Ex $(x^x)' = ?$

$$x^x = \left(e^{\ln x}\right)^x = e^{x\cdot \ln x}$$

$$(x^x)' = \left(e^{x \ln x}\right)' \underset{\uparrow}{=} \underbrace{e^{x\cdot \ln x}}_{f'(g(x))} \overset{= x^x}{} \cdot \underbrace{\left(1\cdot \ln x + x\cdot \frac{1}{x}\right)}_{g'(x)}$$

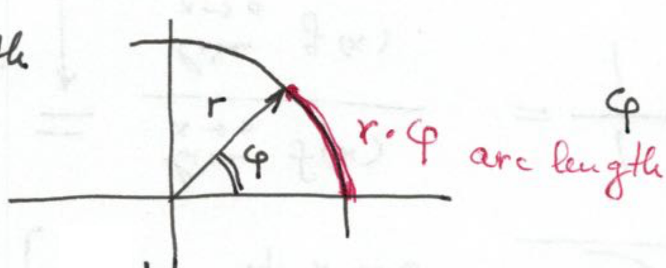
$f(x) = e^x$ chain rule

$g(x) = x \cdot \ln x$

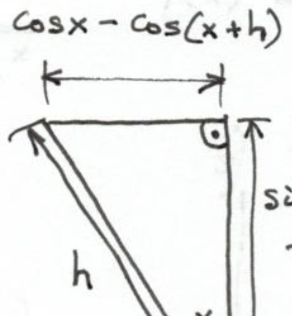
$$= x^x(\ln x + 1)$$

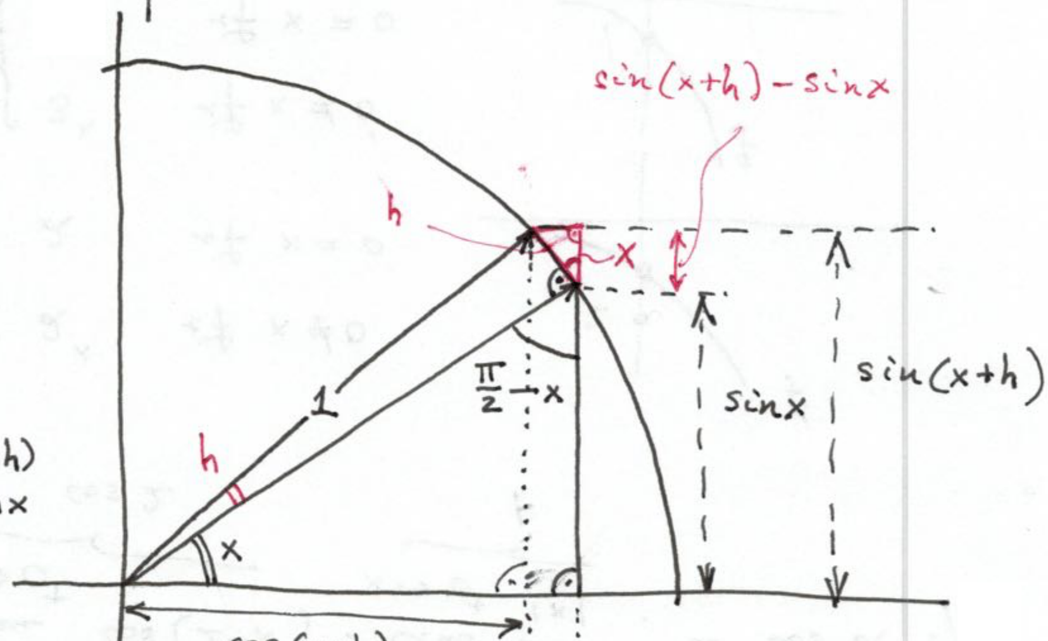
Ex $(\sin x)' = ?$

arc length



φ in radiants.





$$(\sin x)' = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \cos x \quad \Big| \quad -(\cos x)' = \lim_{h\to 0} \frac{\cos x - \cos(x+h)}{h} = \sin x$$

$$\Rightarrow (\cos x)' = -\sin x.$$

Ex $\tan x = \dfrac{\sin x}{\cos x}$. $\quad$ (Recall: $\cos^2 x + \sin^2 x = 1$)

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' \underset{\text{quotient rule}}{=} \frac{(\sin x)' \cdot \cos x - \sin x \cdot (\cos x)'}{\cos^2 x}$$

$$= \frac{\cos^2 x - \sin x\,(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

Ex. $\sin(\arcsin x) = x$

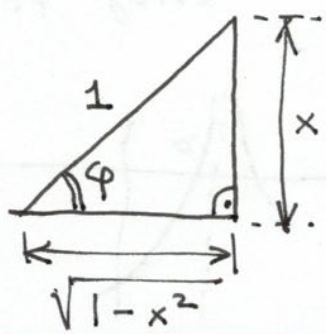
$\arcsin(\sin x) = x$

$(\arcsin x)' = ?$

$$1 = (\underbrace{\sin(\arcsin x)}_{x})' \underset{\text{chain rule}}{=} \overbrace{\underbrace{\cos(\arcsin x)}_{f'(g(x))}}^{\sqrt{1-x^2}} \cdot (\arcsin x)'$$

$f(x) = \sin x$

$g(x) = \arcsin x$



$x = \dfrac{x}{1} = \sin\varphi$

$\varphi = \arcsin x$

$\cos(\arcsin x) = \dfrac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$

$$\Longrightarrow \boxed{(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}}$$

Ex We study $\cos x + i \cdot \sin x$ $\qquad i^2 = -1$

i imaginary unit.

$$f(x) = e^{-ix} \cdot (\cos x + i \sin x)$$

$$f'(x) \underset{\uparrow}{=} -i e^{-ix} \cdot (\cos x + i \sin x) + e^{-ix}(-\sin x + i \cos x)$$

product rule.

$$= -e^{-ix}\left(i\cos x - \sin x\right) + e^{-ix}\left(-\sin x + i\cos x\right)$$

$$= e^{ix}\left(\cancel{-i\cos x} + \cancel{\sin x} + \cancel{(-\sin x)} + \cancel{i\cos x}\right)$$

$$= 0$$

$\Rightarrow$ $f(x)$ is a constant.

$$f(0) = \underbrace{e^0}_{1}\,(\underbrace{\cos 0}_{1} + i \underbrace{\sin 0}_{0}) = 1 \quad \Big\} f(x) = 1$$

$$\Rightarrow f(x) = 1 = e^{-ix}(\cos x + i \sin x). \qquad \left(\text{use } \frac{1}{e^{-ix}} = e^{ix}\right)$$

$$\Rightarrow \boxed{e^{ix} = \cos x + i \sin x} \quad \text{Euler's formula.}$$

$$e^{ix} \cdot e^{iy} = e^{i(x+y)} = \cos(x+y) + i \sin(x+y).$$

$$= (\cos x + i \sin x)(\cos y + i \sin y)$$

$$= \cos x \cos y + \cos x \cdot i \sin y + i \sin x \cdot \cos y - \sin x \cdot \sin y$$

$$= (\cos x \cos y - \sin x \cdot \sin y) + i\,(\cos x \cdot \sin y + \sin x \cdot \cos y)$$

$i \cdot i = -1$

$\underline{x = y}$: $\quad \cos 2x = \cos^2 x - \sin^2 x \qquad \sin 2x = 2\cos x \cdot \sin x$