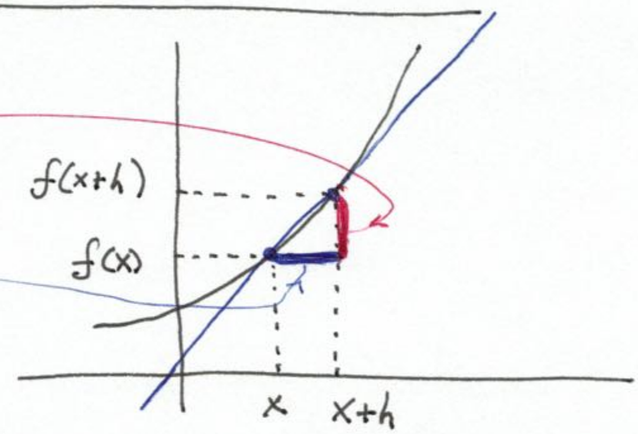
Derivatives: Product, quotient, chain rule.

Recall: Derivative of f at x

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

slope of tangent line at $f(x)$.



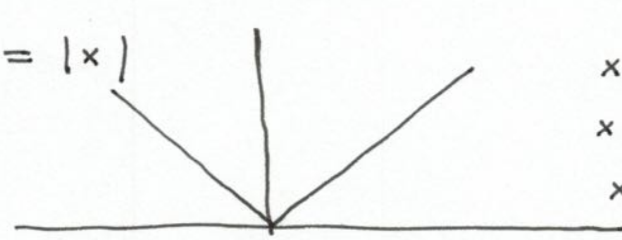
if $f'(x)$ exists, we say that f is differentiable at x.

Def Left derivative of f at x: $f_-'(x) = \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h}.$

Right —— " ——: $f_+'(x) = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}.$

If those one-sided limits exist $\Rightarrow$ f is left/right differentiable at x.

Ex $f(x) = |x|$



$x > 0 \Rightarrow f'(x) = 1$

$x < 0 \Rightarrow f'(x) = -1$.

$x = 0 \Rightarrow f'$ does not exist at $x = 0$

$f_-'(0) = -1$

$f_+'(0) = 1$

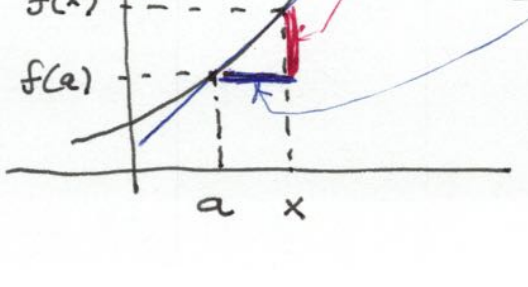
Constant function: $f(x) = c$

$$\Rightarrow f'(x) = \lim_{h \to 0} \frac{\overset{f(x+h)}{c} - \overset{f(x)}{c}}{h} = 0$$

Thm If f is differentiable at $x=a$, then f is continuous at a.

Because: $\lim_{x \to a} f(x) - f(a) = \lim_{x \to a} (f(x) - f(a)) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}(x-a)$

$$= \underbrace{\lim_{x \to a} \frac{f(x) - f(a)}{x - a}}_{f'(a)} \cdot \underbrace{\lim_{x \to a} (x - a)}_{0} = 0$$



$f'(a)$ ← well-defined because f differentiable at a.

Product rule & quotient rule.

Thm: Assume f, g differentiable at x. Then,

Product rule: $\left(f(x)\cdot g(x)\right)' = f'(x)\cdot g(x) + f(x)\cdot g'(x)$.

Quotient rule: $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{(g(x))^2}$.
$(g(x) \neq 0)$

Because:

$$\begin{aligned}
(f(x)\cdot g(x))' &= \lim_{h\to 0} \frac{f(x+h)\cdot g(x+h) - f(x)\cdot g(x)}{h} \\
&= \lim_{h\to 0} \frac{(f(x+h) - f(x))\, g(x+h) + f(x)\cdot g(x+h) - f(x)\cdot g(x)}{h} \\
&= \lim_{h\to 0} \frac{(f(x+h) - f(x))\, g(x+h)}{h} + \lim_{h\to 0} \frac{f(x)\left(g(x+h) - g(x)\right)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \lim_{h\to 0} g(x+h) + f(x) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= f'(x)\cdot g(x) + f(x)\cdot g'(x).
\end{aligned}$$

Ex $(x e^x)' = 1\cdot e^x + x\cdot e^x \qquad \left((e^x)' = e^x\right)$

Ex $\left(\dfrac{1}{g(x)}\right)' = ?$

$$0 = \Big(\underbrace{g(x)\cdot \frac{1}{g(x)}}_{1}\Big)' = g'(x)\cdot \frac{1}{g(x)} + g(x)\cdot \underbrace{\left(\frac{1}{g(x)}\right)'}_{\text{solve for this}}$$

$$\Rightarrow g(x)\cdot\left(\frac{1}{g(x)}\right)' = -\frac{g'(x)}{g(x)}$$

$$\Rightarrow \left(\frac{1}{g(x)}\right)' = -\frac{g'(x)}{(g(x))^2}$$

Quotient rule:

$$\left(\frac{f(x)}{g(x)}\right)' = \left(f(x)\cdot\frac{1}{g(x)}\right)' = f'(x)\cdot\frac{1}{g(x)} + f(x)\cdot\left(\frac{1}{g(x)}\right)'$$

$$= \frac{f'(x)}{g(x)} + f(x)\cdot\frac{-g'(x)}{(g(x))^2} = \frac{f'(x)\cdot g(x) - f(x)g'(x)}{(g(x))^2}$$

Ex $\left(\frac{e^x}{x}\right)' = \frac{e^x\cdot x - e^x\cdot 1}{x^2}$

Ex $\left(\frac{x^2+1}{x-2}\right)' = \frac{2x\cdot(x-2) - (x^2+1)\cdot 1}{(x-2)^2} = \ldots$ (simplify).

Chain rule

Thm Assume $g(x)$ is differentiable at x, and f differentiable at $g(x)$. Then,

$$\left(f(g(x))\right)' = f'(g(x))\cdot g'(x).$$

Ex: $f(x) = e^x \Rightarrow f'(x) = e^x$

$g(x) = x^2 \Rightarrow g'(x) = 2x$.

$f(g(x)) = e^{x^2}$

$$(e^{x^2})' = \underbrace{e^{x^2}}_{f'(g(x))}\cdot\underbrace{2x}_{g'(x)}$$

Ex: $(\ln x)' = ?$

$$1 = (\underbrace{e^{\ln x}}_{x})' = \underbrace{e^{\ln x}}_{f'(g(x))}\cdot\underbrace{(\ln x)'}_{g'(x).} = x\cdot\underbrace{(\ln x)'}_{\text{solve for this}}$$

$e^{\ln x} = x$

$f(x) = e^x$

$g(x) = \ln x$.

$$\Rightarrow \boxed{(\ln x)' = \frac{1}{x}}$$

Ex $(x^r)' = ?$ for r between $-\infty$ and ∞, $x > 0$

$$x^r = (e^{\ln x})^r = \underbrace{e^{r \cdot \ln x}}$$

$$(x^r)' = (e^{r \cdot \ln x})' = \underbrace{e^{r \cdot \ln x}}_{f'(g(x))} \cdot \underbrace{r \cdot \frac{1}{x}}_{g'(x)}$$

$= x^r$

$f(x) = e^x$

$g(x) = r \cdot \ln x$

$$= x^r \cdot r \cdot \frac{1}{x} = r \cdot x^{r-1}$$