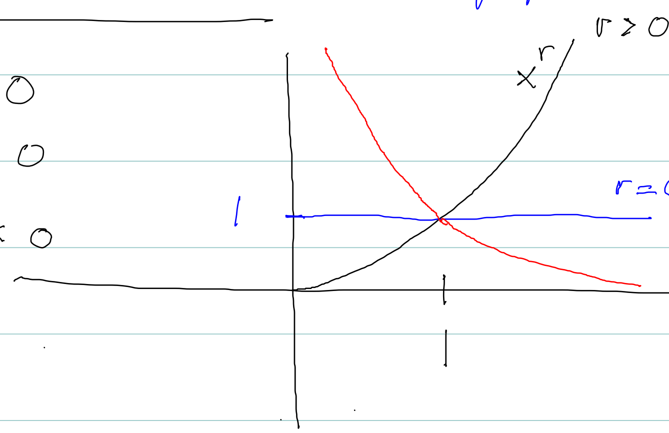
$\Rightarrow$ by squeeze Thm, $\lim_{x\to 0} x^2 \sin\frac{1}{x}$ exists, and equals 0.

Limits at infinity, horizontal asymptotes.

Def $\lim_{x \to \pm\infty} f(x) = L \iff f(x)$ approaches L with arbitrary precision as x grows to $+\infty$ (respectively, to $-\infty$).

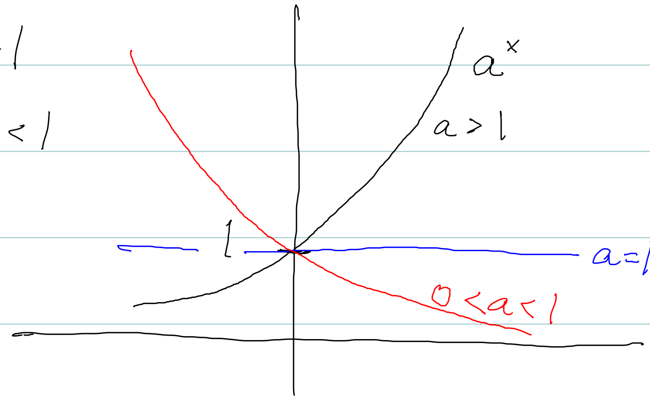


Ex $\lim_{x \to \infty} x^r = \begin{cases} +\infty & \text{if } r > 0 \\ 1 & \text{if } r = 0 \\ 0 & \text{if } r < 0 \end{cases}$



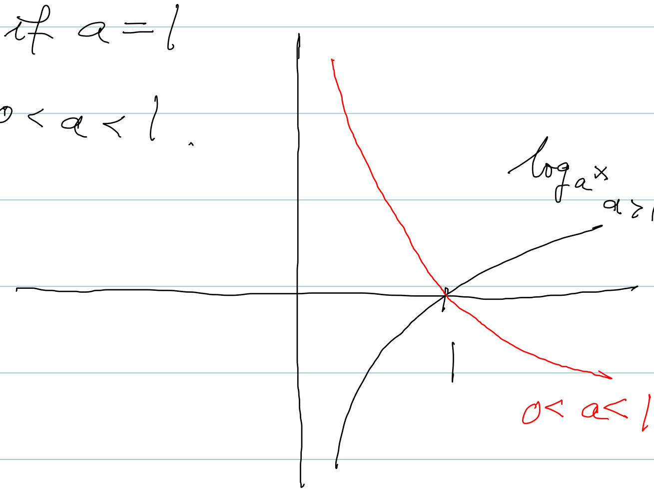
Ex $\lim_{x\to\infty} a^x = \begin{cases} +\infty & \text{if } a>1 \\ 1 & \text{if } a=1 \\ 0 & \text{if } 0<a<1 \end{cases}$

$a>0$



Ex $\lim_{x\to\infty} \log_a x = \begin{cases} +\infty & \text{if } a>1 \\ \text{undefined} & \text{if } a=1 \\ -\infty & \text{if } 0<a<1. \end{cases}$

$a>0$



$\underline{\underline{Ex}}$ $\lim_{x\to\infty} \dfrac{3x^3+2x^2+1}{4x^3-x} = \lim_{x\to\infty} \dfrac{\cancel{x^3}\left(3+\frac{2}{x}+\frac{1}{x^3}\right)}{\cancel{x^3}\left(4-\frac{1}{x^2}\right)}$

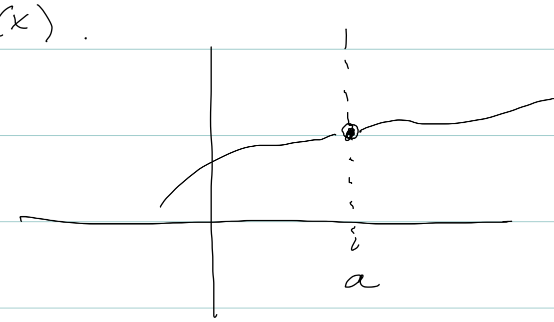
$$= \frac{\lim_{x\to\infty}\left(3+\frac{2}{x}+\frac{1}{x^3}\right)}{\lim_{x\to\infty}\left(4-\frac{1}{x^2}\right)} = \underline{\underline{\frac{3}{4}}}$$

Continuity of functions.

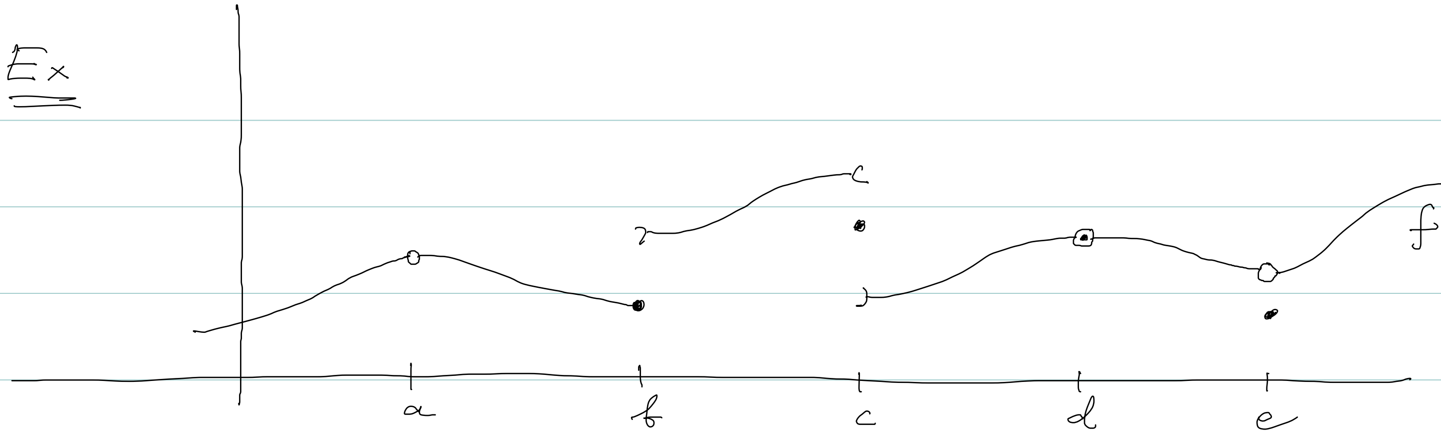
<u>Def</u>: f is continuous at $x = a$ if:

1) f has a fct value $f(a)$ at $x = a$.
2) f has a well-defined limit $\lim_{x \to a} f(x)$
3) they have to be equal,

$$f(a) = \lim_{x \to a} f(x).$$



Ex



	Left limit	Right limit	Limit	Fct value	Continuous
$x = a$:	Y	Y	Y	N	N
$x = b$:	Y	Y	N	Y	N
$x = c$:	Y	Y	N	Y	N
$x = d$:	Y	Y	Y	Y	Y
$x = e$:	Y	Y	Y	Y	N

Ex $f(x) = x^2 \sin \frac{1}{x}$

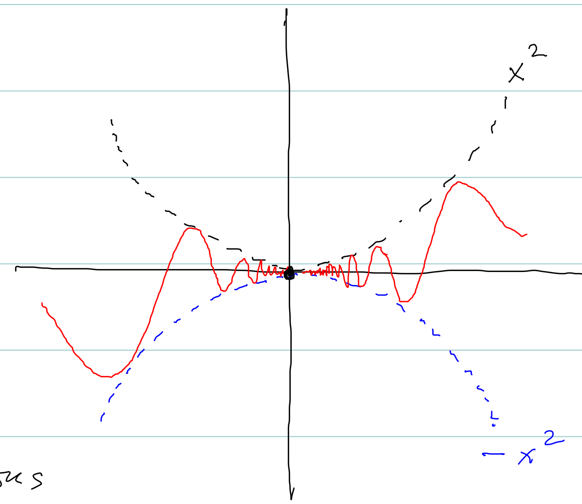
$$\lim_{x \to 0} f(x) = 0$$

but: f has no fct value at $x = 0$.

$\Longrightarrow$ not continuous at $x=0$.

Cook up a new fct.

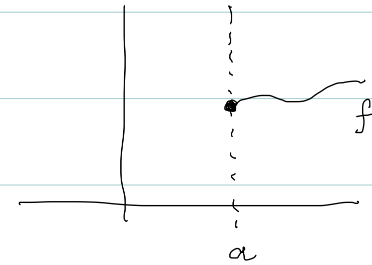
$$g(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$



$g(0) = 0 = \lim_{x \to 0} g(x)$ continuous at $x=0$.

↑ fct value

Left and right continuity

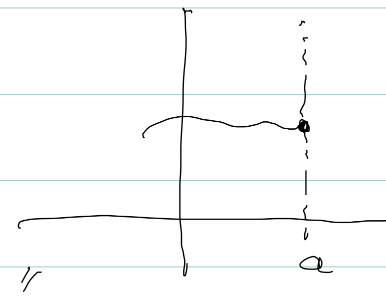


Def f is right continuous at $x = a$ if

$$f(a) = \lim_{x \to a^+} f(x).$$

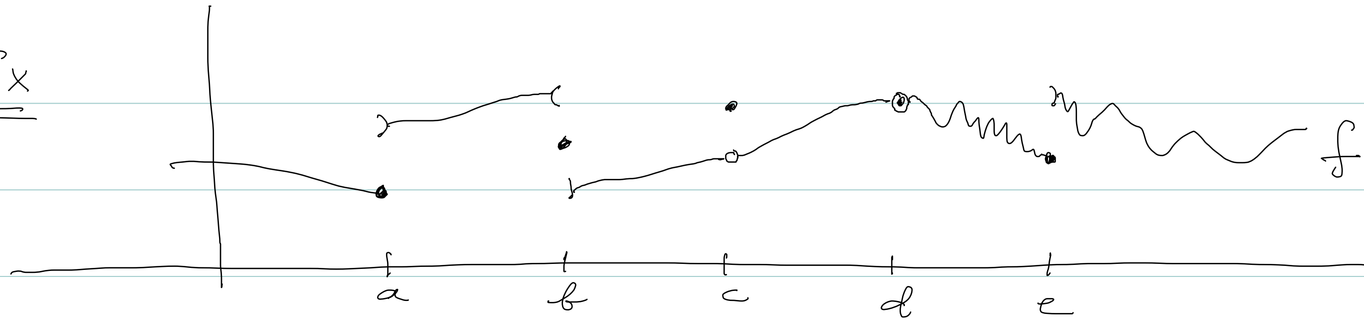
f is left continuous at $x = a$ if

$$f(a) = \lim_{x \to a^-} f(x).$$



" function value = $\begin{matrix}\text{right} \\ \text{left}\end{matrix}$ limit ".

Ex

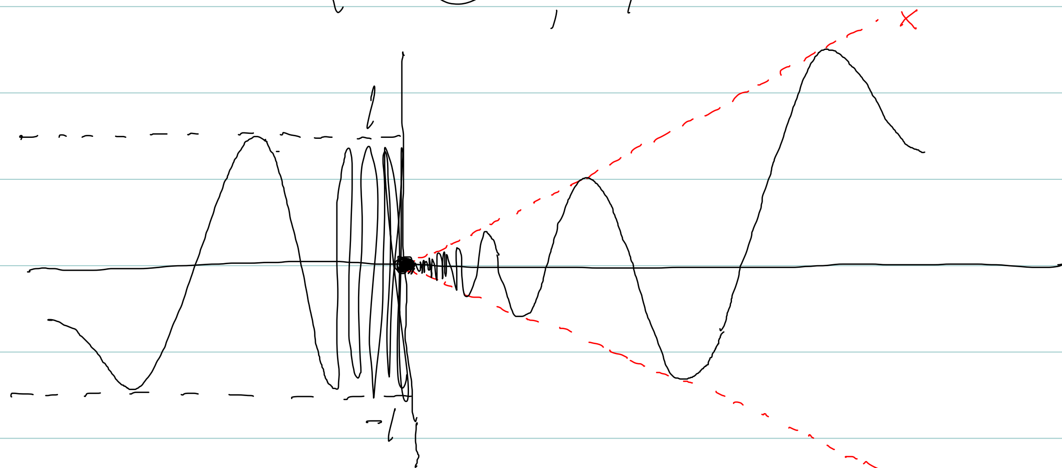


	left contin.	right contin.	continuous.
$x=a$	Y	N	N
$x=b$	N	N	N
$x=c$	N	N	N
$x=d$	Y	Y	Y
$x=e$	Y	N	N

Thm f is continuous at $x=a$ if and only if it is both left and right continuous at $x=a$.

$\underline{\text{Ex}}$:

$$f(x) = \begin{cases} \sin\frac{1}{x}, & \text{if } x < 0 \\ x \sin\frac{1}{x}, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \end{cases}$$



$\lim\limits_{x \to a^-} f(x)$ does not exist $\Rightarrow$ <u>not</u> left continuous at 0.

$\lim\limits_{x \to a^+} f(x) = 0 = f(0) \Rightarrow$ right continuous at 0.

<u>Not</u> continuous at 0.

Thm (continuity laws).

If f and g are continuous at $x=a$, then so are

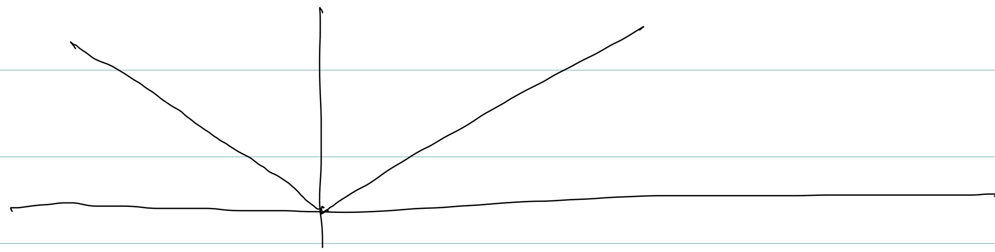
$$\alpha f \pm \beta g \quad (\alpha, \beta \text{ some numbers})$$

$$f \cdot g$$

$$\frac{f}{g}, \text{ if } g(a) \neq 0.$$

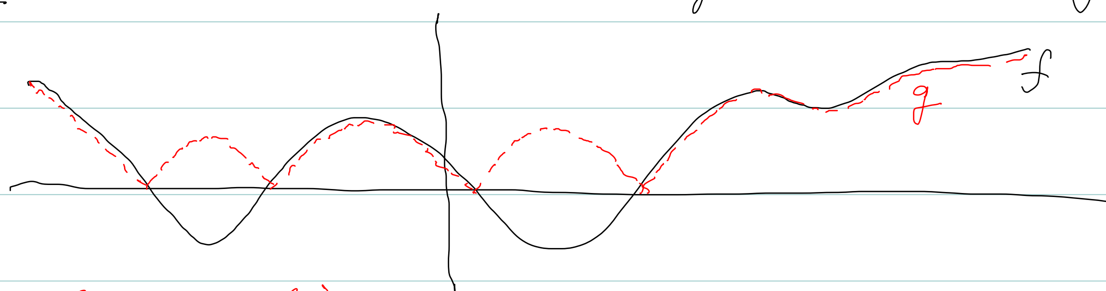
Also true with left / right continuous instead of continuous.

Ex $f(x) = |x|$ where is f continuous ?



continuous at $x=0$. $\Longrightarrow$ $|x|$ continuous for all x.

Ex Assume $f(x)$ continuous everywhere. Where is $g(x) = |f(x)|$ continuous?

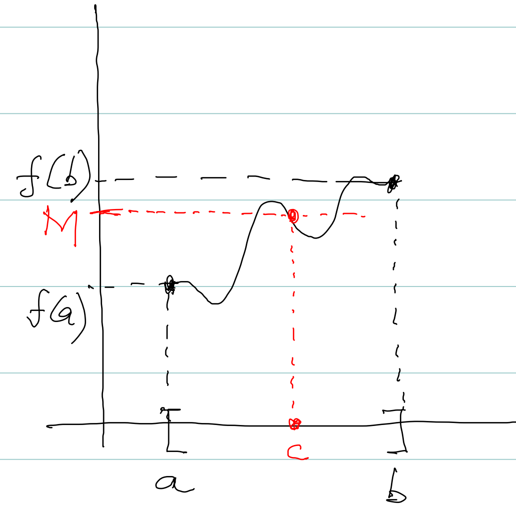


$g(x)$ is continuous everywhere.

Thm (intermediate value theorem)

Assume f is continuous on $[a, b]$.

Then, for any number M between $f(a)$ and $f(b)$, there is a number c in $[a, b]$ such that $f(c) = M$.



Note: This guarantees at least one such c, but there could be more.

Ex Show that the equation

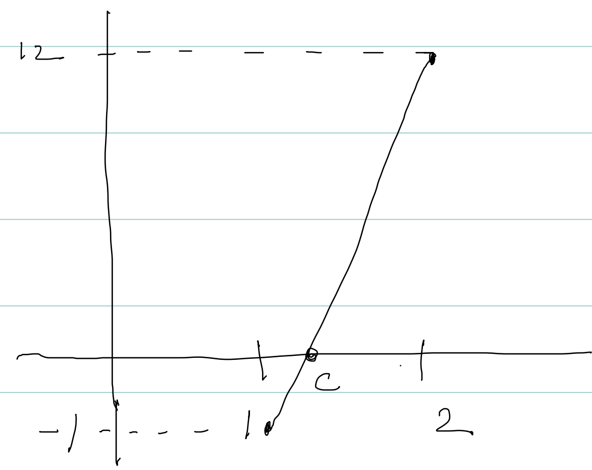
$$f(x) = 4x^3 - 6x^2 + 3x - 2 = 0$$

has a root between $x=1$ and $x=2$.

$$f(1) = 4 - 6 + 3 - 2 = -1$$

$$f(2) = 4\cdot 2^3 - 6\cdot 2^2 + 3\cdot 2 - 2 = 32 - 24 + 6 - 2$$

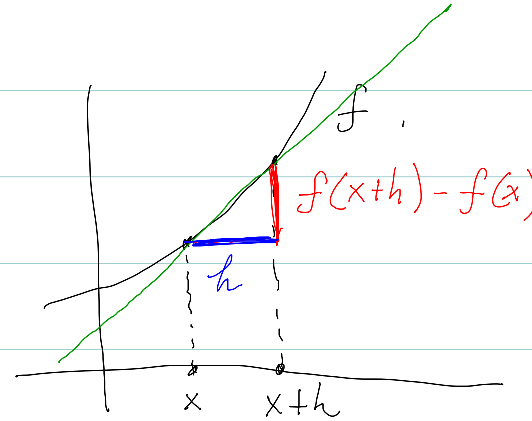
$$= 12$$



Left and right derivatives.

Recall: Derivative of f at x

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



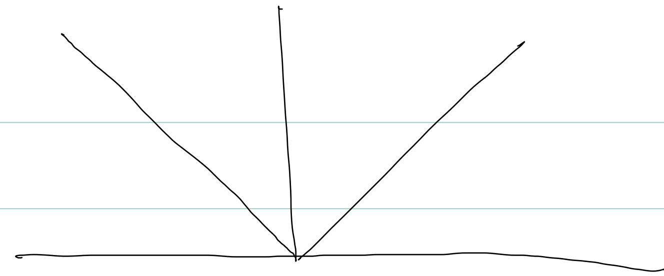
slope of tangent line at $f(x)$.

If $f'(x)$ exists, we say that f is differentiable at x.

Def Left derivative of f at x: $f'_-(x) = \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h}$

Right — " — : $f'_+(x) = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$

Ex $f(x) = |x|$



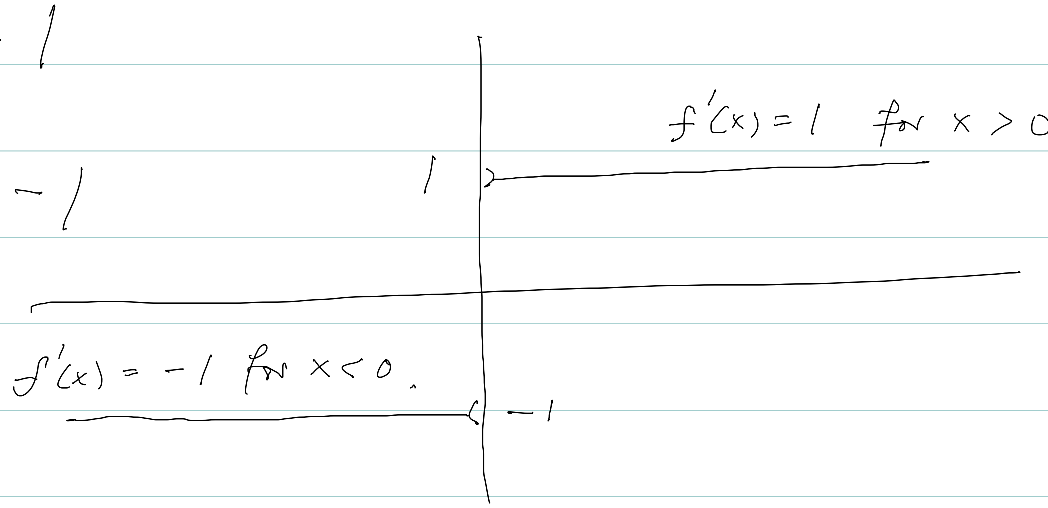
$x > 0 \implies f'(x) = 1$

$x < 0 \implies f'(x) = -1$

$x = 0 \implies f'$ does not exist.

$f'_+(0) = 1$

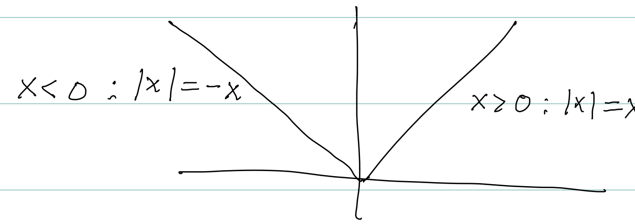
$f'_-(0) = -1$



Ex Check $f'_+(0)=1$ for $f(x)=|x|$.

$$f'_+(0)=\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}\frac{\cancel{h}-0}{\cancel{h}}=1$$

$h>0$ in right limit



$$f'_-(0)=\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}$$

$$=\lim_{h\to 0^-}\frac{-\cancel{h}-0}{\cancel{h}}=-1$$

$\underline{h<0:}$ $f(0+h)=f(h)=|h|$
$=-h$

Thm If f is differentiable at $x=a$, then f is continuous at $x=a$.

Ex Check this!

Need to check that $\lim_{x\to a} f(x) = f(a) \iff \lim_{x\to a} f(x) - f(a) = 0$

$\iff \lim_{h\to 0} f(\underbrace{a+h}_{x}) - f(a) = 0$ want to show this.

$$\lim_{h\to 0} f(a+h) - f(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} h$$

$$= \underbrace{\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}}_{f'(a)} \cdot \underbrace{\lim_{h\to 0} h}_{=0} . = 0$$

because f is assumed differentiable at $x=a$.

Derivatives: Product rule, quotient rule, chain rule.

Thm Assume f, g differentiable at x. Then,

Product rule: $\left(f(x) \cdot g(x)\right)' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

Quotient rule: $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)}$

$g(x) \neq 0$

Ex Check product rule.

$$\boxed{(f(x)\cdot g(x))'} = \lim_{h\to 0}\frac{f(x+h)\cdot g(x+h)-f(x)\cdot g(x)}{h}$$

$$= \lim_{h\to 0}\frac{(f(x+h)-f(x)+f(x))\cdot g(x+h)-f(x)\cdot g(x)}{h}$$

$$= \lim_{h\to 0}\frac{(f(x+h)-f(x))\cdot g(x+h)+f(x)\cdot g(x+h)-f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\frac{(f(x+h)-f(x))}{h}\cdot g(x+h) + \lim_{h\to 0} f(x)\frac{(g(x+h)-g(x))}{h}$$

$$= \left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right)\cdot \lim_{h\to 0} g(x+h) + f(x)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$\boxed{= f'(x)\cdot g(x)+f(x)\cdot g'(x)}$$

Ex $$(x e^x)' = \underbrace{1}_{f'(x)} \cdot \underbrace{e^x}_{g(x)} + \underbrace{x}_{f(x)} \cdot \underbrace{e^x}_{g'(x)}$$

$f(x) = x$

$g(x) = e^x$

$$= (1 + x) \cdot e^x$$

<u>Ex</u> $\left(\dfrac{1}{g(x)}\right)' = ?$

$$1 = g(x) \cdot \frac{1}{g(x)}$$ take derivative on both sides.

$$0 = \left(g(x) \cdot \frac{1}{g(x)}\right)' = g'(x) \cdot \frac{1}{g(x)} + g(x) \cdot \left(\frac{1}{g(x)}\right)'$$

$\Rightarrow$ solve for $\left(\dfrac{1}{g(x)}\right)'$ $\Rightarrow -\dfrac{g'(x)}{g(x)} = g(x) \cdot \left(\dfrac{1}{g(x)}\right)'$

$$\Rightarrow -\frac{g'(x)}{g^2(x)} = \left(\frac{1}{g(x)}\right)'$$

Ex Check quotient rule.

$$\boxed{\left(\frac{f(x)}{g(x)}\right)'} = \left(f(x)\cdot\frac{1}{g(x)}\right)' = f'(x)\cdot\frac{1}{g(x)} + f(x)\cdot\left(\frac{1}{g(x)}\right)'$$

$$= \frac{f'(x)}{g(x)} + f(x)\cdot\left(-\frac{g'(x)}{g^2(x)}\right)$$

$$= \boxed{\frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{g^2(x)}}$$

$\underline{\underline{Ex}}$ $$\left(\frac{e^x}{x}\right)' = \frac{e^x \cdot x - e^x \cdot 1}{x^2}$$

$$= \frac{e^x (x-1)}{x^2}$$

$\underline{\underline{Ex}}$ $$\left(\frac{x^2+1}{x-2}\right)' = \frac{2x \cdot (x-2) - (x^2+1) \cdot 1}{(x-2)^2}$$

$= \ldots$ simplify

Chain rule:

Thm Assume $g(x)$ is differentiable at x, and f differentiable at $g(x)$. Then,

$$\big(f(g(x))\big)' = f'(g(x)) \cdot g'(x)$$

Ex $f(x) = e^x$, $g(x) = x^2$

$f(g(x)) = e^{x^2}$

$(f(g(x)))' = (e^{x^2})'$

$$= \underbrace{e^{x^2}}_{f'(g(x))} \cdot \underbrace{2x}_{g'(x)}.$$

$f'(x) = e^x$

$g'(x) = 2x$

$\underline{\underline{Ex}}$ $(\ln x)' = ?$

$x = e^{\ln x}$ take derivative on both sides.

$= f(g(x))$ where $g(x) = \ln x$, $f(x) = e^x$

$$1 = (e^{\ln x})' = \underbrace{e^{\ln x}}_{f'(g(x))} \cdot \underbrace{(\ln x)'}_{g'(x)}$$

$$= x \cdot (\ln x)'$$

$$\Rightarrow \frac{1}{x} = (\ln x)'$$

$\underline{\underline{Ex}}$ $(x^x)' = ?$

$$x^x = (\underbrace{e^{\ln x}}_{x})^x = e^{x \cdot \ln x} = f(g(x)).$$

$f(x) = e^x$

$g(x) = x \cdot \ln x.$

$$(x^x)' = (e^{x \ln x})' = \underbrace{e^{x \cdot \ln x}}_{f'(g(x))} \cdot \underbrace{(x \cdot \ln x)'}_{g'(x)}$$

$$= x^x \cdot \left(1 \cdot \ln x + x \cdot \frac{1}{x}\right).$$

$$= x^x \cdot (\ln x + 1)$$

Ex $(a^x)' = ?$ $\qquad a > 0$

$$a^x = (\underbrace{e^{\ln a}}_{a})^x = e^{x \cdot \ln a} = f(g(x))$$

$f(x) = e^x$, $g(x) = x \cdot \ln a$

$$(a^x)' = (e^{x \cdot \ln a})' = \underbrace{e^{x \cdot \ln a}}_{f'(g(x))} \cdot \underbrace{(x \cdot \ln a)'}_{g'(x)}.$$

$$= a^x \cdot \ln a$$

<u>Note:</u> $(x \cdot \beta)' = 1 \cdot \beta$.

(where $\beta = \ln a$).

Ex $(x^r)' = ?$, $r \in (-\infty, \infty)$

$$x^r = \left(e^{\ln x}\right)^r = e^{r \cdot \ln x} = f(g(x))$$

$f(x) = e^x$, $g(x) = r \cdot \ln x$.

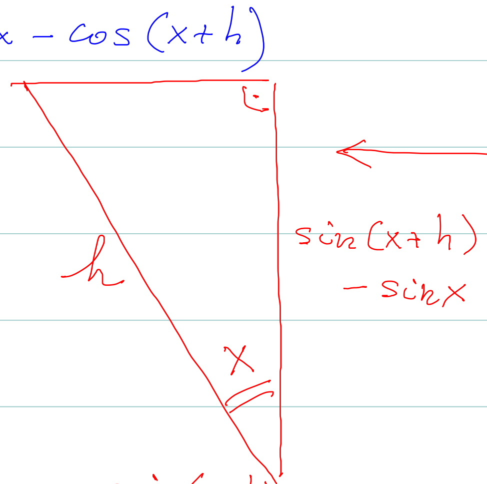
$$(x^r)' = \left(e^{r \cdot \ln x}\right)' = \underbrace{e^{r \cdot \ln x}}_{f'(g(x))} \cdot \underbrace{(r \cdot \ln x)'}_{g'(x)}$$

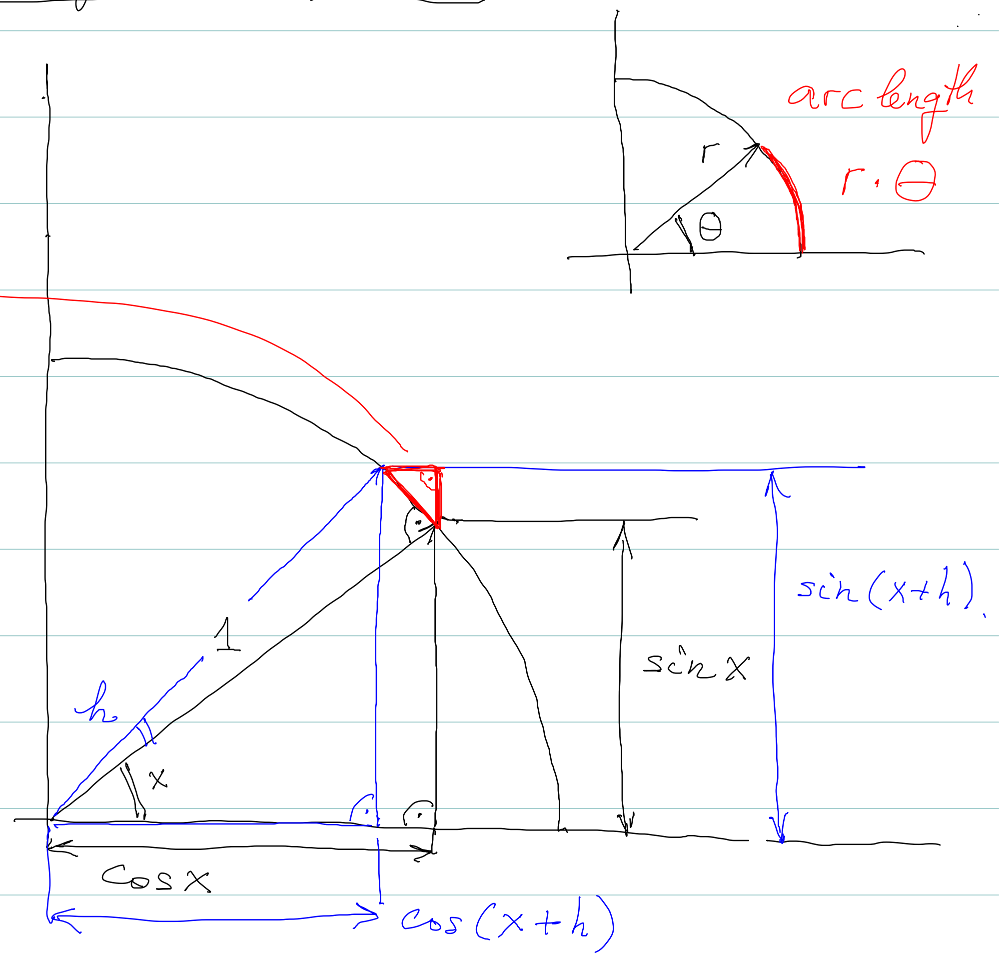
$$= x^r \cdot \frac{r}{x} = x^r \cdot r x^{-1}$$

$$= r x^{r-1}$$

Derivatives of trigonometric functions

<u>Ex</u>: $(\sin x)' = ?$





$$\cos x = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= (\sin x)'$$

$$\sin x = \lim_{h \to 0} \frac{\cos x - \cos(x+h)}{h}$$

$$= -(\cos x)'$$

Pythagoras: $\cos^2 x + \sin^2 x = 1$.

Ex $\tan x = \dfrac{\sin x}{\cos x}$

$$(\tan x)' = \frac{(\sin x)' \cdot \cos x - \sin x \cdot (\cos x)'}{\cos^2 x}$$

$$= \frac{\cos^2 x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}.$$

Ex $f(x) = e^{rx} = f(g(x))$, $f(x) = e^x$, $g(x) = r \cdot x$

$$f'(x) = \underbrace{e^{rx}}_{f'(g(x))} \cdot \underbrace{r}_{g'(x)} = r f(x)$$

$$f(0) = 1$$

Thm If f satisfies $f(0) = 1$ and $f'(x) = r f(x)$ then $f(x) = e^{rx}$

<u>Ex</u> $i^2 = -1$, i imaginary unit.

Look at $f(x) = \cos x + i \cdot \sin x$

$$f'(x) = -\sin x + i \cos x$$

$$= i(\cos x + i \sin x) = i f(x)$$

$$f(0) = \cos 0 + i \sin 0 = 1$$

$$\Rightarrow f(x) = e^{ix} = \cos x + i \sin x$$

Euler's formula.

Ex $\quad e^{i\alpha} \cdot e^{i\beta} = e^{i(\alpha+\beta)}$

$$(\cos\alpha + i\sin\alpha)\cdot(\cos\beta + i\sin\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta + i(\sin\alpha\cdot\cos\beta + \cos\alpha\cdot\sin\beta)$$

$$= \cos(\alpha+\beta) + i\sin(\alpha+\beta)$$

$$\left.\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta\\ \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\cdot\sin\beta\end{aligned}\right\}\ \text{trigonometric identities.}$$

Why is the chain rule correct? Recall: $\left(f(g(x))\right)' = f'(g(x))\,g'(x)$

Check: $$\left(f(g(x))\right)' = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

$$= \lim_{h\to 0} \frac{f\left(g(x) + \frac{g(x+h)-g(x)}{h}\cdot h\right) - f(g(x))}{h}$$

$$= \lim_{h\to 0} \frac{f(g(x) + g'(x)\cdot h) - f(g(x))}{\underbrace{g'(x)\cdot h}_{H = g'(x)\cdot h}} \cdot g'(x)$$

$$= \lim_{H\to 0} \frac{f(g(x)+H) - f(g(x))}{H} \cdot g'(x)$$

$$= f'(g(x)) \cdot g'(x)$$

Ex True or False:

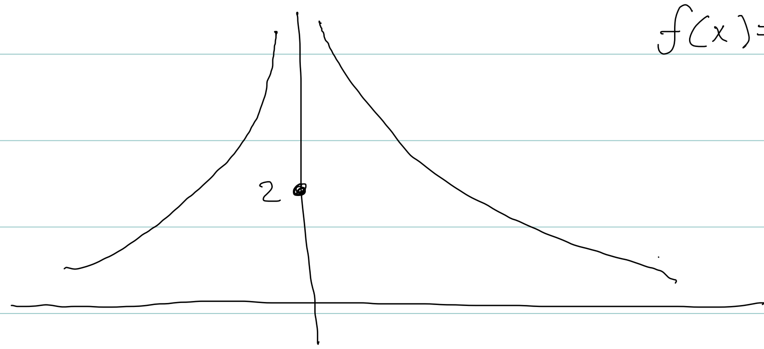
If $\lim_{x \to 6} \left(f(x) \cdot g(x) \right)$ exists, then it equals $f(6) \cdot g(6)$

(Only true if both f, g are continuous at $x=6$)

Ex T or F:

If the line $x=0$ is a vertical asymptote of $f(x)$, then f is not defined at $x=0$

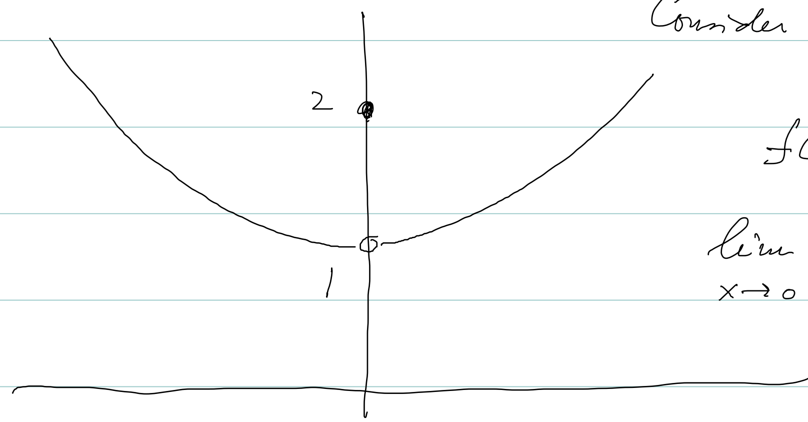
$$f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$$



Ex T or F:

If $f(x) > 1$ for all x, and $\lim_{x \to 0} f(x)$ exists,

then $\lim_{x \to 0} f(x) > 1$.



Consider $f(x) = \begin{cases} 1 + x^2 & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$

$f(x) > 1$

$\lim_{x \to 0} f(x) = 1$.

Ex T or F ?

If $p(x)$ is a polynomial, then $\lim_{x \to a} p(x) = p(a)$.

(Polynomial: $p(x) = 4x^6 + 2x^5 + 10x^3 + 2x + 1$

or $p(x) = 12x^{100} + 15x^{97} + 3x^2 + x + 2$

or $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$)

A polynomial is continuous everywhere.

$\underline{\underline{Ex}}$ $\lim_{x \to 0} \dfrac{\sqrt[3]{1+x} - \sqrt[3]{1}}{x}$

Painful way: $t := \sqrt[3]{1+x}$, $t \to 1$ when $x \to 0$

$t^3 = 1+x \implies x = t^3 - 1$

$\implies \dots = \lim_{t \to 1} \dfrac{t-1}{t^3-1} = \lim_{t \to 1} \dfrac{\cancel{t-1}}{(t^2+t+1)\cancel{(t-1)}} = \dfrac{1}{3}$.

Long division: $t^3 + 0 \quad -1 \div (t-1) = t^2 + t + 1$

$$\begin{array}{l} t^3 - t^2 \\ \hline \quad t^2 + 0 \\ \quad t^2 - t \\ \hline \qquad t - 1 \end{array}$$

Painfree way: $\lim_{x\to 0} \frac{\sqrt[3]{1+x} - \sqrt[3]{1}}{x} = \lim_{h\to 0} \frac{\sqrt[3]{1+h} - \sqrt[3]{1}}{h}$.

$$f(x) = \sqrt[3]{1+x} \qquad = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$$= f'(0) = \frac{1}{3}.$$

$$f(x) = (1+x)^{1/3}$$

$$f'(x) = \frac{1}{3}(1+x)^{-2/3} \cdot 1$$

$$f'(0) = \frac{1}{3}.$$

Ex $\left(2^x \cdot \ln x\right)' = (2^x)' \cdot \ln x + 2^x \frac{1}{x}$

$= 2^x \cdot \ln 2 \cdot \ln x + \frac{2^x}{x}$

Remember: $2^x = e^{x \ln 2}$ (because $2^x = \left(e^{\ln 2}\right)^x = e^{x \ln 2}$)

Ex $f(x) = \dfrac{3x^3 + 2x^2 + x + 1}{x^2+2}$

$$f'(x) = \frac{(9x^2+4x+1)(x^2+2) - (3x^3+2x^2+x+1)\cdot 2x}{(x^2+2)^2}$$

$= \dots = \text{(simplify)}.$

Ex $\lim_{x\to 1} \frac{x^2-1}{|x-1|}$. does it exist? If yes, what is it?

Do left and right limit separately.

$$\lim_{x\to 1^+} \frac{x^2-1}{|x-1|} = \lim_{x\to 1^+} \frac{x^2-1}{x-1}$$

because $|x-1| = x-1$ for $x>1$

$$= \lim_{x\to 1^+} \frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}} = 2$$

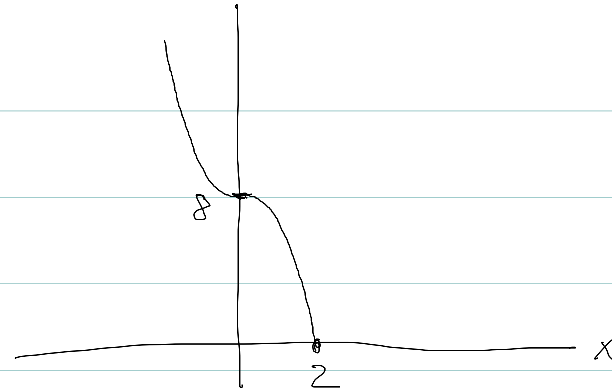
$$\lim_{x\to 1^-} \frac{x^2-1}{|x-1|} = \lim_{x\to 1^-} \frac{x^2-1}{-(x-1)}$$

because $|x-1| = -(x-1)$ for $x<1$

$$= \lim_{x\to 1^-} \frac{(x+1)\cancel{(x-1)}}{-\cancel{(x-1)}} = -2$$

Limit does not exist.

Ex $\lim_{x \to 2^-} \sqrt{8 - x^3}$



$\lim_{x \to 2^-} \sqrt{\underbrace{8 - x^3}_{>0}} = 0$

$\lim_{x \to 2^+} \sqrt{8 - x^3}$ does not exist.

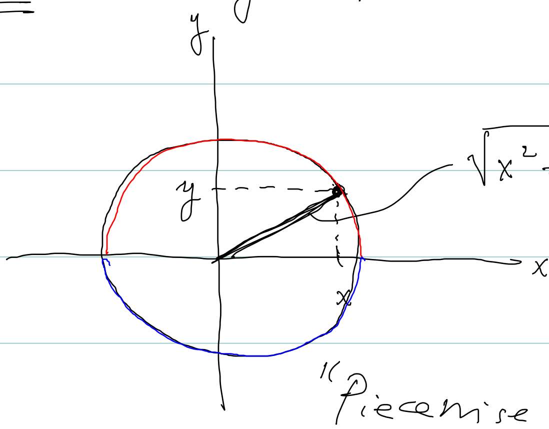
$\lim_{x \to -2} \sqrt{8 - x^3} = 4$

Implicit differentiation

Sometimes, "solving y for x" is difficult.

Ex $x^2+y^2=9$ circle of radius 3

$\Rightarrow y^2 = 9 - x^2$



$\sqrt{x^2+y^2}=3$

$\Rightarrow y = \pm\sqrt{9-x^2}$

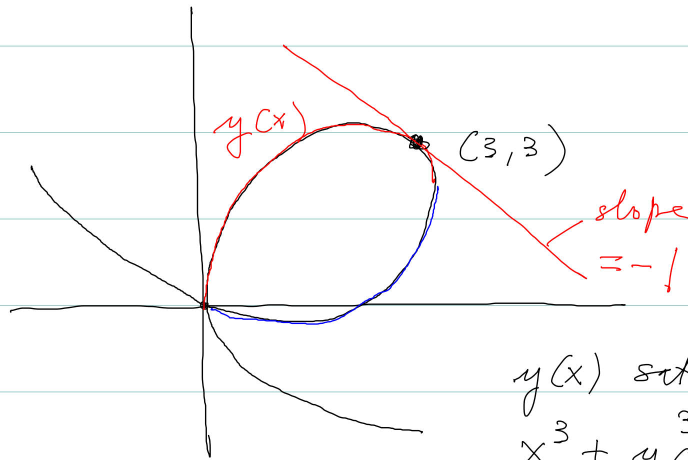
The circle is the combination of graphs of 2 functions.

"Piecewise representable as a graph".

$\underline{\underline{Ex}}$ $x^3 + y^3 = 6x \cdot y$

For instance $(3,3)$ lies on this curve:

$$3^3 + 3^3 = \underbrace{6 \cdot 3 \cdot 3}_{2 \cdot 3 \cdot 3 \cdot 3}$$



what is the slope of the tangent line at $(3,3)$? $\Rightarrow y'(3) = ?$

$y(x)$ satisfies

$$x^3 + y(x)^3 = 6x \cdot y(x).$$

Differentiate both sides in x:

$$3x^2 + 3y(x)^2 \cdot y'(x) = 6y(x) + 6x \cdot y'(x).$$

(note: use chain rule)

Solve for $y'(x)$:

$$3y(x)^2 \cdot y'(x) - 6x \cdot y'(x) = 6y(x) - 3x^2$$

$$(3y(x)^2 - 6x) \cdot y'(x) = 6y(x) - 3x^2$$

$$\Rightarrow \quad y'(x) = \frac{6y(x) - 3x^2}{3y(x)^2 - 6x} \Rightarrow y'(3) = \frac{6 \cdot 3 - 3 \cdot 3^2}{3 \cdot 3^2 - 6 \cdot 3} = -1$$

Ex Find $y'(x)$ if

$$\sin(x^2+y^2) = y \cdot \cos x .$$

Differentiate in x:

$$\left(\cos(x^2+y^2)\right) \cdot (2x + 2y \cdot y') = y' \cdot \cos x - y \cdot \sin x$$

$$\cos(x^2+y^2) \cdot 2y \cdot y' - y' \cdot \cos x = -y \cdot \sin x - \cos(x^2+y^2) \cdot 2x$$

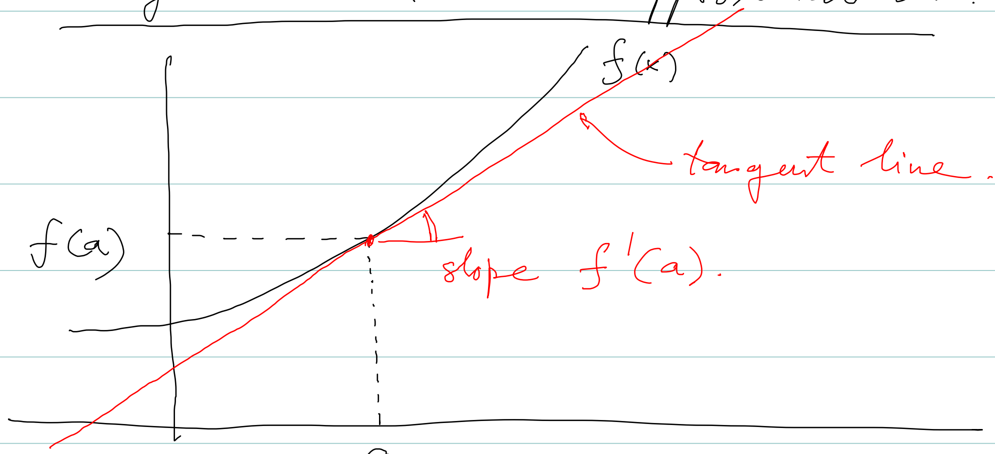
$$\Rightarrow \left(\cos(x^2+y^2) \cdot 2y - \cos x\right) \cdot y' = -y \cdot \sin x - \cos(x^2+y^2) \cdot 2x$$

$$\Rightarrow y' = \frac{-y \cdot \sin x - \cos(x^2+y^2) \cdot 2x}{\cos(x^2+y^2) \cdot 2y - \cos x}$$

Observe: $(0,0)$ lies on this curve.

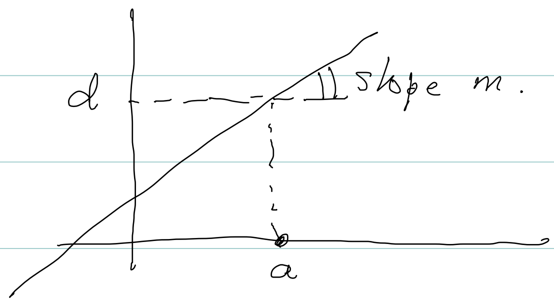
$$\Rightarrow y'(0) = \frac{-0 \cdot \sin 0 - \cos 0 \cdot 2 \cdot 0}{\cos 0 \cdot 2 \cdot 0 - \cos 0} = \frac{0}{-1} = 0 .$$

Tangent line & linear approximation.



Formula for the tangent line: $y = f'(a)\cdot(x-a) + f(a)$

Recall formula for a straight line.

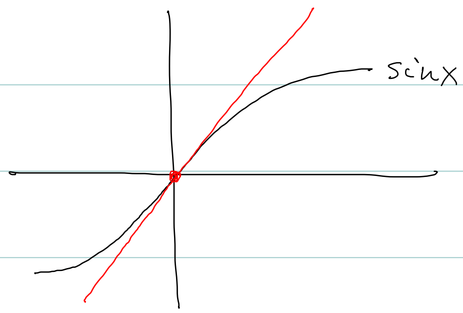


$$y = m(x-a) + d$$

Def The function $y(x) = f'(a)\cdot(x-a) + f(a)$ is called the linearization (= linear approximation) of f at a.

For x close to a, the linearization is a good approx. to $f(x)$.

Ex Find the linearization of $\sin x$ at 0.



$\sin 0 = 0$ $\qquad a = 0$

$\sin' 0 = \cos 0 = 1$

$y = 1\cdot(x-0) + 0 = x$

Ex Calculate $\sqrt{4.02}$ without a calculator.

$f(x) = \sqrt{x}$

$x = 4.02$

pick a: close to x; $f(a)$, $f'(a)$ easy to determine.

$a = 4$. $\quad f(4) = \sqrt{4} = 2$, $f'(4) = \dfrac{1}{2\sqrt{4}} = \dfrac{1}{4}$.

$\Rightarrow$ linearization:

$$y = \frac{1}{4}(\underset{4.02}{\underset{\shortparallel}{x}} - 4) + 2 = \underbrace{\frac{1}{4} \cdot 0.02}_{0.005} + 2 \doteq 2.005$$

Ex Calculate $\ln 3$ without a calculator.

(with $e = 2.72$)

$f(x) = \ln x$

$x = 3$

$a = e \sim 2.72$

$f(a) = \ln e = 1$, $f'(a) = \frac{1}{e} \approx \frac{1}{2.72}$.

Linearization: $y = \frac{1}{2.72}(\underset{\underset{3}{\shortparallel}}{x} - 2.72) + 1$.

$$= \underbrace{\frac{1}{2.72} \cdot 0.28}_{\approx 0.1} + 1. \approx 1.1$$

Ex: $e^{1.1}$ in linear approximation.

$f(x) = e^x$, $x = 1.1$, $a = 1$

$f(a) = e^1 = e$, $f'(a) = e^1 = e$.

Linearization

$$y = e(\underset{1.1}{x} - 1) + e = \underbrace{2.72 \cdot 0.1}_{0.272} + 2.72 = 2.992$$

Ex Find $3^{3.01}$ in linear approximation.

$$f(x) = 3^x, \quad f'(x) = 3^x \cdot \ln 3$$

$$x = 3.01, \quad a = 3.$$

$$f(a) = 3^3 = 27, \quad f'(a) = 27 \cdot \ln 3$$

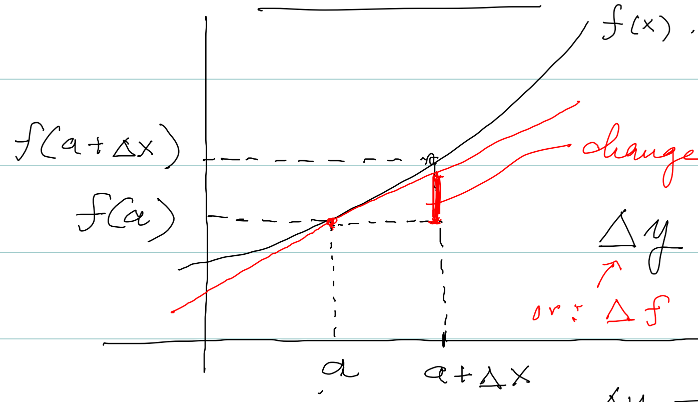
Linearization

$$y = 27 \cdot \ln 3 (\underset{\substack{\shortparallel \\ 3.01}}{x} - 3) + 27$$

$$\approx 27 \cdot 1.1 \cdot 0.01 + 27$$

$$= 29.7 \cdot 0.01 + 27 = 0.297 + 27 = 27.297$$

Differentials.



change of value of f in linear approximation.

$$\Delta y = \underbrace{f'(a)\left(\cancel{a}+\Delta x - \cancel{a}\right) + \cancel{f(a)}}_{\text{linearization}} - \cancel{f(a)}$$

or: Δf

$$\Delta y = f'(a)\cdot \Delta x$$ <u>differential</u>

Δy is, in linear approximation, the change of f when x changes by Δx.

Ex Find ln(1.02) in linear approximation, and determine the differential.

$$f(x) = \ln x, \quad f'(x) = \frac{1}{x}$$

$$x = 1.02, \quad a = 1, \quad f(a) = \ln 1 = 0, \quad f'(a) = \frac{1}{1} = 1$$

Linearization: $y = 1 \cdot (\underset{1.02}{\underset{\shortparallel}{x}} - 1) + 0 = 1 \cdot 0.02 = 0.02$

coincidentally equal because $f(a) = \ln 1 = 0$

Differential: $\Delta x = 0.02$

$$\Delta y = f'(a) \cdot \Delta x = 1 \cdot 0.02 = 0.02$$

Note: Notation for derivatives.

$$f'(x), \quad \frac{df}{dx}$$

Ex $\ln 8$ in linear approximation, and differential ?

$f(x) = \ln x$, $f'(x) = \frac{1}{x}$ | $f(a) = \ln e^2 = 2$

$x = 8$, $a = e^2$, use $e = 2.72$ | $f'(a) = \frac{1}{e^2} \approx \frac{1}{7.32}$

$e^2 = (2.72)^2$ use linearization: $g(x) = x^2$, $g'(x) = 2x$.

$a = 3$, $x = 2.72$.

$g(a) = 3^2 = 9$, $g'(a) = 2 \cdot 3 = 6$

Linearization: $y = 6 \cdot (\underset{2.72}{\underset{\shortparallel}{x}} - 3) + 9$

$= 6(-0.28) + 9$

$= -1.68 + 9 = 7.32 \approx e^2$

Linearization of ln: $y = \frac{1}{e^2}(x - e^2) + 2 \approx \frac{1}{7.32}(\underbrace{8 - 7.32}_{0.68}) + 2$

$= \frac{0.68}{7.32} + 2 \approx 0.09 + 2 = 2.09$

Related rates.

Q: How does one quantity change when another quantity that it depends on changes?

Ex: Ball-shaped balloon of volume $V = \frac{4\pi}{3} r^3$, r = radius.

When air is pumped so that

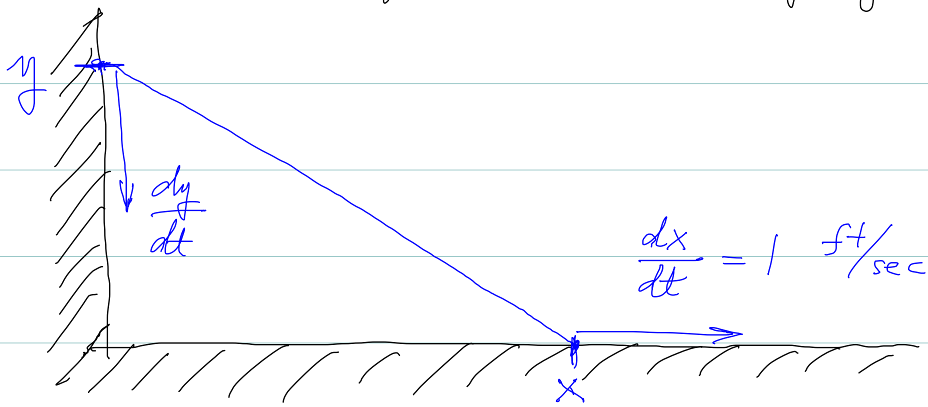
$$\frac{dV}{dt} = V'(t) = 100 \ \text{cm}^3/\text{sec}$$

then, what is the rate of change per sec of the radius, once $r = 25$ cm? Want to know $\frac{dr}{dt} = \frac{100}{4\pi(25)^2} = \frac{1}{25\pi}$ cm/sec.

$\Rightarrow$ differentiate volume formula $\Rightarrow$ $\frac{dV}{dt} = \frac{4\pi}{3} 3 \cdot r^2 \cdot \frac{dr}{dt}$ (chain rule).

$$r = 25\text{cm} \Rightarrow \underbrace{\frac{dV}{dt}}_{=100} = 4\pi \cdot (25)^2 \cdot \frac{dr}{dt}$$

$\underline{\text{Ex}}$ A ladder of length 10 ft standing against a wall.



What is $\frac{dy}{dt}$ when $x = 6$ ft ?

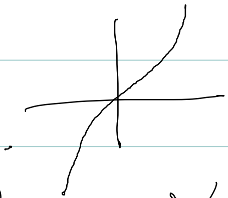
$$x^2 + y^2 = 10^2 = 100.$$

derivative: $\cancel{2}x \cdot \frac{dx}{dt} + \cancel{2}y \cdot \frac{dy}{dt} = 0$

$\Rightarrow \frac{dy}{dt} = -\frac{x}{y} \cdot \frac{dx}{dt}$ when $x = 6$ ft $\Rightarrow y = \sqrt{\underbrace{100-36}_{64}} = 8$ ft.

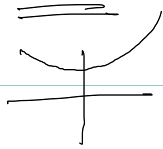
$$= -\frac{6}{8} \cdot \frac{dx}{dt} = -\frac{3}{4} \text{ ft/sec}.$$

Hyperbolic functions.



Def

$$\sinh x = \frac{e^x - e^{-x}}{2}$$



$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\cosh(-x) = \cosh(x)$$

$$\sinh(-x) = -\sinh(x)$$

Similar to trigon. fcts, because:

Euler's formula, $i^2 = -1$

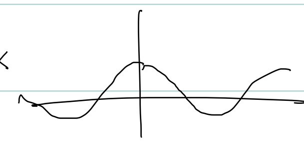
$$e^{ix} = \cos x + i \sin x$$

$$e^{-ix} = \cos(-x) + i\sin(-x)$$

$$= \cos x - i \sin x$$

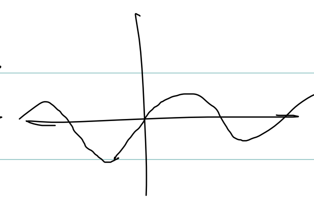
$$\Rightarrow e^{ix} + e^{-ix} = 2\cos x$$

$$\Rightarrow \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

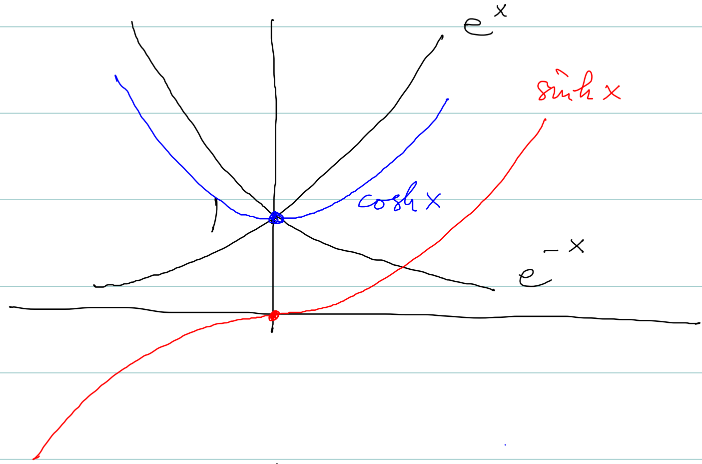


$$\Rightarrow e^{ix} - e^{-ix} = 2i \sin x$$

$$\Rightarrow \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$



$$(\sinh x)' = \cosh x \quad , \quad (\cosh x)' = \sinh x .$$



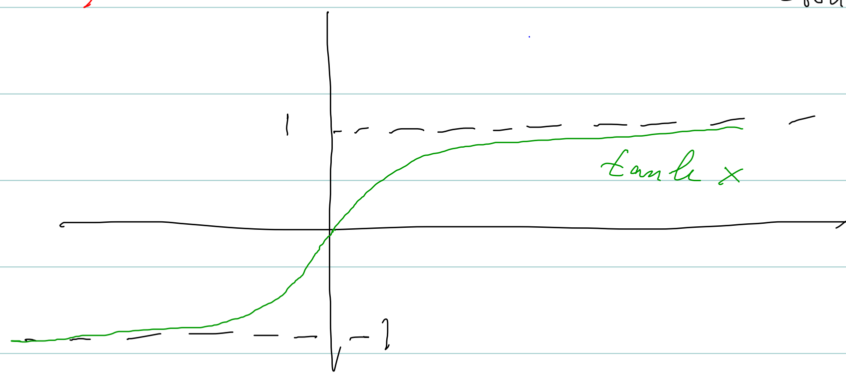
$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\sinh(-x) = \frac{e^{-x} - e^{x}}{2} = -\frac{e^x - e^{-x}}{2} = -\sinh x$$

odd.

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$\cosh(-x) = \dots = \cosh x$ even.



Thm $\sinh^{-1}x = \ln\left(x+\sqrt{x^2+1}\right)$. $\qquad \sinh^{-1}(\sinh x) = x$

$\cosh^{-1}x = \ln\left(x+\sqrt{x^2-1}\right)$. $\qquad \sinh(\sinh^{-1}x) = x$

$\tanh^{-1}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$.

Check: $\sinh y = \frac{1}{2}(e^y - e^{-y}) = x$ solve y for x.

$e^y - e^{-y} = 2x \implies e^y - 2x - e^{-y} = 0$

$\implies (e^y)^2 - 2x\cdot e^y - 1 = 0$ quadratic eq for e^y.

Writing $Y = e^y$, this means $Y^2 - 2x\cdot Y - 1 = 0$

$Y = -\frac{-2x}{2} \pm \frac{1}{2}\sqrt{4x^2 - 4(-1)}$ quadratic formula.

$= x \pm \frac{1}{2}\sqrt{4x^2+4} = x \pm \sqrt{x^2+1}$

+ solution for $Y = e^y > 0$

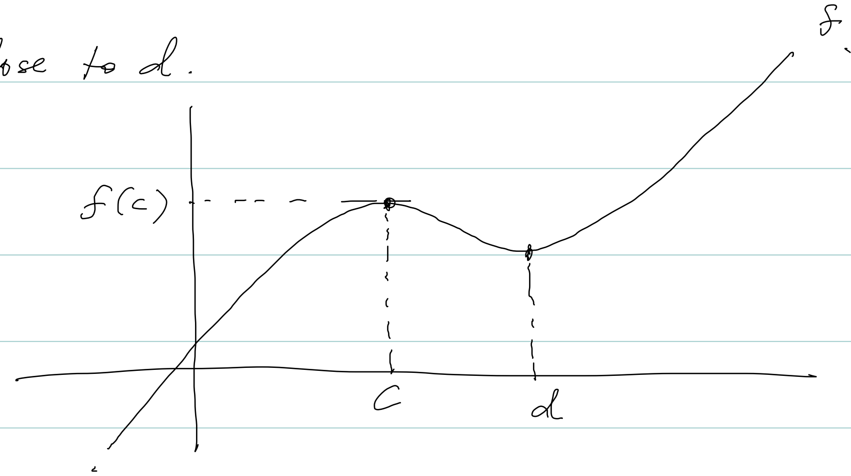
$\implies e^y = x + \sqrt{x^2+1} \implies y = \ln\left(x+\sqrt{x^2+1}\right) = \sinh^{-1}x$

Applications of differentiation.

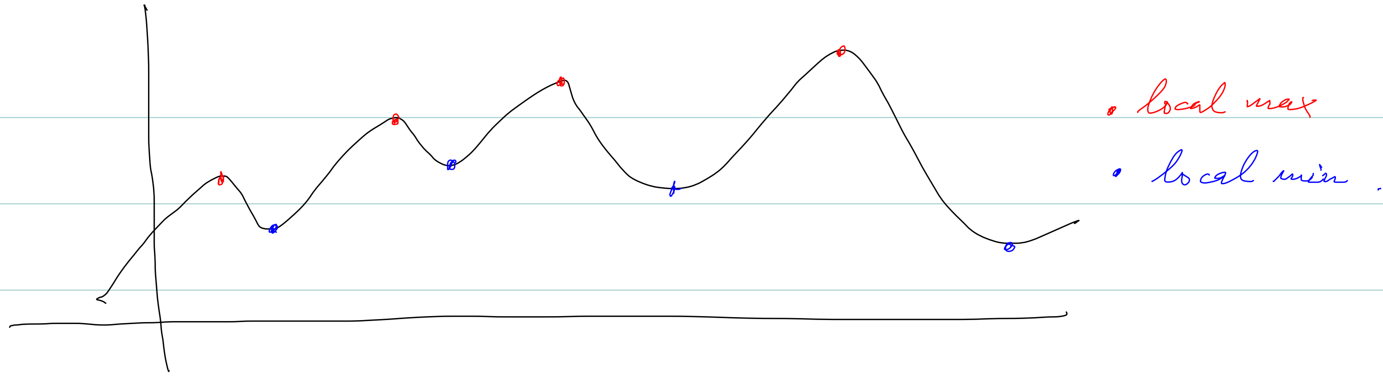
Maximum and minimum values.

Def f has a local max at c if $f(c) \geq f(x)$ for all x sufficiently close to c.

f has a local min at d if $f(d) \leq f(x)$ for all x sufficiently close to d.



Ex

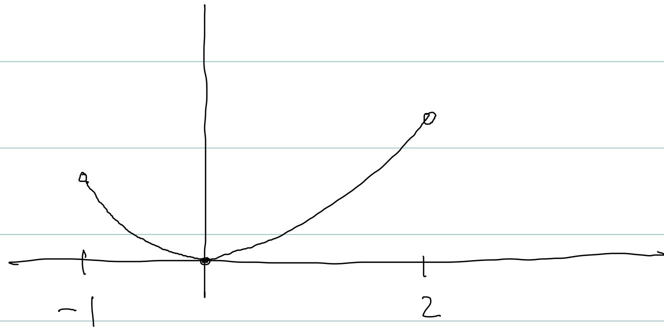


Def: A point c in the domain D of f is a global max if $f(c) \geq f(x)$ for all x in D.

A point d in D is a global min if $f(d) \leq f(x)$ for all x in D.

(global min/max = absolute min/max).

Ex $f(x) = x^2$, for $-1 < x < 2$.

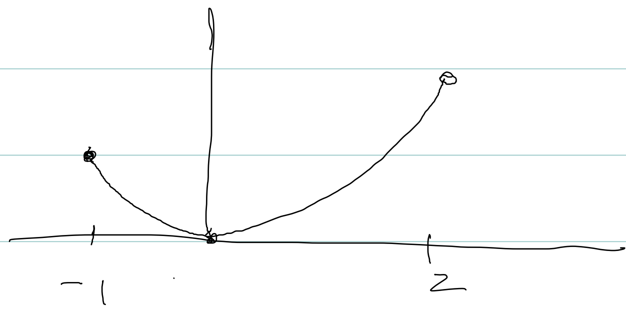


local, global min at 0

no local max

no global max.

Ex $f(x) = x^2$, for $-1 \le x < 2$

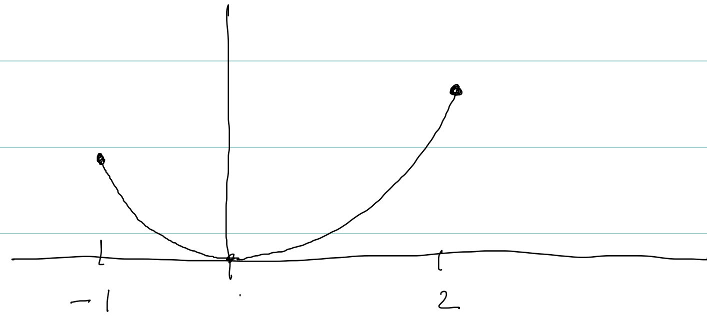


local, global min at 0

local max at -1

no global max.

Ex $f(x) = x^2$, for $-1 \le x \le 2$

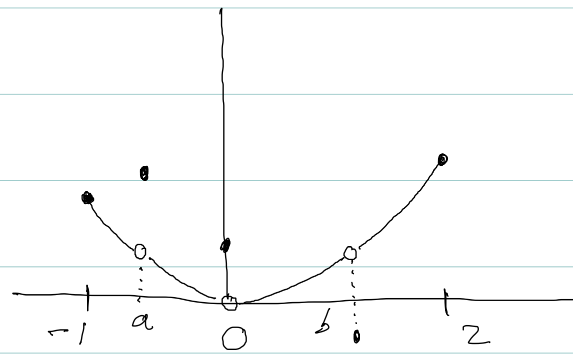


local, global min at 0

local max at -1

local, global max at 2.

Ex



local max at -1

local max at a.

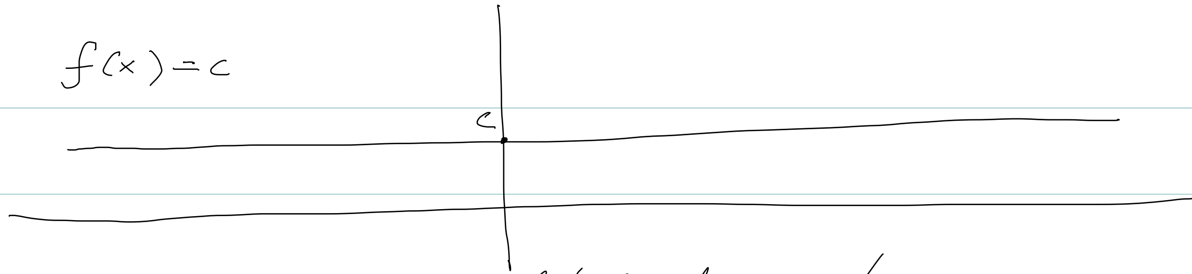
local max at 0

local, global min at b

local, global max at 2.

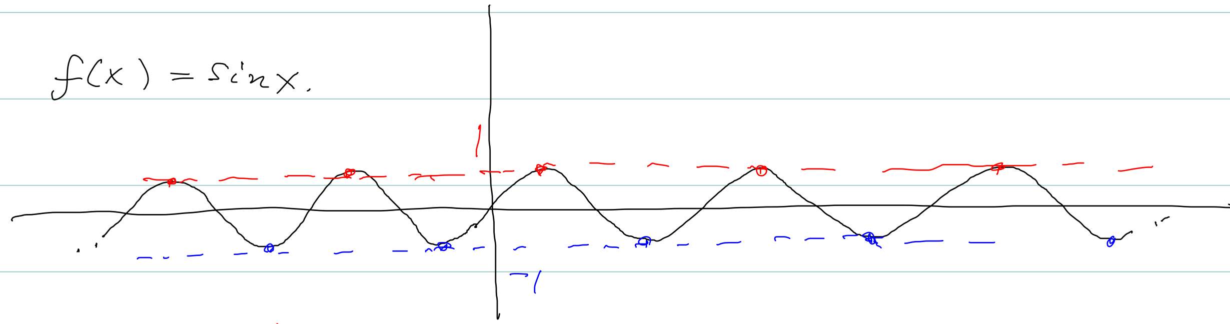
Every global min/max is also a local min/max.

Ex $f(x) = c$



every point x is a local/global min/max.

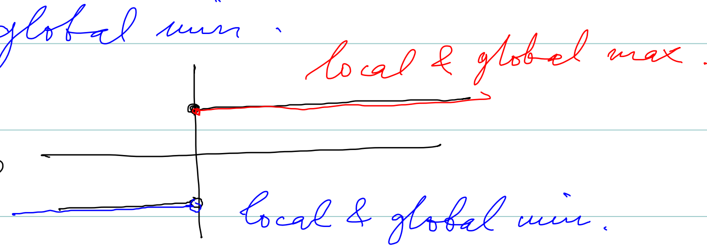
Ex $f(x) = \sin x$.



∘ local and global max

∘ local and global min.

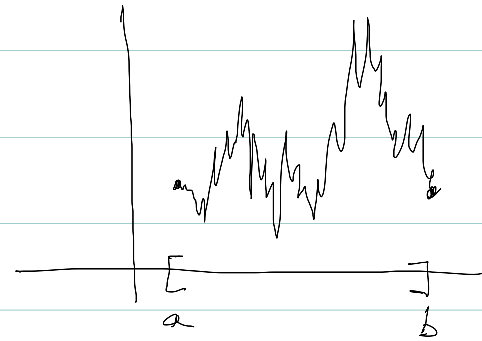
Ex $f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 1, & x = 0 \end{cases}$



Thm (Extreme value thm).

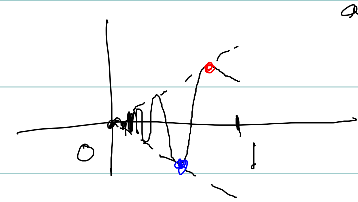
Assume f is continuous on closed interval $[a,b]$.

Then, f has a global max and a global min in $[a,b]$.



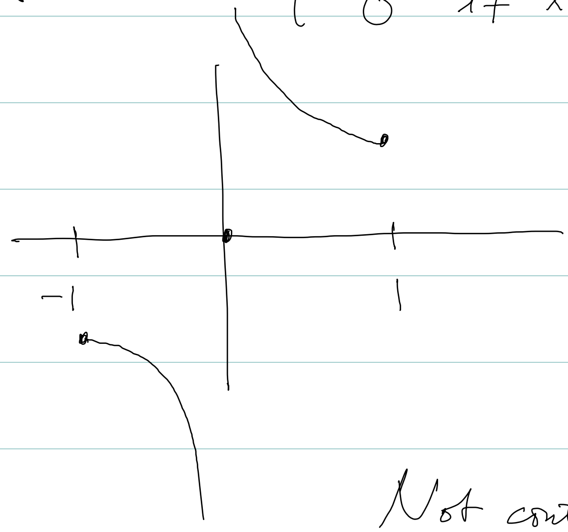
Ex $f(x) = \begin{cases} x \sin \frac{1}{x}, & x > 0 \\ 0, & x = 0 \end{cases}$

$0 \leq x \leq 1$.



f continuous on $[0,1] \Longrightarrow$ can apply extreme value thm.

<u>Ex</u> $f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ for $-1 \leq x \leq 1$.



local max at -1

neither local min nor max at 0

local min at 1

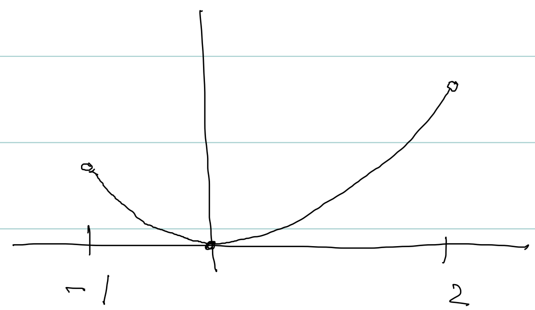
no global min nor max.

Not continuous $\Rightarrow$ extreme value thm does not apply.

Thm (Fermat)

Assume f has a local max or local min at c, and that $f'(c)$ exists. Then, $f'(c) = 0$

Ex $f(x) = x^2$, $-1 < x < 2$

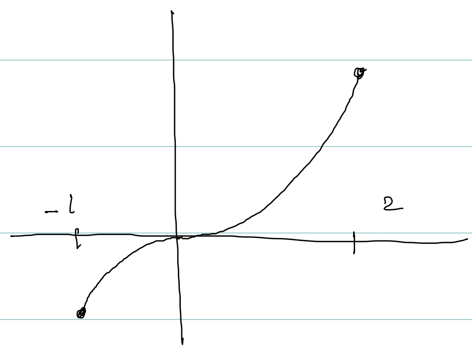


f differentiable everywhere on this domain

local, global min at $x = 0$

$f'(0) = 2 \cdot 0 = 0$

$\underline{\underline{Ex}}$ $f(x) = x^3$, $-1 \le x \le 2$.



① local min: $x = -1$ (global min)

right derivative: $f'_+(-1) = 3(-1)^2 = 3$

(no derivative here $\Longrightarrow$ ~~Fermat~~)

② local max: $x = 2$ (global max)

left derivative: $f'_-(2) = 3 \cdot 2^2 = 12$.

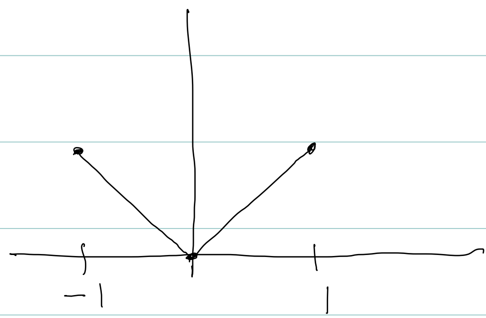
(no derivative here $\Rightarrow$ ~~Fermat~~).

③ $f'(0) = 3 \cdot 0^2 = 0$

neither local min nor local max.

(<u>careful</u>: Fermat implies local min/max and differentiable at $c \Rightarrow f'(c) = 0$ <u>but</u> not the other way around!)

Ex. $f(x) = |x|, \quad -1 \le x \le 1.$



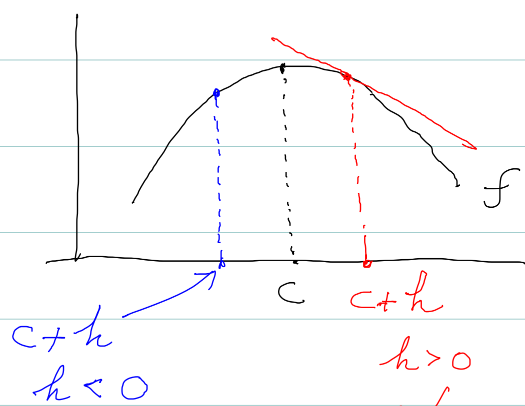
local max: $x = -1, 1$; global max

local, global min: $x = 0$

but f not differentiable at 0

$\Rightarrow$ Fermat not applicable.

Check: Explanation for Fermat's theorem.



local max at c; by assumption, $f'(c)$ exists

$$\Rightarrow f(c+h) - f(c) \leq 0$$

divide by $h>0$, and let $h \to 0^+$

$$f'_+(c) = \lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h} \leq 0$$

$$f(c+h) - f(c) \leq 0$$

ineq. flips because we divide by neg. number.

divide by $h<0$ and let $h \to 0^-$: $f'_-(c) = \lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h} \geq 0$

$$0 \leq f'_-(c) = f'(c) = f'_+(c) \leq 0 \Rightarrow f'(c) = 0$$

f differentiable at c!

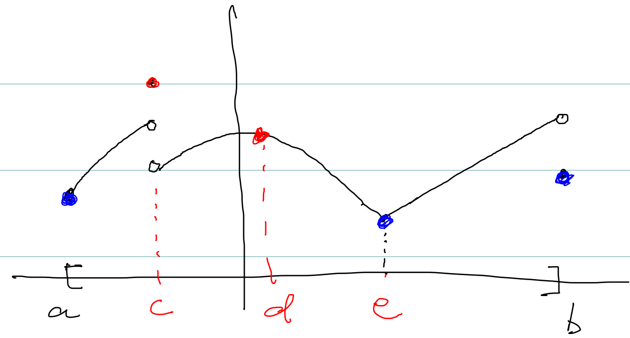
Def (critical number)

A point c in the domain of f is a critical number if:

either $f'(c) = 0$, or $f'(c)$ does not exist.

Thm If f has a local min/max at c, then c is a critical number.

Ex



local max

local min

Critical points:

a because derivative does not exist

b ——— " ———

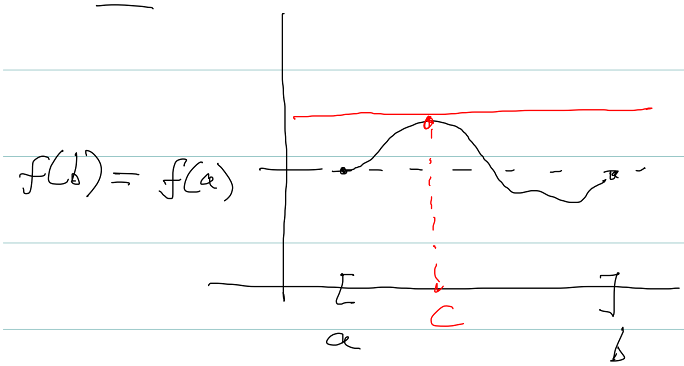
c ——— " ———

d derivative is zero

e derivative does not exist

The mean value theorem.

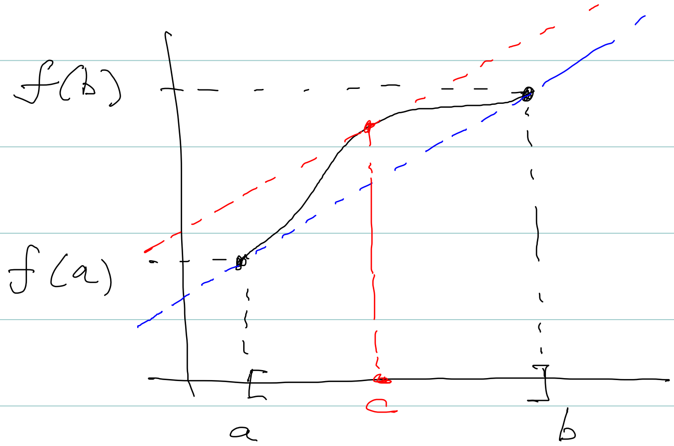
Thm (Rolle).



Assume f is continuous on $[a, b]$ and differentiable in (a, b), and that $f(a) = f(b)$. Then there is a point c in (a, b) such that $f'(c) = 0$

Proof: See the other notes.

Thm (mean value thm)



Assume f continuous on $[a,b]$ and differentiable in (a,b). Then, there is a point c in (a,b) such that

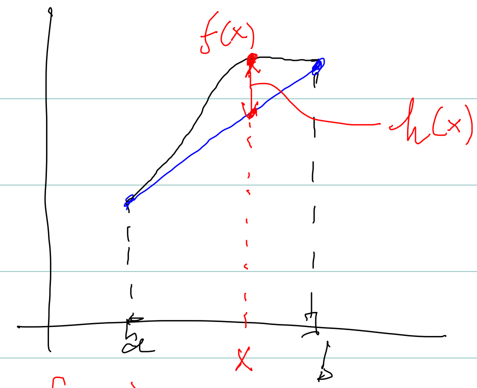
$$f'(c) = \underbrace{\frac{f(b)-f(a)}{b-a}}_{\text{slope of blue line.}}$$

(the tangent line at $f(c)$ is parallel to the straight line connecting $f(a)$ and $f(b)$).

Check (mean value thm).

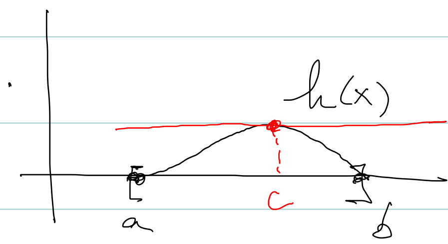
Connecting line:

$$y = \underbrace{\frac{f(b)-f(a)}{b-a}}_{\text{slope}} \cdot (x-a) + f(a)$$



$$h(x) = f(x) - \left(\frac{f(b)-f(a)}{b-a}\cdot(x-a) + f(a)\right).$$

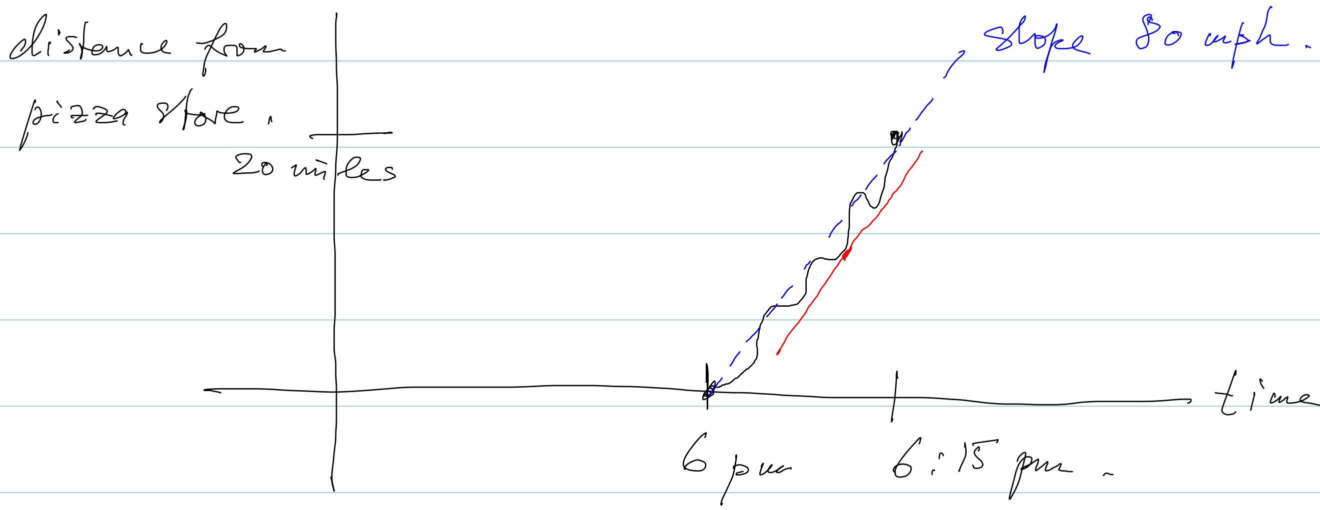
Rolle: There is c in (a,b) where $h'(c) = 0$.



$$\Rightarrow h'(c) = 0 = f'(c) - \frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Ex The local police station orders pizza at 6 pm from a pizza store 20 miles away. At 6:15 pm, the delivery arrives. Is the delivery person in trouble, if the max speed on highway is 65 mph?



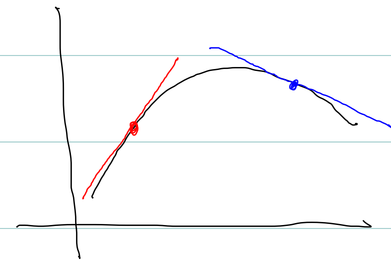
Mean value thm: **Yes**. He/she drove 80 mph at some point.

Derivatives & shape of graphs.

What do we learn from f' about f?

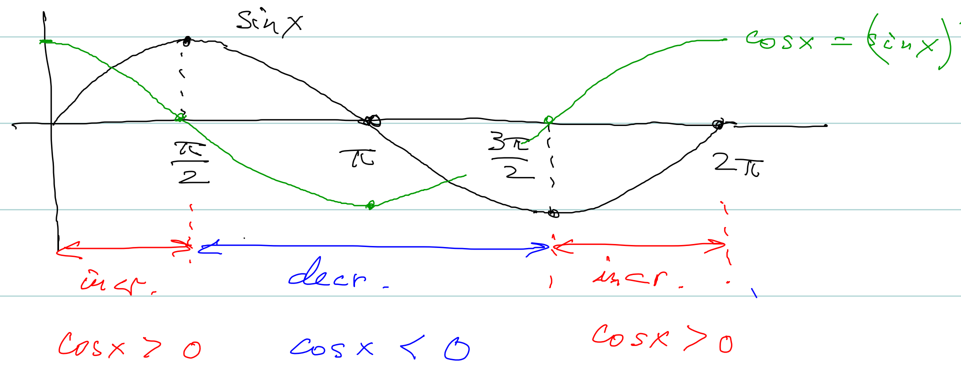
Increasing/decreasing:

- if $f'(x) > 0$, then f is increasing at x
- if $f'(x) < 0$, then f is decreasing at x

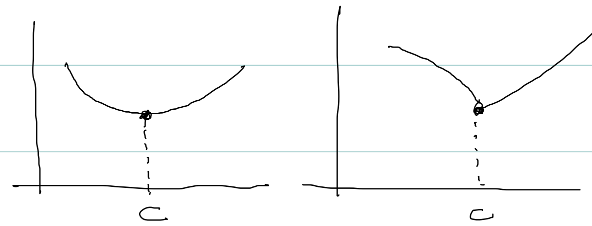


Ex $f(x) = \sin x$

$0 \le x \le 2\pi$.



If f is continuous, and c is a critical number ($f'(c)=0$, or $f'(c)$ does not exist)

- 

if f changes from decreasing ($f'<0$) to increasing ($f'>0$) at c, then f has a local min at the critical number c.

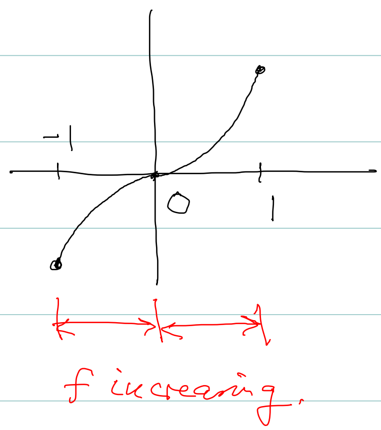
- 

if f changes from increasing ($f'>0$) to decreasing ($f'<0$) at c, then f has a local max at the critical number c.

–

if f does not change from increasing ($f' > 0$) to decreasing ($f' < 0$) or vice versa at c, then the critical point c is neither a local min nor a local max.

Ex $f(x) = x^3$, for $-1 \le x \le 1$.



Crit pts : $x = -1,\ 0,\ 1$.

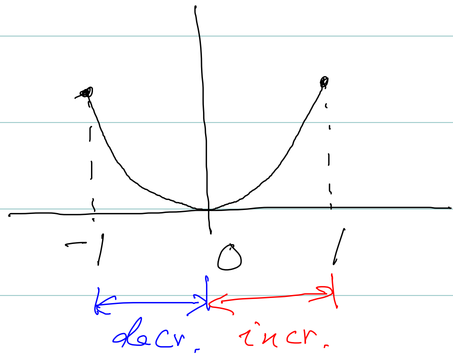
$f'(0) = 0$

Increasing: $f'(x) = 3x^2 > 0$ for $-1 < x < 0$, $0 < x < 1$

Decreasing: $f'(x) = 3x^2 < 0$ nowhere

neither local min nor local max at 0.

Ex $f(x) = x^4$, $-1 \le x \le 1$



Critical points: $x = -1, 0, 1$

$f'(0) = 0$.

Increasing: $f'(x) = 4x^3 > 0$ for $0 < x < 1$

Decreasing: $f'(x) = 4x^3 < 0$ for $-1 < x < 0$

change from decreasing to increasing at 0

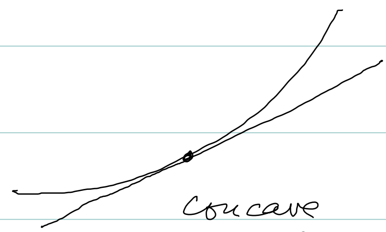
$\Longrightarrow$ 0 local min.

Second derivative.

f' ~ slope of the tangent line.

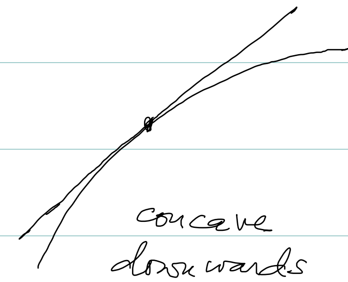
f'' ~ rate of change of steepness of tangent line (slope)

Def



concave upwards

(~ convex)



concave downwards

(~ concave)

graph of f is **above** the tangent

graph of f is **below** the tangent.

Concavity test: Assume f is twice differentiable on (a,b).

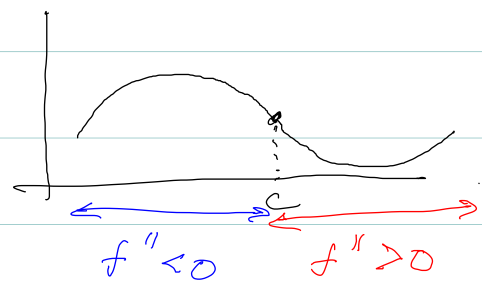
Then: 1) if $\underbrace{f''(x) > 0}$ for x in (a,b) $\Rightarrow$ f concave up in (a,b)

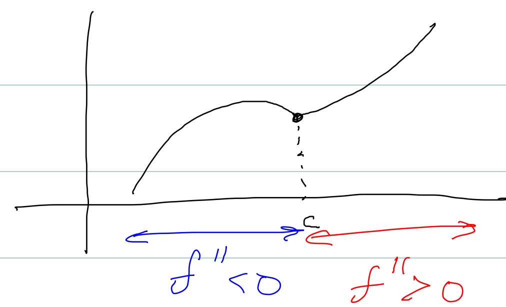
steepness of tangent line increases.

2) if $\underbrace{f''(x) < 0}$ for x in (a,b) $\Rightarrow$ f concave down in (a,b)

steepness of tangent line decreases.

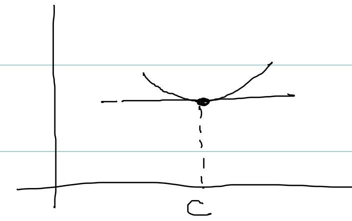
Def: A point c is an inflection point if f changes concavity at c ($\sim f''$ changes sign), and f is continuous at c.



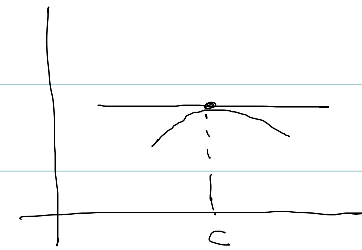


Thm (2nd derivative test) Assume f'' is continuous near c.

1) if $f'(c) = 0$ and $f''(c) > 0$
then f has a local min at c.



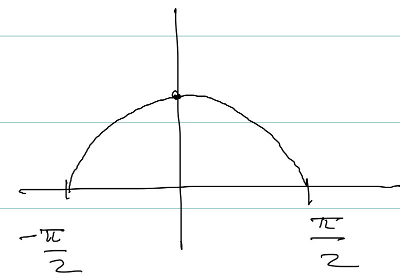
2) if $f'(c) = 0$ and $f''(c) < 0$
then f has a local max at c.



Remark: If $f'(c)$ does not exist, then $f''(c)$ also doesn't exist
$\Rightarrow$ can't use 2nd derivative test.

Ex $f(x) = \cos x, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}.$

Crit numbers: $x = -\frac{\pi}{2}, 0, \frac{\pi}{2}.$



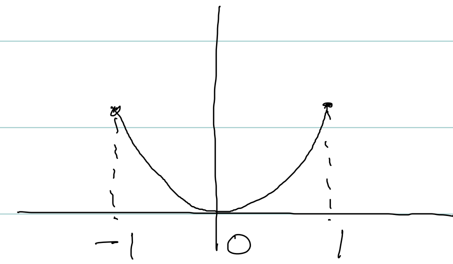
$f'(0) = -\sin 0 = 0$

$f''(0) = -\cos 0 = -1$

$\Rightarrow$ 2nd derivative test: concave down, local max.

Ex $f(x) = x^4$, $-1 \le x \le 1$.

Crit numbers: $x = -1, 0, 1$



At $x = 0$:

$$f'(0) = 4 \cdot 0^3 = 0$$

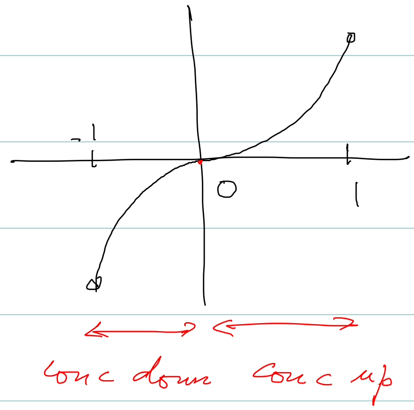
$$f''(x) = (4x^3)' = 12x^2$$

$f''(0) = 0 \Rightarrow$ 2nd derivative test does not apply (because neither $f''(0) > 0$ nor $f''(0) < 0$).

Have to use 1st derivative; verify change from decreasing ($f' < 0$) to increasing ($f' > 0$)

Also, the concavity doesn't change at $0 \Rightarrow 0$ not an inflection pt.

Ex $f(x) = x^3$, $-1 < x < 1$.



$f'(x) = 3x^2$

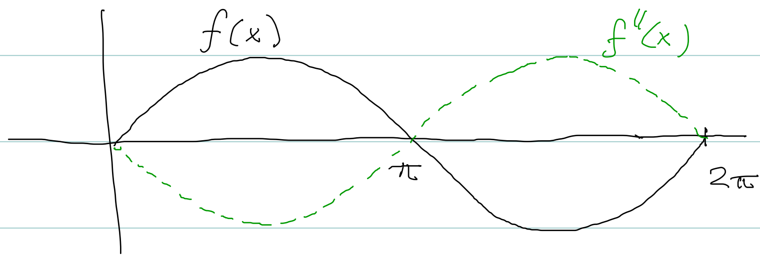
$f''(x) = 6x$

$\begin{cases} f''(x) > 0, & 0 < x < 1 \\ f''(x) < 0, & -1 < x < 0 \end{cases}$

$\Rightarrow$ $x = 0$ is an inflection pt.

In addition, since $f'(0) = 0$, $x = 0$ is also a critical point.

Ex $f(x) = \sin x, \quad 0 \le x \le 2\pi$.



$f'(x) = \cos x$

$f''(x) = -\sin x$

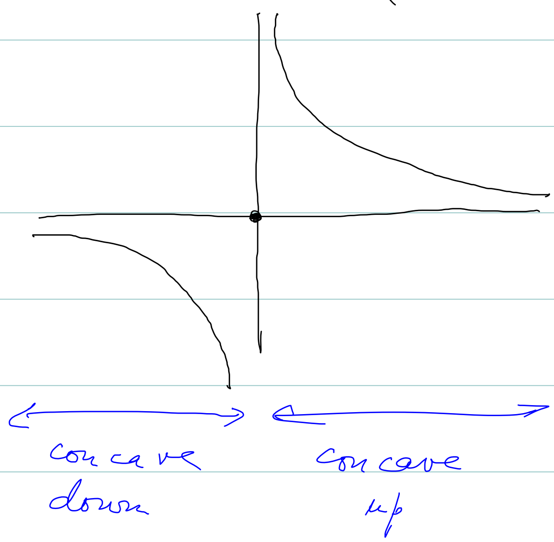
$\begin{cases} f'' > 0, & \pi < x < 2\pi \\ f'' < 0, & 0 < x < \pi \end{cases}$

Inflection pt: $\underline{\underline{x = \pi}}$

Note: $x = \pi$ is not a critical point, $f'(\pi) = \cos\pi \neq 0$

Remark: An inflection point does not need to be a critical point.

Ex $f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$



$f'(x) = -\frac{1}{x^2}$

$f''(x) = \frac{2}{x^3}$ $\quad \begin{cases} f'' < 0, & x < 0 \\ f'' > 0, & x > 0 \end{cases}$

Is $x=0$ an inflection point?

NO because f not continuous at $x=0$.

Ex: $f(x) = x^4 - 4x^3$, $-\infty < x < \infty$,

Draw the graph (min, max, concavity, inflection pts, ...)

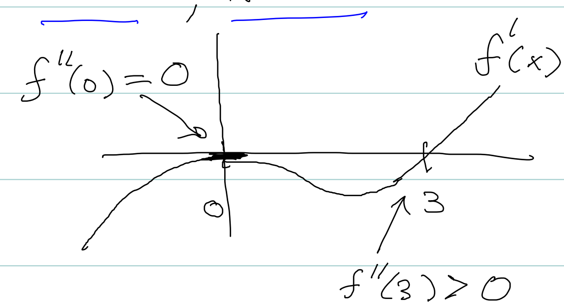
1) Zeros of $f(x) = x^4 - 4x^3 = 0 \Rightarrow$ $\underline{x = 0}$, $\underline{x = 4}$

2) Derivative $f'(x) = 4x^3 - 12x^2$

Critical points: $4x^3 - 12x^2 = 0 \Rightarrow$ $\underline{x = 0}$, $\underline{x = 3}$

f decreasing: $(-\infty, 0), (0, 3)$

f increasing: $(3, \infty)$



3) Second derivative $f''(x) = 12x^2 - 24x$

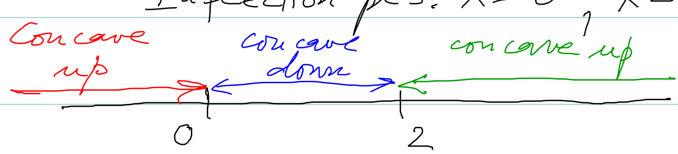
Candidates for inflection pts: $12x^2 - 24x = 0 \Rightarrow$ $\underline{x = 0}$, $\underline{x = 2}$

$f''(x) = 12x(x-2)$

$f''(-0.1) = 12\cdot(-0.1)(-0.1-2) > 0$

Inflection pts: $x = 0$, $x = 2$

$f''(0.1) = 12\cdot(0.1)(0.1-2) < 0$



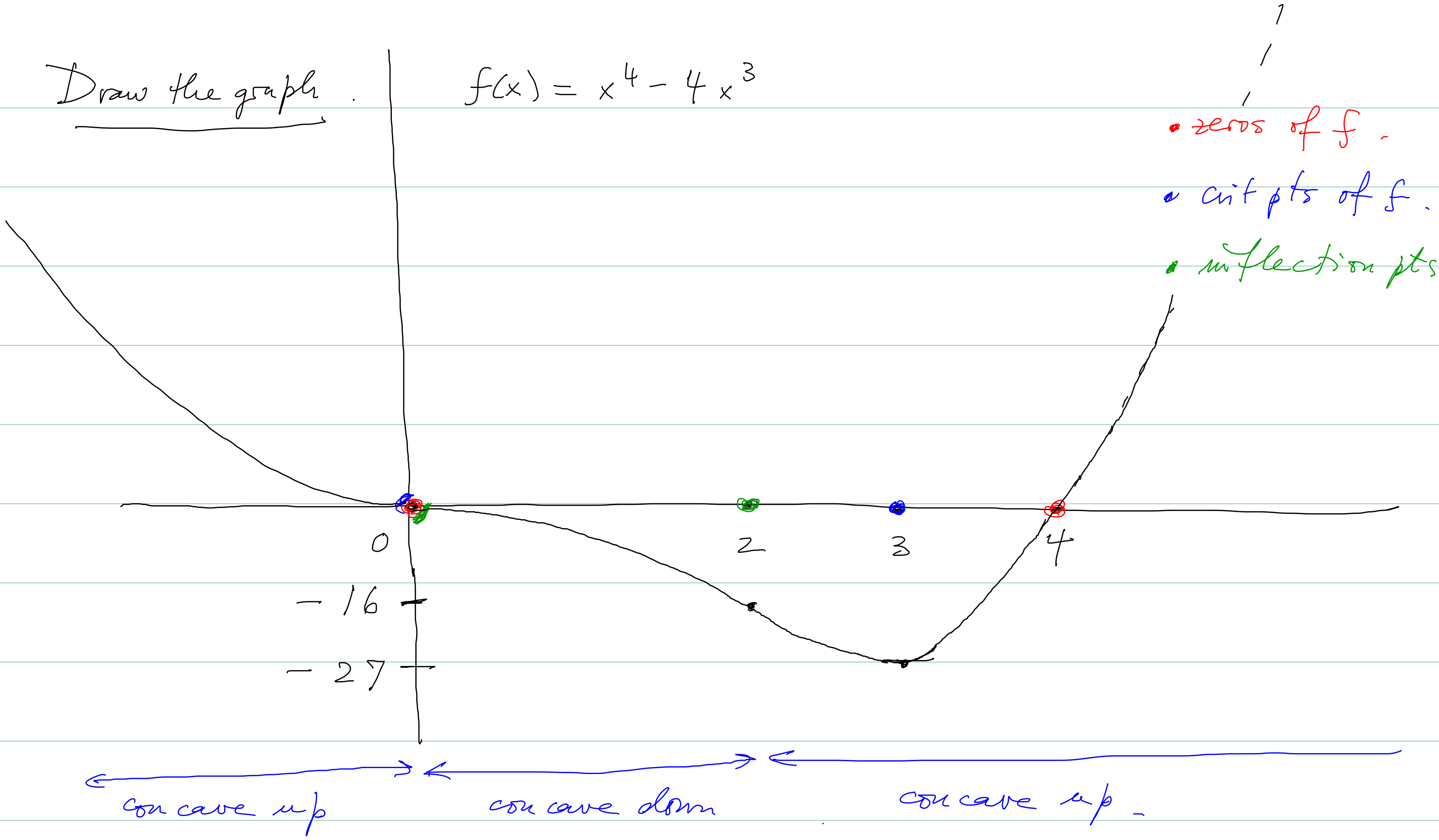
$f''(1.9) = 12\cdot 1.9(1.9-2) < 0$

$f''(2.1) = 12\cdot 2.1(2.1-2) > 0$

$f'(x)$ doesn't change sign at $x=0 \Rightarrow x=0$ is neither local max nor local min.

$f'(x)$ changes from <0 to >0 at $x=3 \Rightarrow x=3$ is a local min.

Draw the graph. $f(x) = x^4 - 4x^3$



$f(3) = 3^4 - 4\cdot 3^3 = -27$

$f(2) = 2^4 - 4\cdot 2^3 = -16$

<u>Ex:</u> $f(x) = x^3 + 3x + 2$.

How many roots does $f(x) = 0$ have?

Intermediate value theorem

$f(0) = 2 > 0$

$f(-1) = (-1)^3 + 3\cdot(-1) + 2 = -1 - 3 + 2 < 0$

$\Rightarrow$ There must be a root in $(-1, 0)$.

$f'(x) = 3x^2 + 3 > 0$ for all x.

$\Rightarrow$ everywhere increasing.

$\Rightarrow$ f can intersect x-axis only once.

$\Rightarrow$ exactly one root.

Indeterminate forms and de l'Hôpital's rule.

Thm (de l'Hôpital's rule).

Assume f, g differentiable near a, and

$$\lim_{x \to a} f(x) = 0, \quad \lim_{x \to a} g(x) = 0$$

$$\left(\text{or } \lim_{x \to a} f(x) = \pm\infty, \quad \lim_{x \to a} g(x) = \pm\infty \right).$$

Then,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Remark: Similarly for left / right limits and left / right derivatives of f, g.

$$\underline{\underline{Ex}} \quad \lim_{x \to 0} \frac{\sin x}{x} \underset{\text{de l'H}}{=} \lim_{x \to 0} \frac{\cos x}{1} = 1$$

$$\underline{\underline{Ex}} \quad \lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1} \frac{2x}{1} = 2.$$

$$\underline{\underline{Ex}} \quad \lim_{x \to 0} \frac{1-\cos x}{x^2} = \lim_{x \to 0} \frac{\sin x}{2x} = \lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2}.$$

Ex Compare x^r, $r>0$, with $\ln x$ as $x \to \infty$.

$$\lim_{x\to\infty} \ln x = \infty, \quad \lim_{x\to\infty} x^r = \infty$$

$$\lim_{x\to\infty} \frac{\ln x}{x^r} \underset{\text{de l'H.}}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{r x^{r-1}}$$

$$= \lim_{x\to\infty} \frac{1}{r x \cdot x^{r-1}} = \lim_{x\to\infty} \frac{1}{r x^r} \underset{r>0}{=} 0$$

$\Rightarrow$ x^r tends to ∞ much faster than $\ln x$, for any $r > 0$

$\Rightarrow$ $x^{0.00000000000000000000000000000000001}$ tends to ∞ faster than $\ln x$.

Calculator: $r = 0.01$. $\quad x = 10^{10,000} \Rightarrow x^{0.001} = 10^{10,000 \cdot 0.001}$

$= 10^{100}$

$\log_{10} x = \log_{10} 10^{10,000} = 10,000 = 10^4$

Ex Let $n > 0$ be an integer. Compare e^x with x^n, as $x \to \infty$

$$\lim_{x\to\infty} e^x = \infty, \quad \lim_{x\to\infty} x^n = \infty$$

$$\lim_{x\to\infty} \frac{e^x}{x^n} \underset{\text{de l'H.}}{=} \lim_{x\to\infty} \frac{e^x}{n x^{n-1}} = \lim_{x\to\infty} \frac{e^x}{n(n-1)x^{n-2}}$$

$$= \dots = \lim_{x\to\infty} \frac{e^x}{n(n-1)(n-2)\cdots 2x} = \lim_{x\to\infty} \frac{e^x}{n!}$$

$$= \infty$$

$\Rightarrow$ e^x tends to ∞ faster than x^n, for any $n > 0$.

$\Rightarrow$ for example, e^x tends to ∞ faster than $x^{1,000,000,000,000,000,000}$

Indeteminate powers.

Assume $f(x) \geq 0$ for x near a.

$$\lim_{x \to a} \left(f(x)\right)^{g(x)}$$

where

1) $f \to 0^+$, $g \to 0$
2) $f \to \infty$, $g \to 0$
3) $f \to \infty$, $g \to \pm\infty$

If $f(x) \geq 0$ for x near a

$$\left(f(x)\right)^{g(x)} = \left(e^{\ln f(x)}\right)^{g(x)} = e^{g(x) \cdot \ln f(x)}$$

$$\Rightarrow \lim_{x \to a} \left(f(x)\right)^{g(x)} = \lim_{x \to a} e^{g(x) \cdot \ln f(x)} = e^{\lim_{x \to a} g(x) \cdot \ln f(x)}$$

if the exponent has a finite limit.

Ex Find $\lim\limits_{x \to \infty} x^{\frac{1}{x}}$

$$\lim_{x \to \infty} x^{\frac{1}{x}} = e^{\lim\limits_{x \to \infty} \frac{1}{x} \ln x} = e^0 = 1$$

$$\lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = 0$$

de l'H.

Ex Find $\lim_{x \to 0^+} x^x$

$$\lim_{x \to 0^+} x^x = e^{\lim_{x \to 0^+} x \ln x} = e^0 = 1$$

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \to 0^+} (-x) = 0$$

de l'H

$$\lim_{x \to 0^+} \ln x = -\infty$$

$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

Why is de l'Hôpital's rule correct?

Simplified case: f, g differentiable at a, and $f, g \to 0$ as $x \to a$, and $g'(a) \neq 0$ $\Rightarrow$ $f(a) = 0$, $g(a) = 0$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f(x) - \overbrace{f(a)}^{=0}}{g(x) - \underbrace{g(a)}_{=0}}$$

$$= \lim_{x \to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} = \frac{\lim_{x \to a} \frac{f(x)-f(a)}{x-a}}{\lim_{x \to a} \frac{g(x)-g(a)}{x-a}}$$

$$= \frac{f'(a)}{g'(a)}$$

General case: See other notes.

Check for $f, g \to \pm\infty$ as $x \to a$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{\frac{1}{g(x)}}{\frac{1}{f(x)}} \underset{\text{de l'H.}}{=} \lim_{x\to a} \frac{-\frac{g'(x)}{g^2(x)}}{-\frac{f'(x)}{f^2(x)}}$$

numerator + denominator $\longrightarrow 0$

assume limits exist.

$$= \lim_{x\to a} \frac{g'(x)}{f'(x)} \cdot \frac{f^2(x)}{g^2(x)} = \left(\lim_{x\to a} \frac{g'(x)}{f'(x)}\right) \cdot \left(\lim_{x\to a} \frac{f(x)}{g(x)}\right)^2$$

$$\Rightarrow \quad 1 = \left(\lim_{x\to a} \frac{g'(x)}{f'(x)}\right) \cdot \lim_{x\to a} \frac{f(x)}{g(x)} \quad \text{(divide by ....)}$$

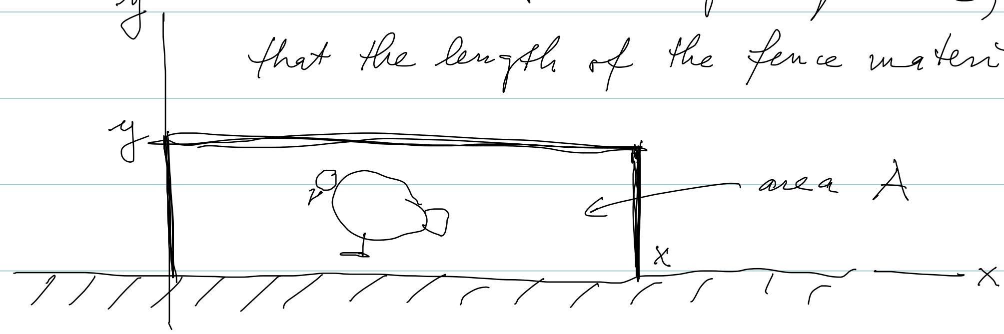
$$\Rightarrow \quad \lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f(x)}{g(x)} \quad \text{de l'Hôpital.}$$

Optimization problems.

Goal: Find best possible outcome, under given constraints.

Ex: Build a fence at one wall of your house.

Make area A as large as possible, under the constraint that the length of the fence material is 100m.



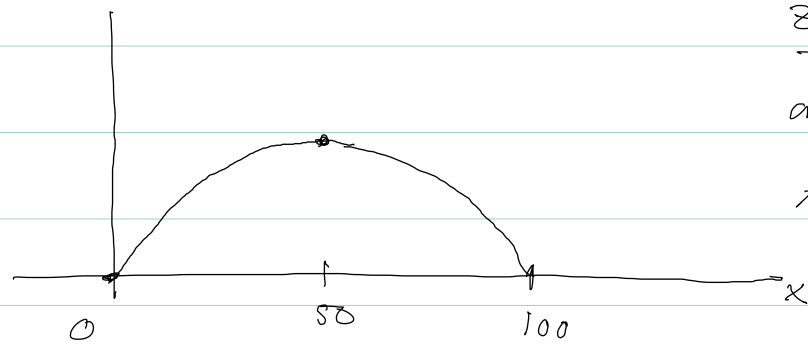
Optimization: $A = x \cdot y$

Constraint: $2y + x = 100$ m length.

solve for $y = \frac{100-x}{2} = 50 - \frac{x}{2}$

Plug into quantity to be optimized:

$$A = x\cdot\left(50 - \frac{x}{2}\right) = 50x - \frac{x^2}{2}.$$



zeros: $x = 0, 100$

critical pts:

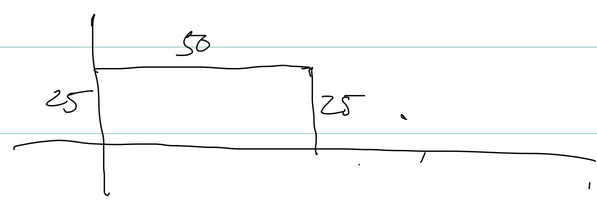
$A'(x) = 50 - x = 0$

$\Rightarrow x = 50, 0, 100$

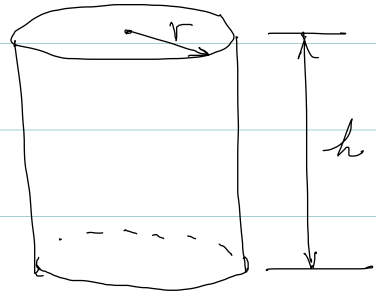
2nd derivative test: $A''(x) = -1 < 0$

domain: $[0, 100]$.

concave down $\Rightarrow$ max.



Ex Build a can (cylindrical) holding 1 ℓ of beer, with the least amount of material.



Volume: $V = \pi r^2 h = 1\ell$ constraint

Area: $A = 2\pi r^2 + 2\pi r \cdot h$

top + bottom

Optimization: Minimize the area, with volume constraint.

Solve constraint equation for $h = \frac{1}{\pi r^2}$.

Plug into $A = 2\pi r^2 + \frac{2\pi r}{\pi r^2} = 2\pi r^2 + \frac{2}{r}$.

$$A(r) = 2\pi r^2 + \frac{2}{r}$$

$$\Rightarrow A'(r) = 4\pi r - \frac{2}{r^2} = 0 \Rightarrow 4\pi r = \frac{2}{r^2} \Rightarrow r^3 = \frac{2}{4\pi} = \frac{1}{2\pi}$$

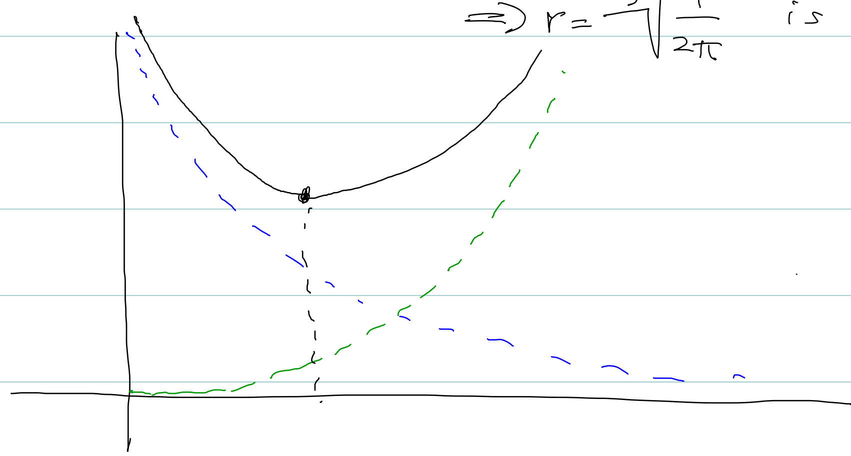
$$\Rightarrow r = \sqrt[3]{\frac{1}{2\pi}}$$

2nd derivative test: $A''(r) = 4\pi - 2\cdot\left(-2\cdot\frac{1}{r^3}\right)$

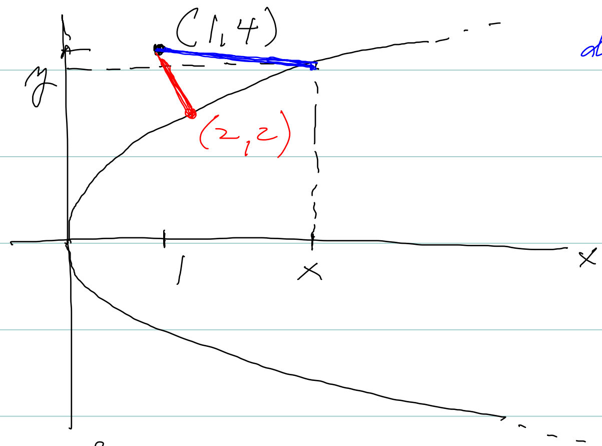
$$= 4\pi + \frac{4}{r^3}.$$

$$A''\left(\sqrt[3]{\frac{1}{2\pi}}\right) = 4\pi + \frac{4}{\frac{1}{2\pi}} = 12\pi > 0$$

$\Rightarrow r = \sqrt[3]{\frac{1}{2\pi}}$ is a local min



Ex Consider the parabola $y^2 = 2x$. Find the closest point on it to the point $(1, 4)$



distance $d = \sqrt{(x-1)^2 + (4-y)^2}$

Optimization: Minimize d.

Constraint: $x = \dfrac{y^2}{2}$

$x = \dfrac{y^2}{2}$ Plug into d:

$$d = \sqrt{\left(\frac{y^2}{2} - 1\right)^2 + (4-y)^2}$$

$$S = d^2 = \left(\frac{y^2}{2} - 1\right)^2 + (4-y)^2$$

square of distance minimal when d is minimal

$$S = \left(\frac{y^2}{2} - 1\right)^2 + (4-y)^2$$

$$S'(y) = 2\left(\frac{y^2}{2} - 1\right) \cdot \frac{\cancel{2}y}{\cancel{2}} + \underbrace{2(4-y)\cdot(-1)}_{2(y-4)}$$

$$= y^3 - \cancel{2y} + \cancel{2y} - 8$$

$$= y^3 - 8 = 0 \Rightarrow y^3 = 8 \Rightarrow y = 2.$$

2nd derivative test:

$$S''(y) = 3y^2 \Rightarrow S''(2) = 3 \cdot 2^2 > 0 \quad \text{local min.}$$

$$x = \frac{y^2}{2} = \frac{2^2}{2} = 2$$

Antiderivatives.

$$F' = f$$ ← f is the derivative of F

F is an antiderivative of f

Ex x^9 is an antiderivative of $9x^8$

x^9+10 —— " —— $9x^8$

Thm If $F(x)$ is an antiderivative of $f(x)$ on (a,b), then any other antiderivative of f on (a,b) has the form

$$F(x) + C$$ ← arbitrary constant.

Therefore, the most general antiderivative of f is

$$F(x) + C$$ ← arbitrary constant.

an arbitrary choice of an antiderivative

<u>Ex</u> Find general antiderivative of $f(x)$:

$f(x) = e^x \implies F(x) = e^x + C$

$f(x) = \frac{1}{x^2} \implies F(x) = -\frac{1}{x} + C$

$f(x) = a^x \implies F(x) = \frac{1}{\ln a} a^x + C$, $a > 0$

$f(x) = \frac{1}{x} \implies F(x) = \ln x + C$, $x > 0$

$$f(x) = x^r \implies F(x) = \begin{cases} \frac{1}{r+1} x^{r+1} + C, & r \neq -1 \\ \ln x + C, & r = -1 \end{cases}$$
$x > 0$

$f(x) = \cos x \implies F(x) = \sin x + C$

$f(x) = \sin x \implies F(x) = -\cos x + C$.

Ex $f(x) = \ln x$, $x > 0 \Longrightarrow$ find general antiderivative.

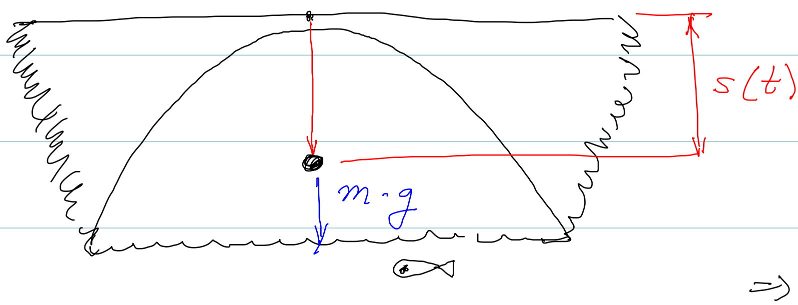
Look at $(x \cdot \ln x)' = \ln x + x \frac{1}{x} = \ln x + 1$

$\Longrightarrow (x \cdot \ln x)' - 1 = \ln x$

$\Longrightarrow (x \cdot \ln x - x)' = \ln x$

$\Longrightarrow \quad F(x) = x \cdot \ln x - x + C$

Ex



Throw a stone at time $t=0$ from $s(0)=0$, with velocity $s'(0)=1\ m/sec$.

$\Rightarrow$ Find $s(t)$

$g = 9.81\ m/sec^2$ gravitational constant.

Newton's law

$$\text{mass} \cdot \text{acceleration} = \text{force}.$$

$$\cancel{m} \cdot s''(t) = \cancel{m} \cdot g$$

$$\Rightarrow \quad s'(t) = g \cdot t + C$$

$$\Rightarrow \quad s(t) = g\frac{t^2}{2} + C \cdot t + D$$

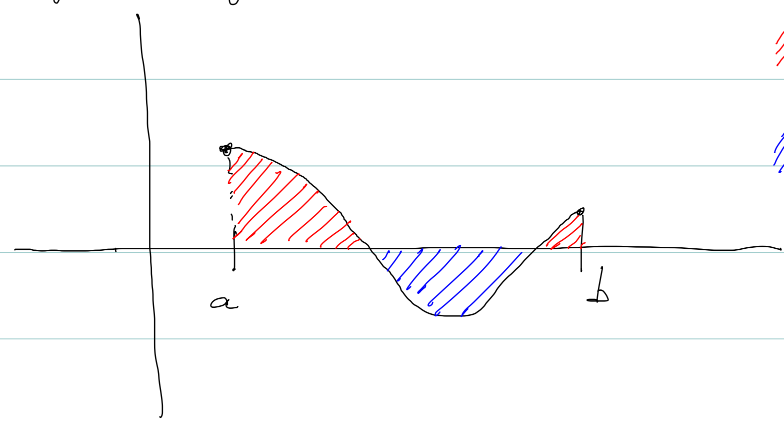
Determine the constants using initial conditions:

$$\left.\begin{array}{l} \Rightarrow s'(0) = 1 = g \cdot 0 + C \quad \Rightarrow \quad C = 1 \\ s(0) = 0 = g \cdot \frac{0^2}{2} + C \cdot 0 + D \quad \Rightarrow \quad D = 0 \end{array}\right\} \quad s(t) = g\frac{t^2}{2} + t.$$

Integrals

Def: The definite integral of a continuous function f on $[a, b]$ is the area enclosed between the graph of f and the x-axis.

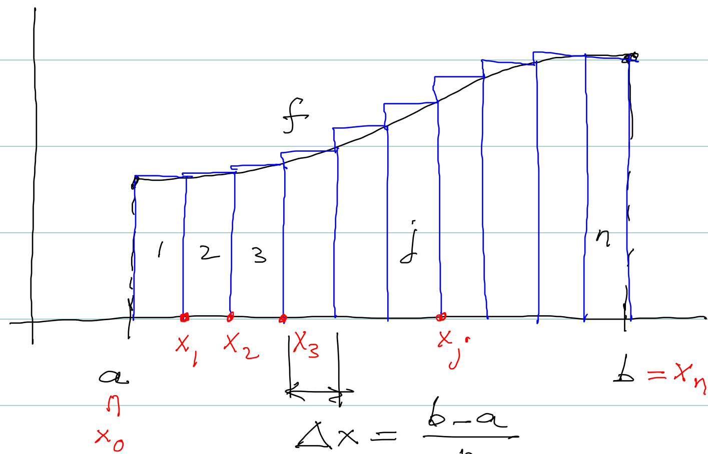
If an area lies below x-axis, it is counted with a negative sign.



//// positive area

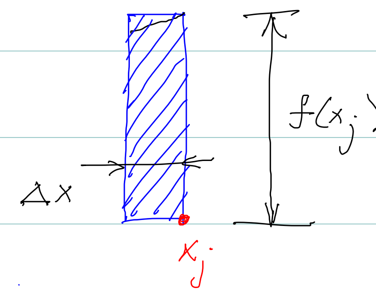
//// negative area.

How to calculate this area?



$\Delta x = \frac{b-a}{n}$.

j-th rectangle.



area $= f(x_j) \cdot \Delta x$

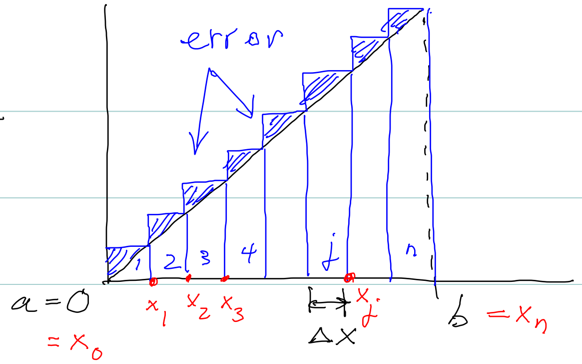
Sum of areas of all n rectangles:

$$\underbrace{f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x}_{\text{approximation of area.}} \xrightarrow{\text{limit } n \to \infty} \text{area under } f(x)$$

$\sim$ integral of f over $[a, b]$

Riemann sum.

Ex



$f(x)=x$

area of j-th rectangle:

$$f(x_j)\cdot\Delta x = j\cdot\frac{b}{n}\cdot\Delta x$$

$$\Delta x = \frac{b}{n}, \quad x_j = j\cdot\Delta x = j\cdot\frac{b}{n}, \quad f(x_j) = x_j = j\cdot\frac{b}{n}$$

Sum of areas of all rectangles:

$$f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x = \left(\frac{b}{n} + 2\frac{b}{n} + 3\frac{b}{n} + \dots + n\frac{b}{n}\right)\underbrace{\frac{b}{n}}_{\Delta x}$$

$$= (1+2+3+\dots+n)\left(\frac{b}{n}\right)^2$$

$$= \frac{n}{2}\cdot(n+1)\cdot\left(\frac{b}{n}\right)^2$$

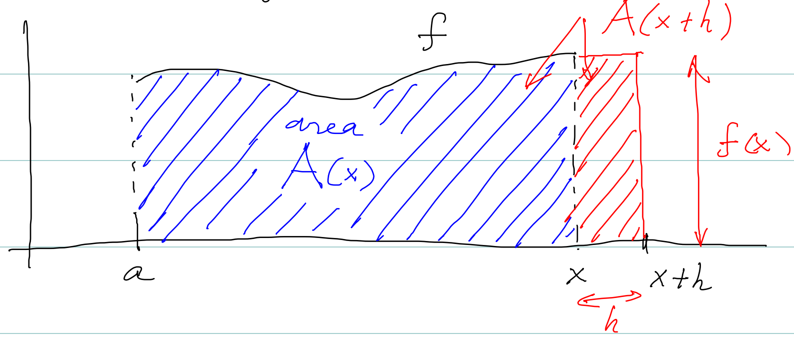
$$1+2+3+4+5+6+7+8+9+10+11+12+13 = 6\tfrac{1}{2}\cdot 14$$

14

$$= \left(\frac{n^2}{2}+\frac{n}{2}\right)\cdot\frac{b^2}{n^2} = \frac{1}{2}b^2 + \frac{b^2}{2n} \Longrightarrow \lim_{n\to\infty}\left(\frac{1}{2}b^2 + \frac{b^2}{2n}\right) = \frac{1}{2}b^2$$

approximation error.

A more elegant way to calculate an integral:



$$\text{////} = A(x+h) - A(x) \approx f(x)\cdot h.$$

$$\Rightarrow f(x) = \lim_{h\to 0} \frac{A(x+h)-A(x)}{h} = A'(x)$$

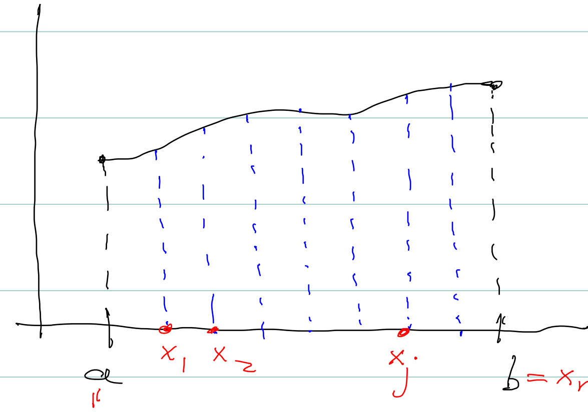
$\Rightarrow$ $A(x)$ is an antiderivative of f !

Def (Definite integrals).

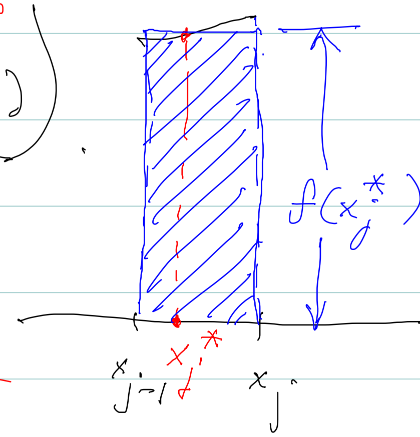
Assume f continuous fct on $[a,b]$.

Let $\Delta x = \frac{b-a}{n}$ and $x_j = a + j \cdot \Delta x$

Pick a sample point x_j^* in $[x_{j-1}, x_j]$



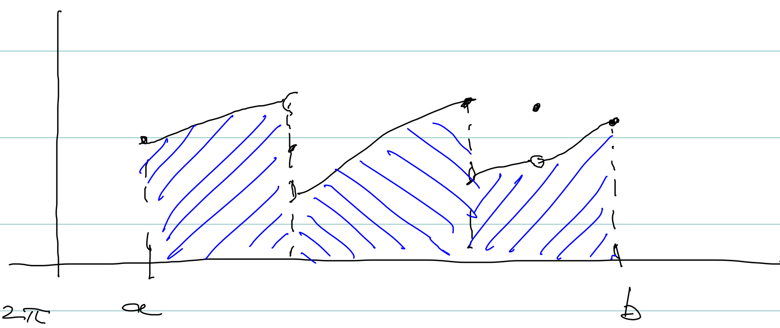
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \underbrace{\left(f(x_1^*)\Delta x + f(x_2^*)\Delta x + \dots + f(x_n^*) \right)}_{\text{Riemann sum.}}$$



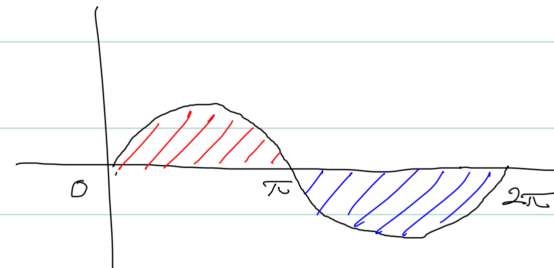
is the definite integral of f from a to b, if this limit exists. If it exists, we call f integrable on $[a,b]$.

"definite": boundary points a, b are specified.

Thm If f is continuous on $[a,b]$, or if it has only finitely many jump discontinuities (left-/right limits exist), then f is integrable.



Ex $\int_0^{2\pi} \sin x \, dx = 0$



Ex $f(x) = \frac{1}{x}$ is not integrable on $[0,1]$ (right limit at 0 does not exist).

Note: Riemann sums are too cumbersome for actual calculations. They are extremely useful for numerical evaluation of integrals using a computer.

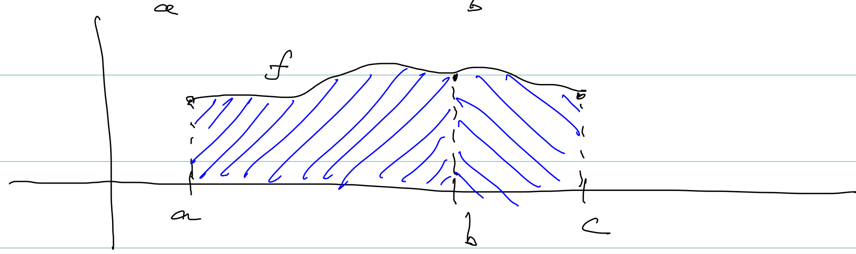
Properties of definite integrals.

Thm.

$$\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$$

$$\int_a^b \big(f(x) \pm g(x)\big)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$



The fundamental theorem of calculus.

Bridge between differentiation and integration.

Discovered by Barrow in 17th century (advisor of Isaac Newton).

Consider: Area $A(x) = \int_a^x f(t)\,dt$

integration variable; dummy variable

$$\int_a^x f(\circ)\,d\circ$$

Already checked:

$$A'(x) = f(x).$$

Thm (Fundamental theorem of calculus, Part I).

Assume f continuous on $[a,b]$. Then,

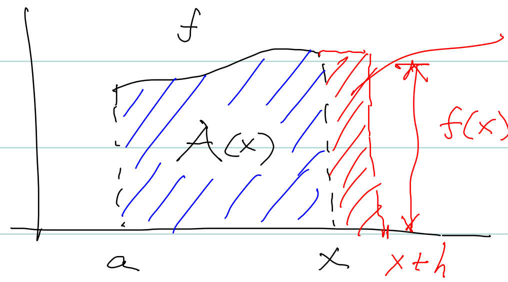
$$A(x) = \int_a^x f(t)\,dt$$

is continuous for x in $[a,b]$, and differentiable for x in (a,b).

$$A'(x) = f(x).$$

(differentiation undoes what integration does).

Recall:



$$A(x+h) - A(x) \approx f(x)\cdot h$$

$$\Rightarrow f(x) = \lim_{h\to 0} \frac{A(x+h) - A(x)}{h}$$

$$= A'(x).$$

$\underline{\underline{Ex}}$ $A(x) = \int_1^x e^{t^2}\,dt \Rightarrow A'(x) = e^{x^2}$

$\underline{\underline{Ex}}$ $A(x) = \int_2^x \left(t^{10} + \ln t - e^t\right) dt \Rightarrow A'(x) = x^{10} + \ln x - e^x$

$\underline{\underline{Ex}}$ $g(x) = \int_2^{x^2} \left(e^{t^2} + \ln t^3\right) dt = A(x^2)$

chain rule.

$$g'(x) = A'(x^2) \cdot 2x$$
$$= \left(e^{x^4} + \ln x^6\right) \cdot 2x$$

Thm (Fundamental theorem of calculus, Part II).

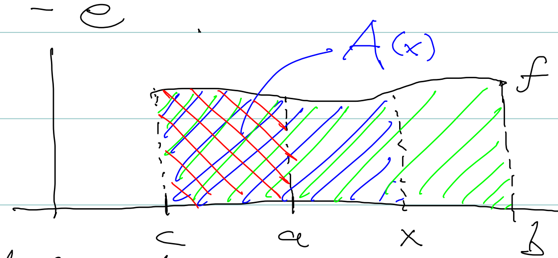
If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b$$

where F is an arbitrary antiderivative of f.

Ex: $\int_1^2 e^x\,dx = e^x\Big|_1^2 = e^2 - e^1$.

WHY? Check:



$A(x) = \int_c^x f(t)\,dt$ is an antiderivative of f.

$$\Longrightarrow \int_a^b f(t)\,dt = A(b) - A(a).$$

$$= (F(b) + \cancel{C}) - (F(a) + \cancel{C})$$

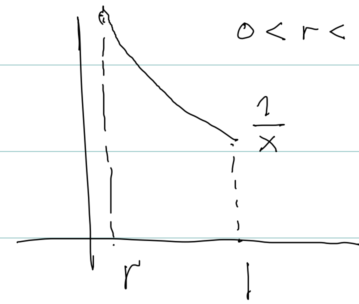
$$= F(b) - F(a)$$

Arbitrary antiderivative:

$$A(x) = F(x) + C$$

Ex $\displaystyle\int_1^{10} \frac{1}{x}\,dx = \ln x \Big|_1^{10} = \ln 10 - \underbrace{\ln 1}_{0}$

Ex $\displaystyle\int_r^{1} \frac{1}{x}\,dx = \ln x \Big|_r^{1} = \underbrace{\ln 1}_{=0} - \ln r$ $\quad 0 < r < 1$



$= -\ln r > 0$

$\longrightarrow \infty$ when $r \to 0^+$

$\Rightarrow \frac{1}{x}$ is not integrable on $[0,1]$.

$\underline{\underline{Ex}}$ $\displaystyle\int_0^{\pi/4} \frac{1}{\cos^2 x}\,dx = \tan x \Big|_0^{\frac{\pi}{4}} = \tan\frac{\pi}{4} - \tan 0$

$= 1 - 0 = 1.$

$\underline{\underline{Ex}}$ $\displaystyle\int_a^b x^r\,dx = \frac{x^{r+1}}{r+1}\Big|_a^b = \frac{b^{r+1} - a^{r+1}}{r+1}.$

$r \neq -1$

$\underline{\underline{Ex}}$ $\displaystyle\int_0^{\pi/4} \frac{\sin x}{\cos x}\,dx = -\ln(\cos x)\Big|_0^{\pi/4} = -\left(\ln\left(\cos\frac{\pi}{4}\right) - \ln(\cos 0)\right).$

$= -\left(\ln\frac{\sqrt{2}}{2} - \underbrace{\ln 1}_{0}\right)$

$= -\ln\frac{\sqrt{2}}{2}.$

Indefinite integrals

Convenient notation for the most general antiderivative.

$$\int f(x)\,dx = F(x) + C$$

Ex $$\int (x^3 + x)\,dx = \frac{x^4}{4} + \frac{x^2}{2} + C$$

Note: A definite integral $\int_a^b f(x)\,dx$ is a number (an area).

An indefinite integral $\int f(x)\,dx$ is a function (general antiderivative).

Properties:

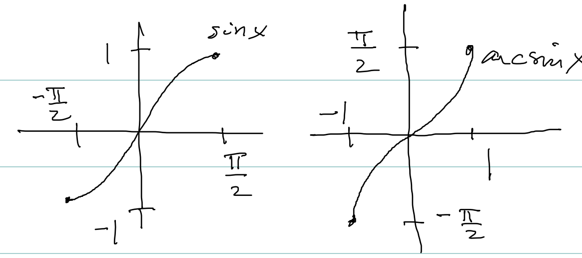
$$\int c f(x)\,dx = c \int f(x)\,dx$$

$$\int \big(f(x) \pm g(x)\big)\,dx = \int f(x)\,dx \pm \int g(x)\,dx.$$

Interlude: Derivatives of inverse trigonometric functions.

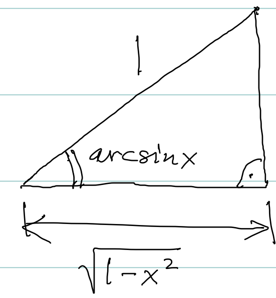
$\arcsin x = \sin^{-1} x$. $\quad \sin(\arcsin x) = x$, $\arcsin(\sin x) = x$.

Find $(\arcsin x)' = ?$



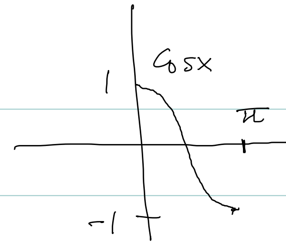
Differentiate:

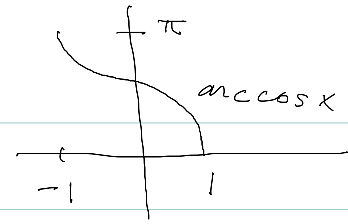
$$x = \sin(\arcsin x)$$
$$\Downarrow$$
$$1 = \cos(\arcsin x) \cdot (\arcsin x)'$$
$$\Downarrow$$
$$(\arcsin x)' = \frac{1}{\cos(\arcsin x)}.$$
$$= \frac{1}{\sqrt{1-x^2}}$$



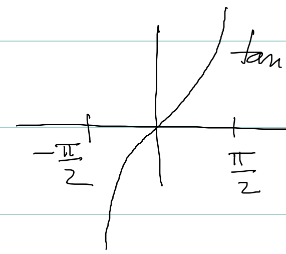
$\Longrightarrow \cos(\arcsin x) = \dfrac{\sqrt{1-x^2}}{1}$

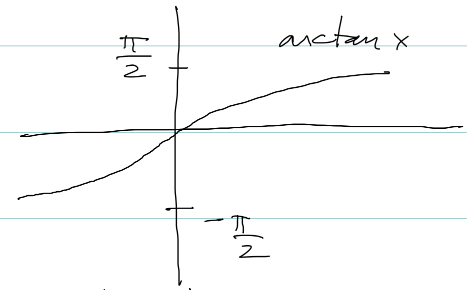
Similarly, $(\arccos x)' = \dfrac{-1}{\sqrt{1-x^2}}$





$$(\arctan x)' = \frac{1}{1+x^2}$$





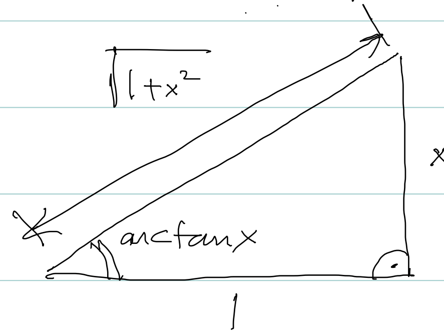
$$x = \tan(\arctan x)$$

differentiate $\Downarrow$

$$1 = \frac{1}{\cos^2(\arctan x)} \cdot (\arctan x)'$$

$$\Downarrow$$

$$(\arctan x)' = \cos^2(\arctan x) = \frac{1}{1+x^2}$$



$$\cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}$$

Ex

$$\int a^x\,dx = \frac{a^x}{\ln a} + C$$

$$\int \frac{1}{x}\,dx = \ln x + C$$

$$\int x^r\,dx = \frac{x^{r+1}}{r+1} + C \qquad r \neq -1$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

$$\int \tan x\,dx = -\ln(\cos x) + C$$

$$\int \ln x\,dx = x\ln x - x + C$$

$$\int \frac{-1}{\sqrt{1-x^2}}\,dx = \arccos x + C$$

Interpretation of FToC, Part II.

Thm (Net change thm)

$$\underbrace{\int_a^b f'(x)\,dx}_{\text{rate of change, integrated from a to b.}} = f(x)\Big|_a^b = \underbrace{f(b)-f(a)}_{\text{total change of f going from a to b.}}$$

The substitution rule.

Helps to solve an integral where the function was obtained from differentiation using chain rule.

$$\int g'(f(x)) \cdot f'(x)\,dx = g(f(x)) + C$$

because by chain rule, $(g(f(x)))' = g'(f(x)) \cdot f'(x)$.

$$\int_a^b g'(f(x)) \cdot f'(x)\,dx = g(f(x))\Big|_a^b = g(f(b)) - g(f(a)).$$

Ex $\int \frac{\cos x}{\sin^2 x}\,dx = \int g'(f(x))\cdot f'(x)\,dx$

$f'(x) = \cos x \Longrightarrow f(x) = \sin x$

$g'(f(x)) = g'(\sin x) = \frac{1}{\sin^2 x} \Longrightarrow g'(x) = \frac{1}{x^2} \Rightarrow g(x) = \frac{-1}{x}$

$\dots = g(f(x)) + C = \frac{-1}{\sin x} + C$

Ex $\displaystyle\int_2^{10} \frac{1}{x}\,\frac{1}{(\ln x)^3}\,dx = \int_2^{10} g'(f(x))\cdot f'(x)\,dx$

$f'(x) = \frac{1}{x} \Rightarrow f(x) = \ln x$

$g'(f(x)) = g'(\ln x) = \frac{1}{(\ln x)^3} \Rightarrow g'(x) = \frac{1}{x^3} \Rightarrow g(x) = \frac{-1}{2x^2}$

$$\dots = g(f(x))\Big|_2^{10} = -\frac{1}{2(\ln x)^2}\Big|_2^{10}$$

$$= -\left(\frac{1}{2(\ln 10)^2} - \frac{1}{2(\ln 2)^2}\right)$$

Substitution rule.

$$\int g'(\underbrace{f(x)}_{=:u}) \cdot \underbrace{f'(x)\,dx}_{du}$$

$u = f(x)$.

$du = f'(x)\,dx$

$$= \int g'(u)\,du = g(u) + C$$

$$= g(f(x)) + C$$

Ex $\int \sin^7 x \cos x \, dx = \int u^7 \, du = \frac{u^8}{8} + C = \frac{\sin^8 x}{8} + C$

$u = \sin x \implies du = \cos x \, dx$

Ex $\int x^5 \cos(x^6+1) \, dx = \int \cos u \, \frac{1}{6} du = \frac{1}{6} \int \cos u \, du$

$\frac{1}{6} du$

$u = x^6 + 1 \implies du = 6x^5 dx$

$= \frac{\sin u}{6} + C$

$= \frac{\sin(x^6+1)}{6} + C$

$\underline{Ex}$ $\int \sqrt{1+x^2}\, x^5\,dx = \int \sqrt{1+x^2}\, x^4 \cdot \underbrace{x\,dx}_{\frac{1}{2}du} = \frac{1}{2}\int \sqrt{u}\,(u-1)^2\,du$

$u = 1+x^2 \Rightarrow du = 2x\,\underline{dx}$

$\Downarrow$

$x^2 = u-1$

?

$\int \sqrt{u}\,(u-1)^{5/2}\,dx$

$$\cdots = \frac{1}{2}\int u^{\frac{1}{2}}\left(u^2-2u+1\right)du$$

$$= \frac{1}{2}\int \left(u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\right)du$$

$$= \frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{7}u^{7/2} - \cancel{2}\cdot\frac{2}{5}u^{\frac{5}{2}} + \frac{\cancel{2}}{3}u^{3/2}\right) + C.$$

$$= \frac{1}{7}\left(1+x^2\right)^{7/2} - \frac{2}{5}\left(1+x^2\right)^{5/2} + \frac{1}{3}\left(1+x^2\right)^{3/2} + C$$

Substitution rule for definite integrals.

Thm $$\int_a^b g'(\underbrace{f(x)}_{u}) \cdot \underbrace{f'(x)\,dx}_{du} = g(f(x))\Big|_a^b = g(f(b)) - g(f(a)).$$

$$\int_{f(a)=u(a)}^{f(b)=u(b)} g'(u)\,du = g(u)\Big|_{f(a)}^{f(b)} = g(f(b)) - g(f(a))$$

Ex $$\int_0^4 \sqrt{2x+1}\,dx = \int_1^9 \sqrt{u}\ \frac{1}{2}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_1^9$$

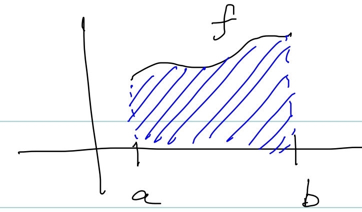
$u = 2x+1 \Rightarrow du = 2dx$

$u(0) = 1, \ u(4) = 9$

$$= \frac{1}{3}\left(9^{3/2} - 1^{3/2}\right) = \frac{1}{3}(27-1)$$

$$= \frac{26}{3}$$

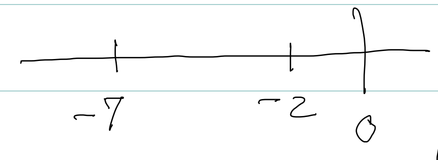
Thm $$\int_a^b f(t)\,dt = -\int_b^a f(t)\,dt$$



Ex $$\int_1^2 \frac{1}{(3-5x)^2}\,dx = \int_{-2}^{-7} \frac{1}{u^2}\,\frac{-1}{5}\,du$$

$$u = 3-5x \Rightarrow du = -5\,dx$$

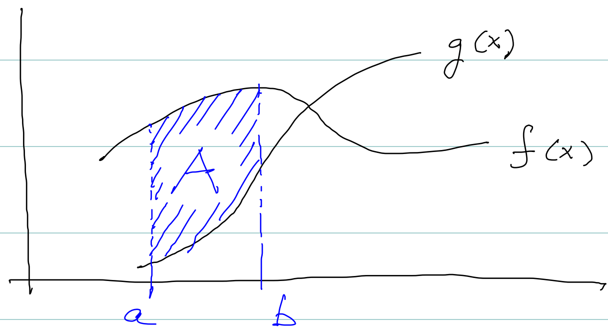
$$u(1) = -2, \quad u(2) = -7$$



$$= \int_{-7}^{-2} \frac{1}{u^2}\,\frac{+1}{5}\,du = \frac{1}{5}\,\frac{-1}{u}\Big|_{-7}^{-2} = \frac{1}{5}\left(\frac{-1}{-2} - \frac{-1}{-7}\right)$$

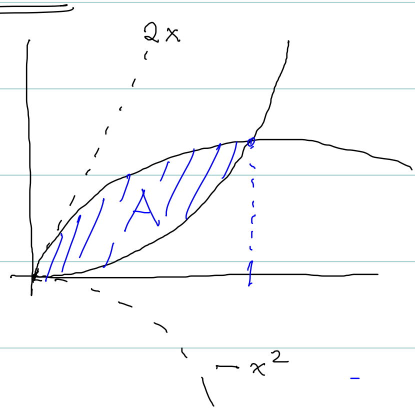
$$= \frac{1}{5}\left(\frac{1}{2} - \frac{1}{7}\right) = \frac{1}{5}\,\frac{\overbrace{7-2}^{5}}{14} = \underline{\underline{\frac{1}{14}}}$$

Areas between curves.



$$A = \int_a^b \big(f(x) - g(x)\big)\,dx.$$

Ex Find the area enclosed by $y = x^2$ and $y = 2x - x^2$, $x \geq 0$



Intersection point: $x^2 = 2x - x^2 \Rightarrow 2x^2 = 2x$

$\Rightarrow x = 0, 1$.

$$A = \int_0^1 \left(2x - x^2 - x^2\right) dx = 2\int_0^1 (x - x^2)\,dx$$

$$= 2\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = 2\left(\frac{1}{2} - \frac{1}{3} - 0\right).$$

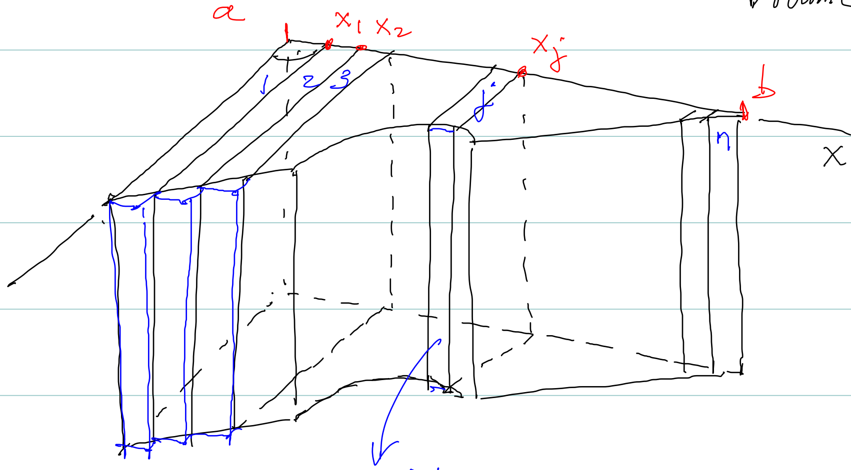
$$= 2\,\frac{3-2}{6} = \underline{\underline{\frac{1}{3}}}$$

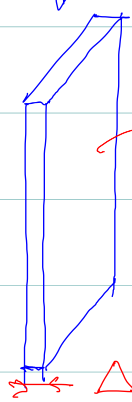
Volumes

$$\text{Volume} = \lim_{n\to\infty}\underbrace{\left(A(x_1)\Delta x + A(x_2)\Delta x + \dots + A(x_n)\Delta x\right)}_{\text{Riemann sum}}$$

$$= \int_a^b A(x)\,dx$$

area of cross-section at x



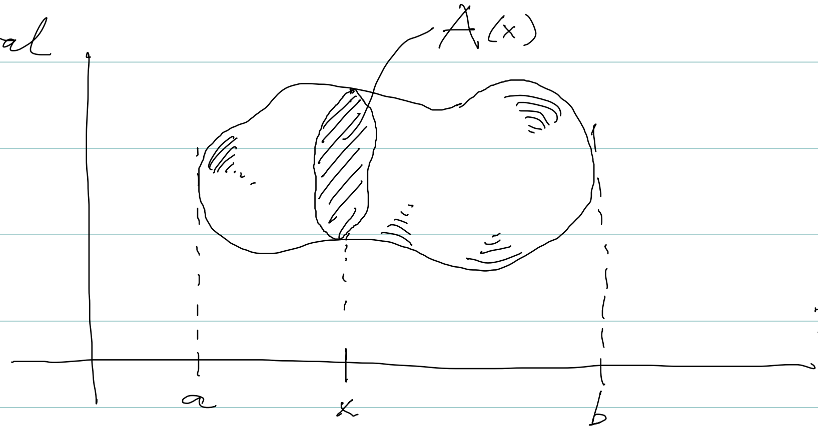


$A(x_j)$

Volume $A(x_j)\cdot\Delta x$

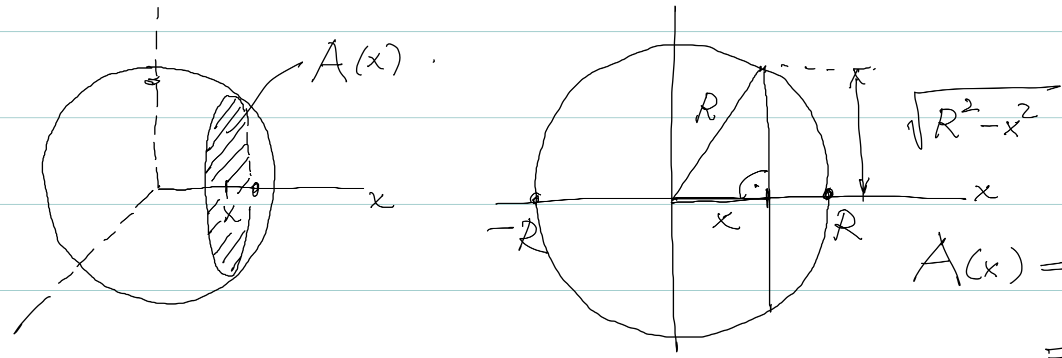
$\Delta x = \dfrac{b-a}{n}$

In general



$$\text{Volume} = \int_a^b A(x)\,dx.$$

Ex Find the volume of a ball of radius R.



$$A(x) = \pi\left(\sqrt{R^2-x^2}\right)^2 = \pi\left(R^2-x^2\right).$$

$$\text{Volume} = \int_{-R}^{R} A(x)\,dx = \int_{-R}^{R} \pi\left(R^2-x^2\right)dx = \pi\int_{-R}^{R}\left(R^2-x^2\right)dx$$

$$= \pi\left(R^2x - \frac{x^3}{3}\right)\Big|_{-R}^{R} = \pi\left(R^2\cdot R - \frac{R^3}{3} - \left(R^2(-R) - \frac{(-R)^3}{3}\right)\right)$$

$$= \pi\left(R^3 - \frac{R^3}{3} + R^3 - \frac{R^3}{3}\right) = \pi R^3\left(2-\frac{2}{3}\right) = \frac{4\pi}{3}R^3.$$

$\underline{\underline{Ex}}$ $\lim_{x\to\infty} (e^x + x)^{\frac{1}{x}} = \lim_{x\to\infty} e^{\frac{1}{x}\ln(e^x+x)}$

$= e^{\lim_{x\to\infty} \frac{1}{x}\ln(e^x+x)} = e^{1} = e$

if limit exists; exponential is continuous.

$\lim_{x\to\infty} \frac{\ln(e^x+x)}{x} \overset{\text{de l'H}}{=} \lim_{x\to\infty} \frac{\frac{1}{e^x+x}\cdot(e^x+1)}{1} = \lim_{x\to\infty} \frac{e^x+1}{e^x+x}$

$\overset{\text{de l'H}}{=} \lim_{x\to\infty} \frac{e^x}{e^x+1} \overset{\text{de l'H}}{=} \lim_{x\to\infty} \frac{\cancel{e^x}}{\cancel{e^x}} = 1.$

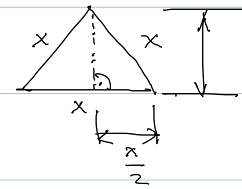
$$\underline{\underline{Ex}} \quad \lim_{t\to\infty} t \ln\left(1+\frac{3}{t}\right) = \lim_{t\to\infty} \frac{\ln\left(1+\frac{3}{t}\right)}{\frac{1}{t}}$$

$$\underset{\text{de l'H}}{=} \lim_{t\to\infty} \frac{\frac{1}{1+\frac{3}{t}}\cdot\left(\frac{-3}{t^2}\right)}{-\frac{1}{t^2}} = \lim_{t\to\infty} \frac{1}{1+\frac{3}{t}}\cdot\left(\frac{+3}{\cancel{t^2}}\right)\cdot\left(+\cancel{t^2}\right)$$

$$= \lim_{t\to\infty} \frac{1}{1+\frac{3}{t}}\cdot 3 = \underline{3}.$$

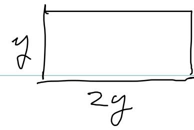
Ex 30 cm wire cut into 2 pieces.

one piece $\longrightarrow$ triangle, equilateral



$\frac{\sqrt{3}}{2}x = \sqrt{x^2 - \left(\frac{x}{2}\right)^2}$

$= \sqrt{\frac{3}{4}x^2}$

other piece $\longrightarrow$ rectangle 

Where to cut the wire to have max area enclosed?

Optimization: Area $A = \frac{\sqrt{3}}{4}x^2 + 2y^2$

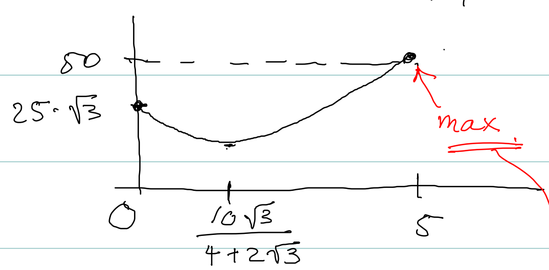
Constraint: $3x + 6y = 30 \Rightarrow x + 2y = 10 \Rightarrow x = 10 - 2y$

Plug into $A = \frac{\sqrt{3}}{4}(10-2y)^2 + 2y^2$. domain of y: $[0,5]$

Derivative $A'(y) = \frac{\sqrt{3}}{4} \cdot 2(10-2y)\cdot(-2) + 4y = -10\sqrt{3} + 2\sqrt{3}y + 4y = 0$

$\Rightarrow -10\sqrt{3} + (4 + 2\sqrt{3})y = 0 \Rightarrow y = \frac{10\sqrt{3}}{4+2\sqrt{3}}$ in $[0,5]$.

Min/max? $A''\left(\frac{10\sqrt{3}}{4+2\sqrt{3}}\right) = 4 + 2\sqrt{3} > 0$ min!



$A(0) = \frac{\sqrt{3}}{4}\cdot 100 + 2\cdot 0 = 25\cdot\sqrt{3}$.

$A(5) = \frac{\sqrt{3}}{4}\underbrace{(10-2\cdot 5)^2}_{0} + 2\cdot 5^2 = 50$

$y = 5 \Rightarrow$ only rectangle.

Ex Assume x, y satisfy $6y^2 + x^2 = 2 - x^3 \cdot e^{4-4y}$

Determine y' when $x = -2$, $y = 1$.

Check: $6 \cdot 1^2 + (-2)^2 \stackrel{?}{=} 2 - (-2)^3 \cdot e^{4-4\cdot 1}$

$6 + 4 \stackrel{?}{=} 2 - (-8) \cdot e^0$ ✓

Implicit differentiation. $12y \cdot y' + 2x = -3x^2 e^{4-4y} - x^3 e^{4-4y} \cdot (-4y')$

$\Rightarrow \quad 12 \cdot 1 \cdot y' + 2(-2) = -3(-2)^2 e^{4-4\cdot 1} - (-2)^3 e^{4-4\cdot 1}(-4y')$

$12y' - 4 = -12 - 32y'$

$\Rightarrow \quad 44y' = -8 \Rightarrow y' = \frac{-8}{44} = -\frac{2}{11}$

Ex Find $f(5.01)$ when $f(x) = 3x\, e^{2x-10}$.

$\Rightarrow$ Linear approximation: $a = 5$.

$$f(5) = 3 \cdot 5 \cdot e^{2 \cdot 5 - 10} = 15.$$

$$f'(x) = 3\, e^{2x-10} + 3x\,(2)\, e^{2x-10}$$

$$f'(5) = 3 + 6 \cdot 5 = 33$$

Linearization $y = f(a) + f'(a)(x-a)$

$\Rightarrow$ $y = 15 + 33(x-5)$. $x = 5.01$.

$\Rightarrow$ $f(5.01) \approx 15 + 33 \cdot 0.01 = 15.33$

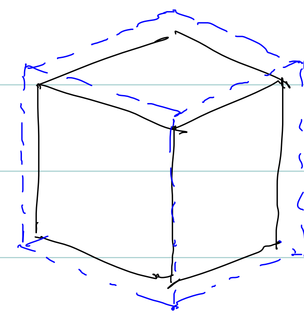
Ex Square box of side lengths a.

By how much does the volume change when all sides increase by Δx when $a = 1\ ft$.

Volume $\quad V = a^3$

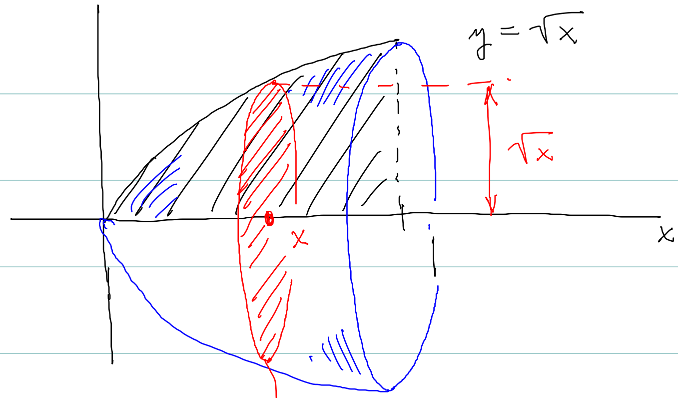
Rate of change as a differential

$$\Delta V = 3a^2 \cdot \Delta x$$



$a = 1\ ft \Rightarrow \quad \Delta V = 3 \cdot \Delta x$

Ex

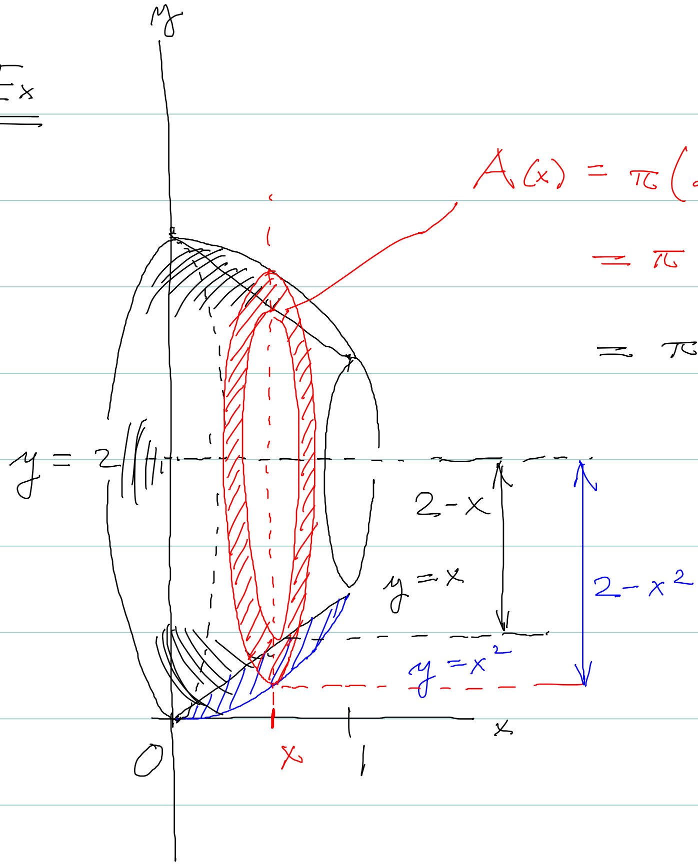


rotate area around x-axis.

$A(x) = \pi\left(\sqrt{x}\right)^2 = \pi x$.

$$\text{Volume} = \int_0^1 \pi x \, dx = \pi \frac{x^2}{2}\Big|_0^1 = \frac{\pi}{2}.$$

Ex



Rotate area around axis $y = 2$.

$$A(x) = \pi(2-x^2)^2 - \pi(2-x)^2$$

$$= \pi\left(\cancel{4} - 4x^2 + x^4 - (\cancel{4} - 4x + x^2)\right).$$

$$= \pi\left(-5x^2 + x^4 + 4x\right)$$

$$\text{Volume} = \int_0^1 \pi(-5x^2 + x^4 + 4x)\,dx$$

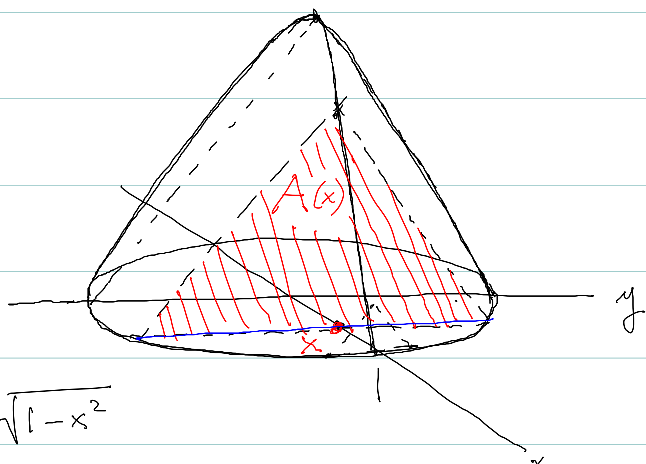
$$= \pi\left(-5\frac{x^3}{3} + \frac{x^5}{5} + 4\frac{x^2}{2}\right)\Big|_0^1$$

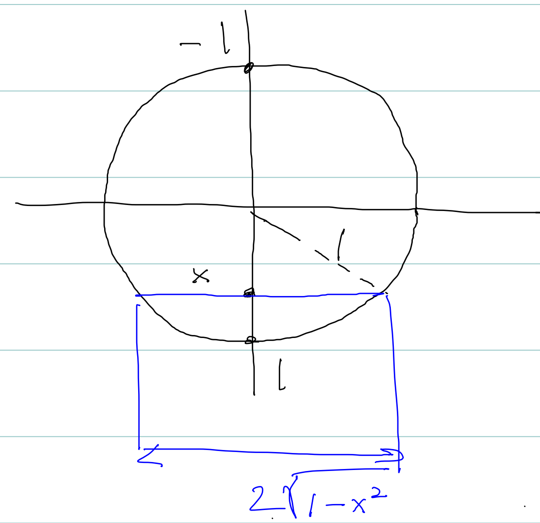
$$= \pi\left(-\frac{5}{3} + \frac{1}{5} + 2\right)$$

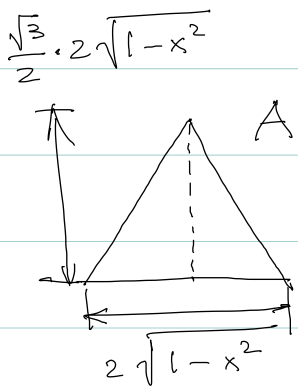
$$= \pi\,\frac{-25+3+30}{15} = \pi\,\frac{8}{15}$$

Ex Solid with a circular base of radius 1.

Cross-sections perpendicular to x-axis are equilateral triangles.







$$A(x) = \frac{1}{2}\sqrt{3}\sqrt{1-x^2}\cdot 2\sqrt{1-x^2}$$
$$= \sqrt{3}\,(1-x^2).$$

$$\text{Volume} = \int_{-1}^{1}\sqrt{3}\,(1-x^2)\,dx = \sqrt{3}\left(x-\frac{x^3}{3}\right)\Big|_{-1}^{1} = \sqrt{3}\left(1-\frac{1}{3}-\left(-1-\frac{-1}{3}\right)\right)$$
$$= \underline{\underline{\sqrt{3}\,\frac{4}{3}}}.$$

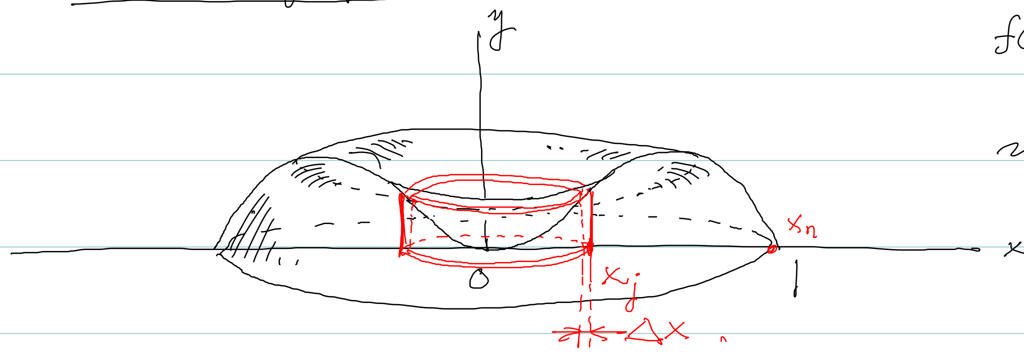
Volumes using cylindrical shells.

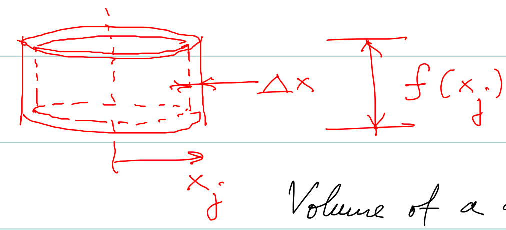
Ex

$f(x) = 3x^2 - 3x^3$.

$0 \le x \le 1$

rotate around y-axis.





$$A(x_j) = \underbrace{2\pi x_j}_{\text{circumference}} \cdot \underbrace{f(x_j)}_{\text{height}}$$

Volume of a cylindrical shell:

$A(x_j) \cdot \Delta x$

Riemann sum.

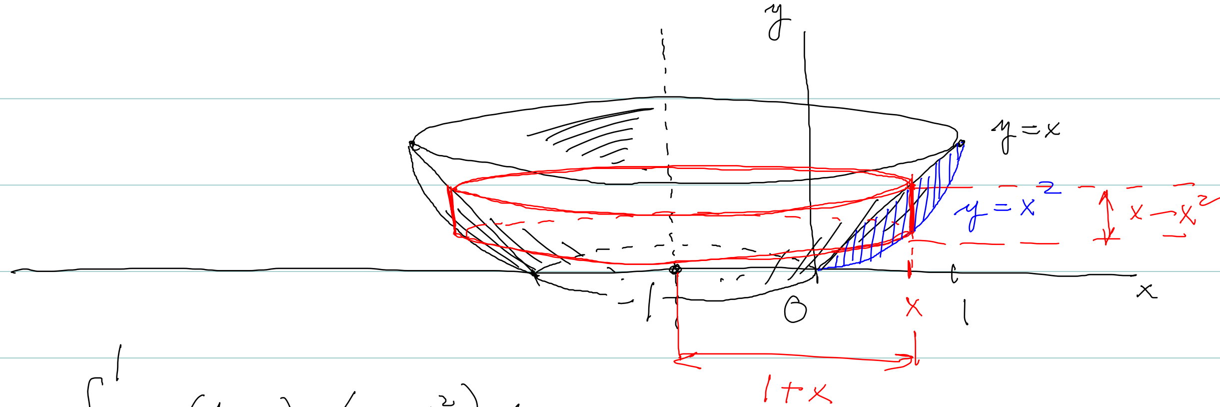
$$\text{Total volume} = \lim_{n\to\infty} \left(A(x_1)\Delta x + A(x_2)\Delta x + \cdots + A(x_n)\Delta x \right)$$

$$= \int_0^1 A(x)\,dx = \int_0^1 2\pi x\, f(x)\,dx$$

$$\text{Volume} = \int_0^1 2\pi x\left(3x^2 - 3x^3\right)dx = 6\pi \int_0^1 \left(x^3 - x^4\right)dx$$

$$= 6\pi\left(\frac{x^4}{4} - \frac{x^5}{5}\right)\Big|_0^1 = 6\pi\left(\frac{1}{4} - \frac{1}{5}\right) = 6\pi\,\frac{1}{20} = \pi\,\frac{3}{10}.$$

Ex



$$\text{Volume} = \int_0^1 2\pi(1+x)\cdot(x-x^2)\,dx$$

$$= 2\pi\int_0^1 (x-\cancel{x^2}+\cancel{x^2}-x^3)\,dx = 2\pi\int_0^1 (x-x^3)\,dx = 2\pi\left(\frac{x^2}{2}-\frac{x^4}{4}\right)\Big|_0^1$$

$$= 2\pi\left(\frac{1}{2}-\frac{1}{4}-0\right) = 2\pi\frac{1}{4} = \underline{\underline{\frac{\pi}{2}}}$$

Ex $\int 5 \underbrace{\sin^3 x}_{\sin x \cdot \sin^2 x \,=\, \sin x \cdot (1-\cos^2 x)} \cos^2 x \, dx$

$$= \int 5 \sin x \,(1-\cos^2 x)\cos^2 x \, dx$$

$u = \cos x \implies du = -\sin x \, dx$

$$= -\int 5(1-u^2)\,u^2 \, du$$

$$= -5\int (u^2 - u^4)\, du$$

$$= -5\left(\frac{u^3}{3} - \frac{u^5}{5}\right) + C = -\frac{5}{3}u^3 + u^5 + C$$

$$= -\frac{5}{3}\cos^3 x + \cos^5 x + C$$

Ex $\int \sqrt{1-\cos\theta}\, d\theta = \ldots\ldots$

use $\cos 2x = 1 - 2\sin^2 x$

$\Rightarrow 2x = \theta \Rightarrow \cos\theta = 1 - 2\sin^2\frac{\theta}{2}$

$\sqrt{1-\cos\theta} = \sqrt{2\sin^2\frac{\theta}{2}}$

Ex $\int \left(5\cos\theta + 3\cos^3\theta\right) d\theta$

$\cos\theta \cdot \cos^2\theta = \cos\theta\left(1-\sin^2\theta\right)$

$= \int\left(5\cos\theta + 3\cos\theta\left(1-\sin^2\theta\right)\right) d\theta$

$= \int 8\cos\theta\, d\theta - 3\int \sin^2\theta \cos\theta\, d\theta$

$u = \sin\theta \Rightarrow du = \cos\theta\, d\theta$

$= \ldots\ldots$

$\underline{\underline{Ex}}$ $\int \sec^8(2x)\tan(2x)\,dx$

$$= \int \frac{1}{\cos^8(2x)} \, \frac{\sin(2x)}{\cos(2x)}\,dx = \int \frac{1}{\cos^9(2x)} \sin(2x)\,dx$$

$u = \cos(2x) \Longrightarrow du = -2\sin(2x)\,dx$

$\Downarrow$

$\sin(2x)\,dx = -\frac{1}{2}\,du$

$$= -\frac{1}{2}\int \frac{1}{u^9}\,du$$

$= \dots$

Ex $\displaystyle\int \frac{\arcsin x}{\sqrt{1-x^2}}\,dx$

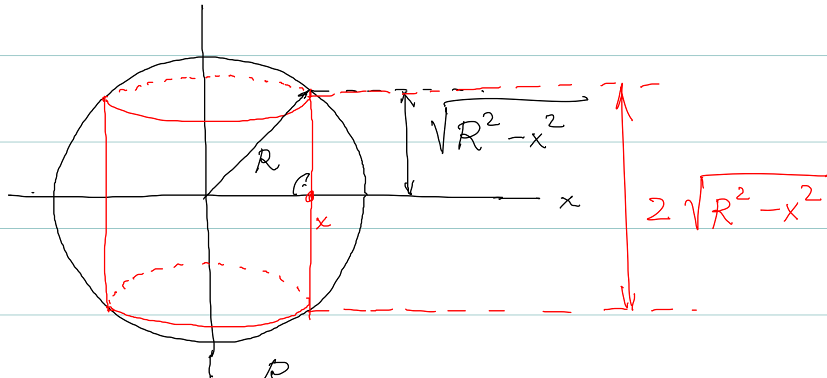
$u = \arcsin x \implies x = \sin u \implies dx = \cos u\,du$

$$= \int \frac{u}{\sqrt{1-\sin^2 u}} \cos u\,du = \int \frac{u}{\sqrt{\cos^2 u}} \cos u\,du$$

$$= \int \frac{u}{\cancel{\cos u}} \cancel{\cos u}\,du = \int u\,du = \dots\,.$$

assume $\cos u > 0$.

Ex Ball of radius R.



$$\text{Volume} = \int_0^R 2\pi x \cdot 2\sqrt{R^2-x^2}\,dx = 4\pi \int_0^R x\sqrt{R^2-x^2}\,dx$$

$u = R^2 - x^2 \Rightarrow du = -2x\,dx$

$$= 4\pi \int_{R^2 = u(x=0)}^{0 = u(x=R)} \sqrt{u}\left(-\frac{1}{2}\right) du$$

$\Rightarrow x\,dx = -\frac{1}{2}\,du$.

$$= \frac{4\pi}{2}\int_0^{R^2} \sqrt{u}\,du, = 2\pi\,\frac{2}{3}\,u^{3/2}\Big|_0^{R^2} = \frac{4\pi}{3}R^3$$

Ex $$K(x) = \int_{g(x)}^{h(x)} f(t)\,dt = F(h(x)) - F(g(x)).$$

$\uparrow$ antiderivative of f.

$$K'(x) = ?$$

$$\boxed{K'(x)} = F'(h(x))\cdot h'(x) - F'(g(x))\cdot g'(x)$$

$$\boxed{= f(h(x))\cdot h'(x) - f(g(x))\cdot g'(x)}.$$

Ex $$K(x) = \int_{x}^{x^2} \ln(\cos t)\,dt.$$

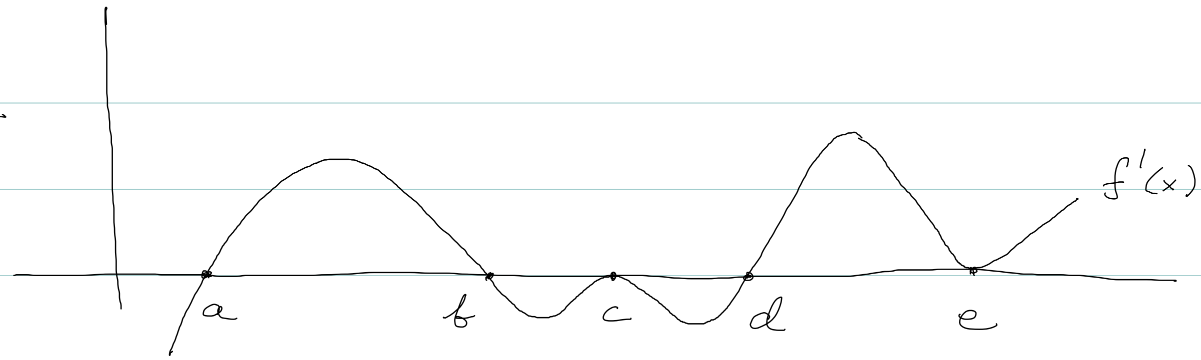
$$\Rightarrow K'(x) = \ln(\cos(x^2))\cdot 2x - \ln(\cos x)\cdot 1.$$

Ex $\lim_{x\to\infty}\left(\sqrt{x^2+1}-\sqrt{x^2+2}\right) = ?$

$$= \lim_{x\to\infty}\left(\sqrt{x^2+1}-\sqrt{x^2+2}\right)\frac{\sqrt{x^2+1}+\sqrt{x^2+2}}{\sqrt{x^2+1}+\sqrt{x^2+2}}.$$

$$= \lim_{x\to\infty}\frac{x^2+1-(x^2+2)}{\sqrt{x^2+1}+\sqrt{x^2+2}} = \lim_{x\to\infty}\frac{-1}{\sqrt{x^2+1}+\sqrt{x^2+2}} = 0$$

Ex



f has the following properties:

$x=d$, $x=a$: crit pt, local min

$x=b$: crit pt, local max.

$x=c$: crit pt, neither loc min nor max. f decreasing

$f''(c)=0$, $f''(x)$ changes sign at $x=c \Rightarrow$ inflection pt

$x=e$: crit pt, neither loc min nor loc max. f increasing.

$f''(e)=0$, $f''(x)$ changes sign at $x=e \Rightarrow$ inflection pt.

Ex $f(x) = x^{x^2} \Rightarrow$ find $f'(x) = ?$

$$f(x) = \left(e^{\ln x}\right)^{x^2} = e^{x^2 \cdot \ln x}$$

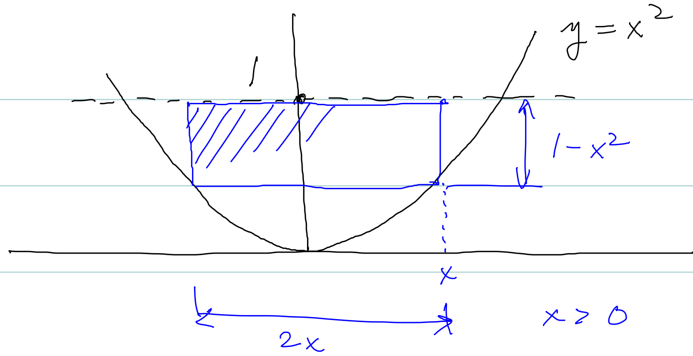
$$f'(x) = e^{x^2 \ln x}\left(x^2 \cdot \ln x\right)'$$

$$= x^{x^2}\left(2x \cdot \ln x + \underbrace{x^2 \cdot \frac{1}{x}}_{x}\right)$$

$$= x^{x^2} \cdot x\left(2 \ln x + 1\right)$$

$$= x^{x^2+1}\left(2 \ln x + 1\right).$$

Ex



Find rectangle with largest area.

Area $A(x) = 2x(1-x^2)$

$A'(x) = 2(1-x^2) + 2x(-2x) = 2 - 2x^2 - 4x^2 = 2 - 6x^2 = 0$

$\Rightarrow x = \pm \frac{1}{\sqrt{3}} \Rightarrow x = \frac{1}{\sqrt{3}} > 0$ for rectangle.

Max or min? $A''(x) = -12x \Rightarrow A''\left(\frac{1}{\sqrt{3}}\right) < 0$

$\Rightarrow$ max.