$$x \cdot y = \underbrace{a^{\log_a x}}_{x} \; \underbrace{a^{\log_a y}}_{y} = \boxed{a^{\log_a x + \log_a y}} \quad , x, y > 0$$

$$a > 0.$$

$$= \boxed{a^{\log_a (x \cdot y)}}$$

$$\Rightarrow \boxed{\log_a (x \cdot y) = \log_a x + \log_a y.}$$

$$\frac{x}{y} = \underbrace{a^{\log_a x}}_{x} \; \underbrace{a^{-\log_a y}}_{\frac{1}{y}.} = a^{\log_a x - \log_a y}.$$

$$= a^{\log_a \frac{x}{y}}$$

$$\Rightarrow \boxed{\log_a \frac{1}{y} = - \log_a y} \qquad (x = 1).$$

<u>Ex</u>. $\log_a (x^{10}) = \log_a (\underbrace{x \cdot x \cdots x}_{10}) = \log_a x + \log_a (\underbrace{x \cdot x \cdots x}_{9})$

$$= \ldots = 10 \cdot \log_a x.$$

More generally: $\boxed{\log_a x^r = r \log_a x.}$, for any $r \in (-\infty, \infty)$

<u>Ex</u>: $\log_a \sqrt{x} = \log_a x^{1/2} = \frac{1}{2} \log_a x$

$\log_a \frac{1}{x} = \log_a x^{-1} = (-1) \cdot \log_a x = -\log_a x$

$\log_a 1 = \log_a x^0 = 0 \cdot \log_a x = 0$

$\log_a 0$ <u>not</u> defined.

$$\boxed{\log_b x = (\log_a x)\cdot(\log_b a)}$$

Ex Check.

$$\underline{\underline{x}} = b^{\log_b x} = b^{(\log_a x)\cdot(\log_b a)}$$

$$b^{\alpha\cdot\beta} = \left(b^{\beta}\right)^{\alpha} = \left(b^{\alpha}\right)^{\beta}$$

$$= \Big(\underbrace{b^{\log_b a}}_{a}\Big)^{(\log_a x)}$$

$$= a^{\log_a x} = \underline{\underline{x}}$$

Ex Find.

$$\log_2 4^{17} - 3^{\log_9 81} = \underline{\underline{25}}.$$

$$\log_2 4^{17} = \log_2 (2^2)^{17} = \log_2 2^{34} = 34\cdot\underbrace{\log_2 2}_{1.} = 34$$

$$3^{\log_9 81} = 3^{\log_9 9^2} = 3^{2\,\overbrace{\log_9 9}^{1}} = 3^2 = 9$$

Notation: $\boxed{\log_e x = \ln x}$ base e, Euler number.

natural logarithm.

Limits

Calculus is very different from algebra in that we do <u>not</u> usually try to solve equations <u>exactly</u>, but we will try to find arbitrarily good approximations with good error control.

Ex

$$1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\dots = \dots$$ arbitrarily close to 2.

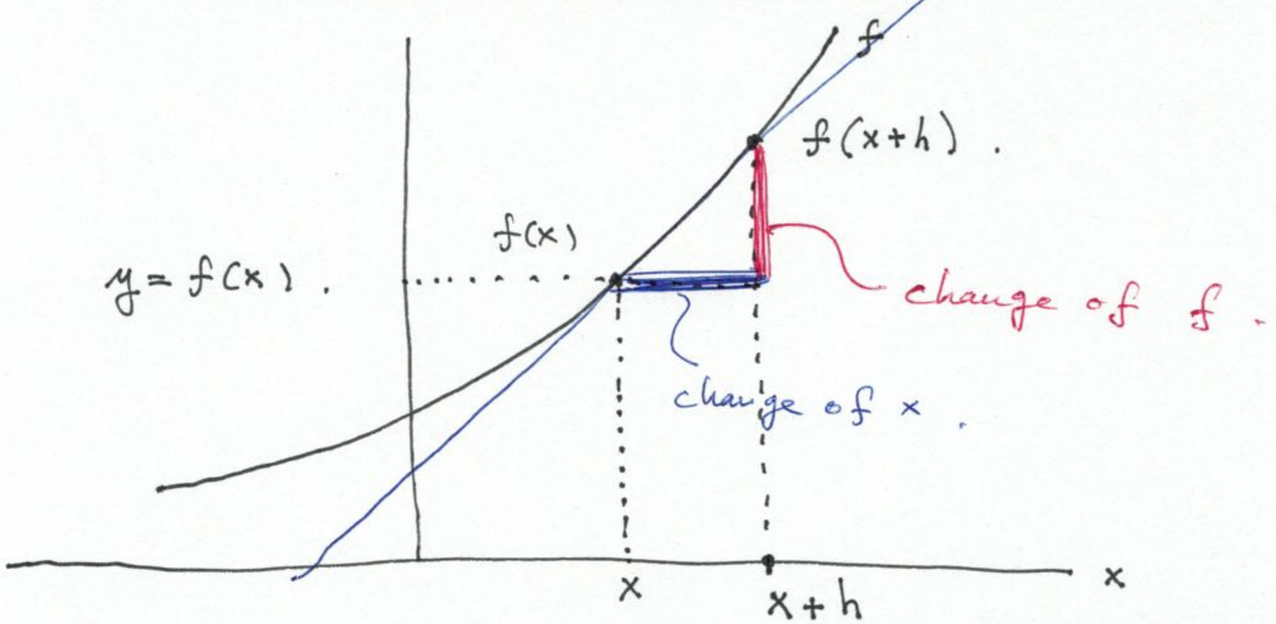
<u>Notation:</u>

$$\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{2^2}+\dots+\frac{1}{2^n}\right)=2$$

<u>Q</u>: How many terms do I have to include so that my error is smaller than a given tolerance, for instance 0.001?

$\Rightarrow$ Calculus gives us a way to determine this.

Rates of change of functions

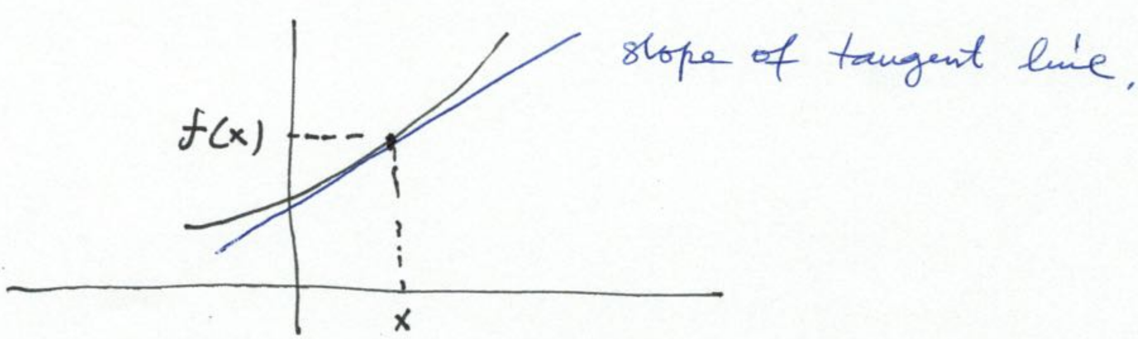


$$\text{Rate of change} := \frac{\text{change of } f}{\text{change of } x} = \frac{f(x+h) - f(x)}{\cancel{x} + h - \cancel{x}} .$$

" defined as "

This is the slope of the blue straight line .

<u>Def:</u> $f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ derivative of f at x.



Ex: $f(x) = x^2$.

$$(x^2)' = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}$$

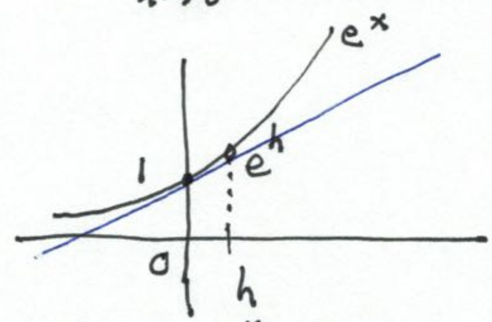
$$= \lim_{h\to 0} (2x + h) = 2x$$

Ex $$(e^x)' = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h}.$$

$$= \lim_{h\to 0} \frac{e^x e^h - e^x}{h}$$

$$= e^x \lim_{h\to 0} \frac{e^h - 1}{h} = e^x \lim_{h\to 0} \overbrace{\frac{e^{0+h} - e^0}{h}}^{1}$$

$$= e^x$$



slope of e^x as it crosses y-axis: 1

Ex: $$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}}$$

$$= \lim_{x\to 1} (x+1) = 2$$