Why is the chain rule correct?

Recall: $\left(f(g(x))\right)' = f'(g(x)) \cdot g'(x)$.

Because: $\left(f(g(x))\right)' = \lim\limits_{h\to 0} \dfrac{f(g(x+h)) - f(g(x))}{h}$

$$= \lim_{h\to 0} \frac{f\left(g(x) + \overbrace{\frac{g(x+h)-g(x)}{h}}^{\to\, g'(x) \text{ as } h\to 0}\cdot h\right) - f(g(x))}{h}$$

$$= \lim_{h\to 0} \frac{f(g(x) + \overbrace{g'(x)\cdot h}^{\to H}) - f(g(x))}{g'(x)\cdot h} \cdot g'(x).$$

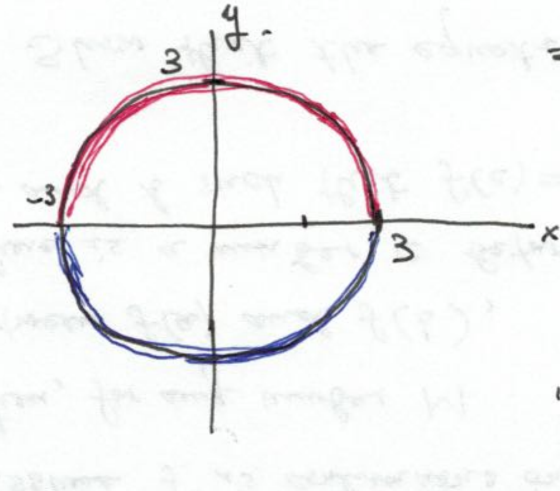
$$= \lim_{H\to 0} \frac{f(g(x)+H) - f(g(x))}{H} \cdot g'(x).$$

$$= f'(g(x)) \cdot g'(x).$$

Implicit differentiation.

Sometimes, "solving y for x" is dificult.

Ex $\quad x^2+y^2 = 9 \quad$ circle of radius 3.



$\Rightarrow y^2 = 9 - x^2$

$\Rightarrow y = \pm\sqrt{9-x^2}$

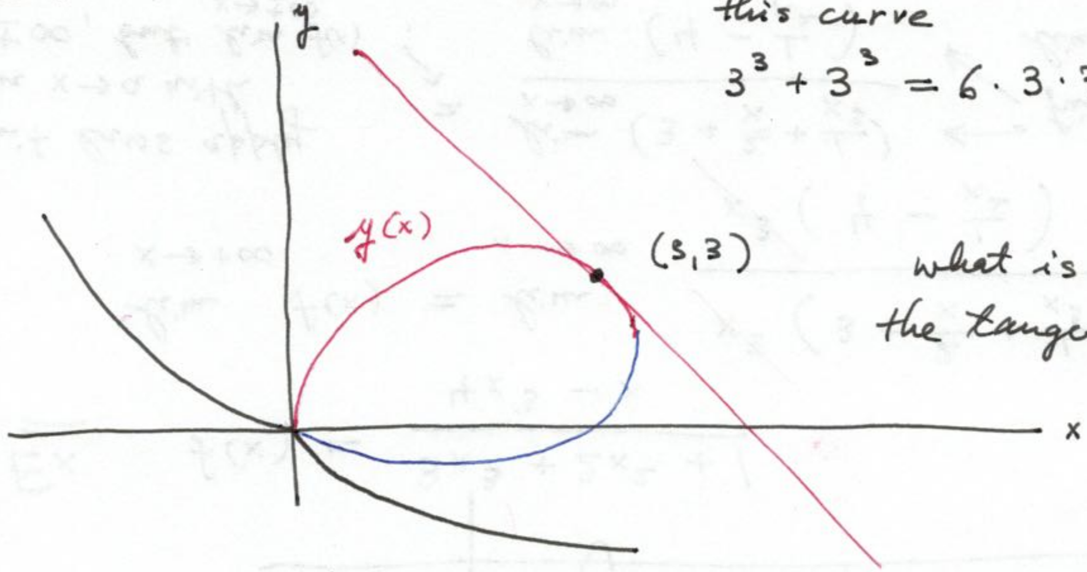
The circle is the combination of graphs of 2 functions.

"Piecewise representable as a graph".

$\underline{\underline{Ex}}$ $x^3 + y^3 = 6x \cdot y$

For instance, $(3,3)$ lies on this curve

$$3^3 + 3^3 = 6 \cdot 3 \cdot 3 = 2 \cdot 3^3 \; \checkmark$$



what is the slope of the tangent line at $(3,3)$?

$y'(3) = ?$

$y(x)$ satisfies

$$x^3 + (y(x))^3 = 6x \cdot y(x)$$

differentiate both sides:

$$3x^2 + 3(y(x))^2 \cdot y'(x) = 6\,y(x) + 6x \cdot y'(x).$$

solve for $y'(x)$:

$$\left(3(y(x))^2 - 6x\right) \cdot y'(x) = 6\,y(x) - 3x^2$$

$$\Rightarrow \quad y'(x) = \frac{6\,y(x) - 3x^2}{3(y(x))^2 - 6x}$$

for $(3,3)$ with $x=3$, $y=3$:

$$y'(3) = \frac{6 \cdot 3 - 3 \cdot 3^2}{3 \cdot 3^2 - 6 \cdot 3} = -1$$

Ex Find $y'(x)$ if

$$\sin(x^2+y^2) = y \cdot \cos x$$

Differentiate in x:

$$\left(\cos(x^2+y^2)\right) \cdot (2x + 2y \cdot y') = y \cdot (-\sin x) + y' \cdot \cos x$$

Solve for y':

$$\left(2y \cdot \cos(x^2+y^2) - \cos x\right) \cdot y' = -y \cdot \sin x - 2x \cdot \cos(x^2+y^2)$$

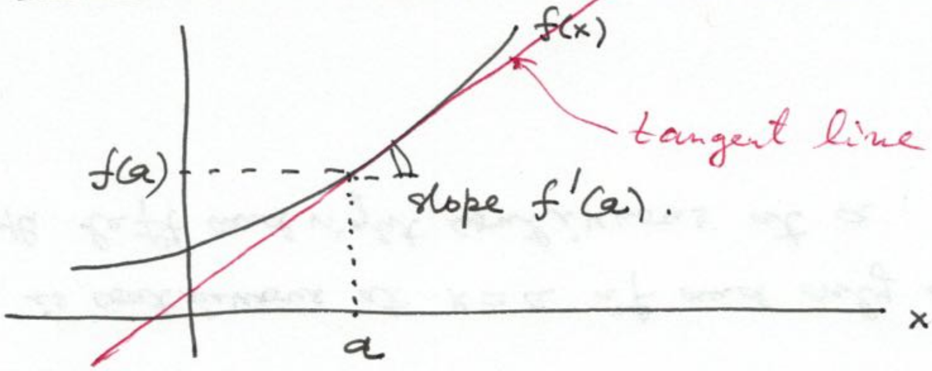
$$\Rightarrow y' = -\frac{y \cdot \sin x + 2x \cdot \cos(x^2+y^2)}{2y \cdot \cos(x^2+y^2) - \cos x}$$

$(0,0)$ solves the equation: $\underbrace{\sin(0^2+0^2)}_{0} = \underbrace{0 \cdot \underbrace{\cos 0}_{1}}_{0}$

Find $y'(0)$

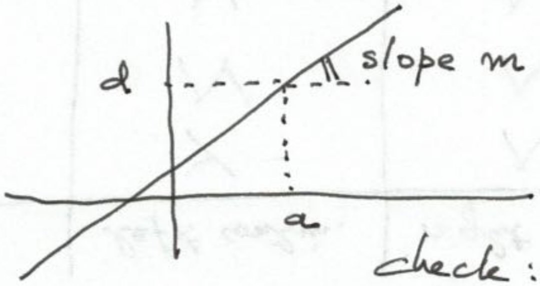
$$y'(0) = -\frac{0 \cdot \sin 0 + 2 \cdot 0 \cdot \cos(0+0)}{\underbrace{2 \cdot 0 \cdot \cos(0+0)}_{0} - \cos 0} = -\frac{0}{-1} = 0$$

Tangent line & linear approximation



Formula for the tangent line:

Recall formula for a straight line.



$$y = m(x-a) + d.$$

check: $y(a) = d$, $y'(x) = m$ ✓

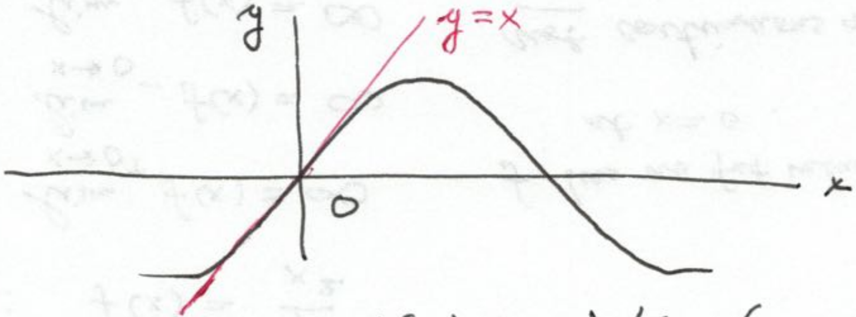
$$\boxed{y = f'(a)\cdot(x-a) + f(a).}$$ equation for the tangent line.

<u>check:</u> $y(a) = f(a)$, $y'(x) = f'(a)$.

The equation for the tangent line gives us an approximation of $f(x)$ when x is close to a.

The function $y(x) = f'(a)(x-a) + f(a)$ is called the linearization (= linear approximation) of f at a.

Ex. Find the linear approximation of $\sin x$ at 0.



$\sin'(0) = \cos 0 = 1.$

$\sin(0) = 0$

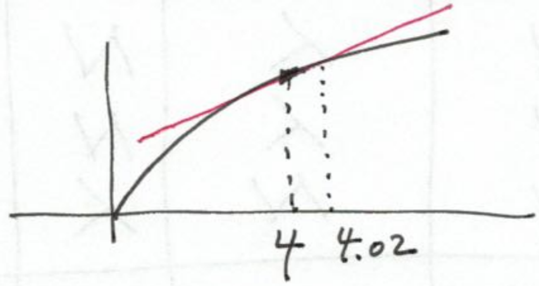
$$y(x) = \sin'(0)\cdot\left(x - 0\right) + \sin(0).$$

$$= 1 \cdot x + 0 = x$$

Ex Calculate $\sqrt{4.02}$ without a calculator.

$f(x) = \sqrt{x}.$

$f'(x) = \dfrac{1}{2\sqrt{x}}$



$a = 4$

$x = 4.02$

$f(a) = \sqrt{4} = 2$

$f'(a) = \dfrac{1}{2\sqrt{4}} = \dfrac{1}{4}.$

Linear approximation: $y = \underbrace{\tfrac{1}{4}}_{f'(a)}\left(\underbrace{4.02}_{x} - \underbrace{4}_{a}\right) + \underbrace{2}_{f(a)}$

$= 0.25 \cdot 0.02 + 2$

$= 0.005 + 2 = 2.005$