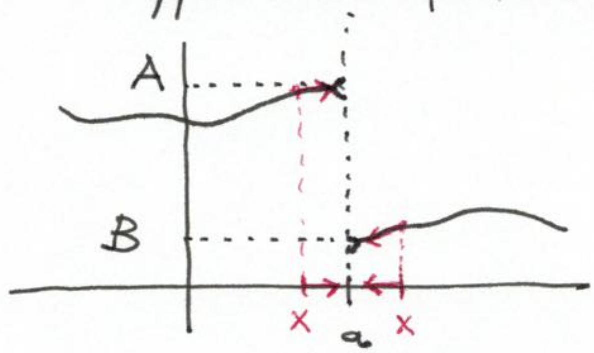
Left- and right limits.

Def Left limit $\lim\limits_{x \to a^-} f(x) = A$ — $f(x)$ approaches A when x approaches a from the left

Right limit $\lim\limits_{x \to a^+} f(x) = B$ — $f(x)$ approaches B when x approaches a from the right

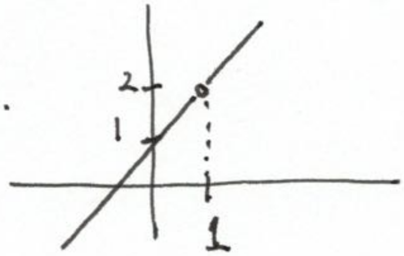


Ex: $\lim\limits_{x \to 0^+} \dfrac{x}{|x|} = 1$

$\lim\limits_{x \to 0^-} \dfrac{x}{|x|} = -1.$

Ex: $\lim\limits_{x \to 1^-} \dfrac{x^2-1}{x-1} = 2$

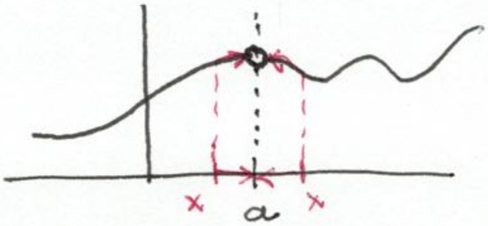
$\lim\limits_{x \to 1^+} \dfrac{x^2-1}{x-1} = 2.$



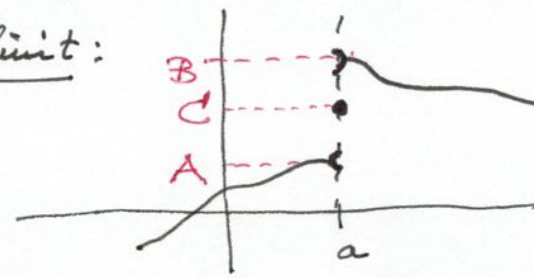
$\lim\limits_{x \to 0^-} \dfrac{x^2-1}{x-1} = 1$

$\lim\limits_{x \to 0^+} \dfrac{x^2-1}{x-1} = 1.$

Theorem: $\lim\limits_{x \to a} f(x)$ exists if and only if both $\lim\limits_{x \to a^-} f(x)$ and $\lim\limits_{x \to a^+} f(x)$ exist, and they have the same value



no limit:



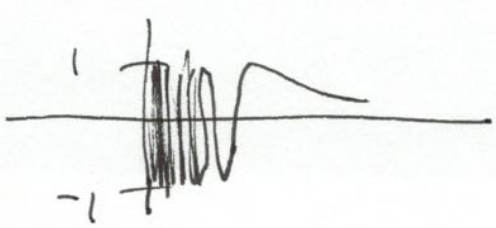
$\lim\limits_{x \to a^-} f(x) = A$, $\lim\limits_{x \to a^+} f(x) = B$

$f(a) = C$

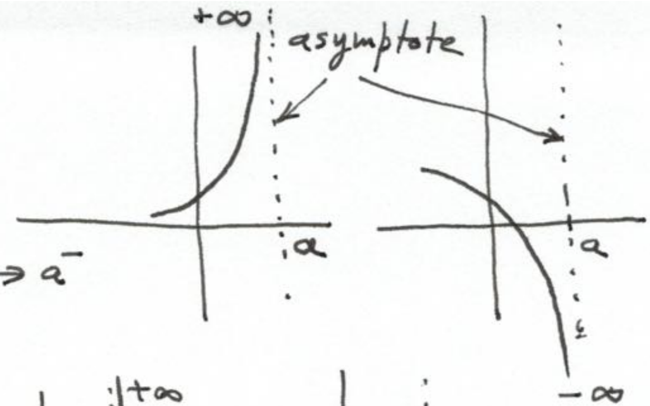
no limit.

another way how a (left/right) limit might fail to exist:

$\lim\limits_{x \to 0^+} \sin\left(\dfrac{1}{x}\right)$

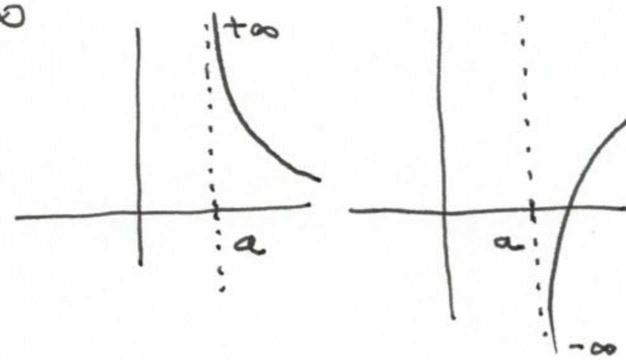


Def "$\lim\limits_{x\to a^-} f(x) = +\infty$ or $-\infty$"

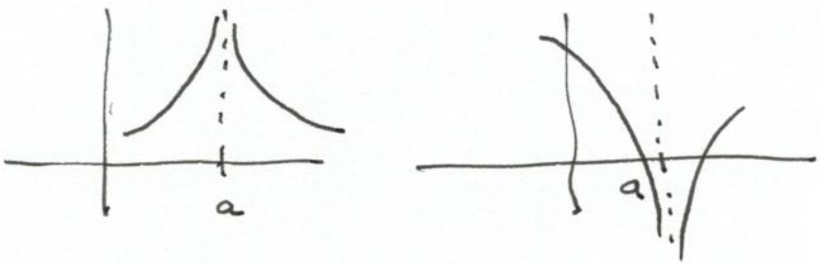


$f(x)$ grows unboundedly to $+\infty$ (or decreases to $-\infty$) when $x \to a^-$

$\lim\limits_{x\to a^+} f(x) = +\infty$ or $-\infty$



$\lim\limits_{x\to a} f(x) = +\infty$ or $-\infty$



Limit laws.

Thm Assume that $\lim\limits_{x\to a^{\pm}} f(x)$, $\lim\limits_{x\to a^{\pm}} g(x)$ both exist and are finite

Then: 1) $\lim\limits_{x\to a^{\pm}} \big(\alpha f(x) \pm \beta g(x)\big) = \alpha \lim\limits_{x\to a^{\pm}} f(x) + \beta \lim\limits_{x\to a^{\pm}} g(x).$

for any numbers α, β.

2) $\lim\limits_{x\to a^{\pm}} \big(f(x)\cdot g(x)\big) = \lim\limits_{x\to a^{\pm}} f(x) \cdot \lim\limits_{x\to a^{\pm}} g(x).$

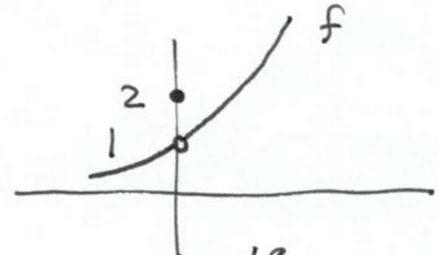
3) $\lim\limits_{x\to a^{\pm}} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a^{\pm}} f(x)}{\lim\limits_{x\to a^{\pm}} g(x)}$ if $\lim\limits_{x\to a^{\pm}} g(x) \neq 0$

(<u>Notation:</u> The statements are true for $\lim\limits_{x\to a}$ everywhere, or $\lim\limits_{x\to a^+}$, or $\lim\limits_{x\to a^-}$ everywhere.)

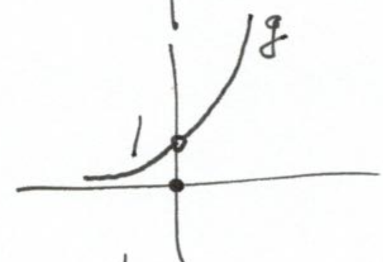
Ex
$$\lim_{x\to 0}(2x+3)\sin x \overset{2)}{=} \underbrace{\lim_{x\to 0}(2x+3)}_{3}\cdot \underbrace{\lim_{x\to 0}\sin x}_{0} = 3\cdot 0 = 0$$
(both finite)

Ex
$$\lim_{x\to 0^+}\left(x\cdot\cos(2+x)\right)\cdot\frac{1}{|x|}$$
$$\overset{2)}{=} \underbrace{\lim_{x\to 0^+}\cos(2+x)}_{\cos 2}\cdot\underbrace{\lim_{x\to 0^+}\frac{x}{|x|}}_{1} = \cos 2 .$$

Ex
$$f(x) = \begin{cases} 2^x & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$$



$$g(x) = \begin{cases} 3^x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$



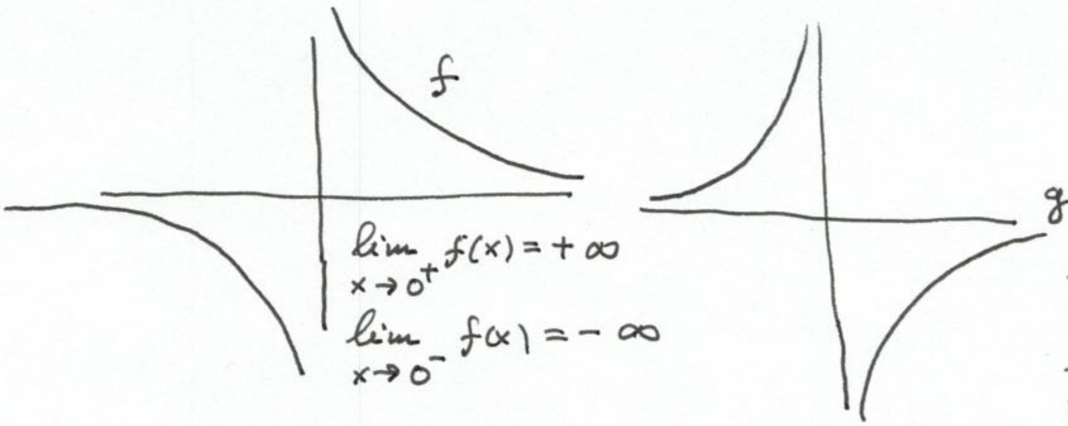
$$\lim_{x\to 0}\frac{f(x)}{g(x)} \overset{3)}{=} \frac{\lim_{x\to 0} f(x)}{\lim_{x\to 0} g(x)} = \frac{1}{1} = 1 .$$

we are allowed to use the limit rules because $\lim_{x\to 0} f(x) = 1$, $\lim_{x\to 0} g(x) = 1 \neq 0$ (both finite).

The fact that the fct value $g(0) = 0$ does not matter.

Function value of $\frac{f(x)}{g(x)}$ at $x = 0$ is <u>not</u> defined.

Ex: $f(x) = \frac{1}{x}$, $g(x) = -\frac{2}{x}$.



$\lim_{x \to 0^+} f(x) = +\infty$

$\lim_{x \to 0^-} f(x) = -\infty$

$\lim_{x \to 0^+} g(x) = -\infty$

$\lim_{x \to 0^-} g(x) = +\infty$.

no fct values, no limits at $x=0$.

$$\lim_{x \to 0^+} \left(f(x) + g(x) \right) = \lim_{x \to 0^+} \left(\frac{1}{x} - \frac{2}{x} \right) = \lim_{x \to 0^+} \left(\frac{-1}{x} \right) = -\infty$$

But compare to

$$\lim_{x \to 0^+} f(x) + \lim_{x \to 0^+} g(x) = \underbrace{\lim_{x \to 0^+} \frac{1}{x}}_{\infty} + \underbrace{\lim_{x \to 0^+} \left(\frac{-2}{x} \right)}_{-\infty} = \underline{\text{not}}\ \underline{\text{defined}}$$

Those are not the same. Limit laws are only applicable if f and g have finite limits.