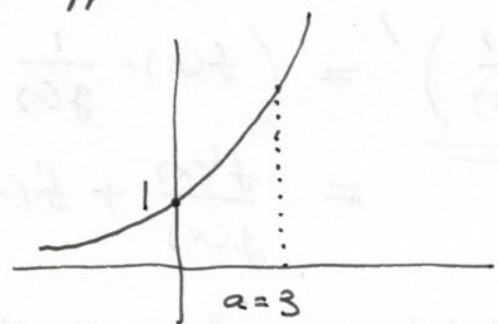
Ex Find $3^{3.01}$ in linear approximation.

$f(x) = 3^x$

$f'(x) = 3^x \cdot \ln 3$



pick $a = 3$

because we can easily evaluate $f(3)$ and $f'(3)$, and 3 is close to 3.01.

$f(3) = 27 \quad , \quad f'(3) = 27 \cdot \ln 3$

Tangent line: $y = \underbrace{27 \cdot \ln 3}_{f'(3)} \cdot (\underset{\substack{\uparrow \\ x}}{3.01} - \underset{\substack{\uparrow \\ a}}{3}) + \underset{\substack{\uparrow \\ f(3)}}{27}$

What about $\ln 3$? Remember: $e \simeq 2.72$

$\ln 3 \approx \ln(\underbrace{e + 0.28}_{x})$, $a = e$

$g(x) = \ln x \qquad g(e) = \ln e = 1$

$g'(x) = \frac{1}{x} \qquad g'(e) = \frac{1}{e} = \frac{1}{2.72}$

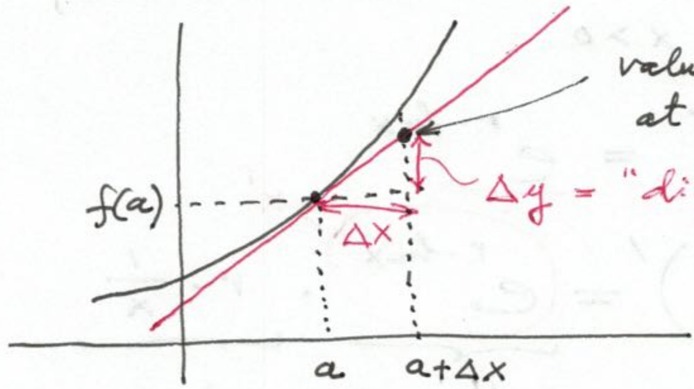
Linear approx: $y = \underset{\substack{\uparrow \\ g'(e)}}{\frac{1}{e}} \cdot (\underset{\substack{\uparrow \\ x}}{3} - \underset{\substack{\uparrow \\ a}}{e}) + \underset{\substack{\uparrow \\ \ln e}}{1} = \underbrace{\frac{1}{2.72} \cdot 0.28}_{\simeq 0.1} + 1$

$\simeq 1.1$

$y = \underbrace{27 \cdot 1.1}_{} \cdot (0.01) + 27 = 27.297 \approx 3^{3.01}$

$$\begin{array}{r} 27 \\ 2.7 \\ \hline 29.7 \end{array} \quad 0.297$$

Differentials.



Tangent line: $y(a+\Delta x) = f'(a)\big(\overbrace{\underbrace{a+\Delta x}_{x} - a}^{\Delta x}\big) + f(a)$

$$\boxed{\Delta y = y(a+\Delta x) - f(a) = f'(a)\cdot\Delta x} \quad \text{Differential.}$$

Ex Find $\ln(1.02)$ in linear approximation, and determine the differential.

$f(x) = \ln x \qquad a = 1 \qquad \Delta x = 0.02$

$f'(x) = \frac{1}{x} \qquad x = 1.02 \quad , \ f(1) = 0,\ f'(1) = 1$

$$y = \underbrace{1}_{f'(1)}\cdot(\underbrace{1.02}_{x} - \underbrace{1}_{a}) + \underbrace{0}_{f(1)} = 1\cdot 0.02 = 0.02 \approx \ln(1.02)$$

differential:

$$\Delta y = f'(1)\cdot\Delta x = 1\cdot 0.02 = 0.02$$

This is how much larger $\ln(1.02)$ is than $\ln(1)$, in linear approximation.

Ex $\ln 8 = \ln(\overline{e^2 + 0.6}) \approx \ln'(e^2)\cdot 0.6 + \ln(e^2)$.

$e^2 = (2.72)^2 = (3-0.28)^2 \approx 9 - 1.68 + 0.09 \approx 7.4$.

$$= \frac{1}{(2.72)^2}\cdot 0.6 + 2\cdot\underbrace{\ln e}_{1} \approx \frac{1}{7.4}\cdot 0.6 + 2 \approx 0.08 + 2 \approx 2.08$$

Related rates.

Q: How does one quantity change when another quantity that it depends on changes?

Ex Ball-shaped ballon of volume $V = \frac{4\pi}{3} r^3$, r = radius

When air is pumped so that

$$\frac{dV}{dt} = V'(t) = 100\ \mathrm{cm^3/sec}$$

then what is the rate of change per sec of the radius, once $r = 25$ cm?

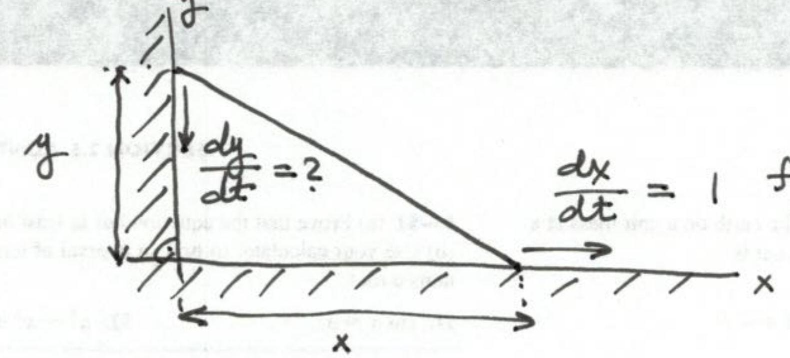
$\Rightarrow$ want to know $r'(t) = \frac{dr}{dt}$

$\Rightarrow$ differentiate formula for V in terms of t:

$$\frac{dV}{dt} = \frac{4\pi}{\cancel{3}}\,\cancel{3}\,r^2\,\frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \underbrace{\frac{dV}{dt}}_{=\cancel{100}} \cdot \underbrace{\frac{1}{4\pi r^2}}_{\frac{1}{\cancel{4}\pi\, 25^{\cancel{2}}}} = \frac{1}{25\pi}\ \mathrm{cm/sec}$$

Ex: A ladder of length 10 ft standing against a wall.



What is $\frac{dy}{dt}$ when $x = 6$ ft ?

Pythagoras: $x^2 + y^2 = 10^2 = 100$.

$x = 6$ ft $\Rightarrow y = \sqrt{100 - 6^2} = 8$ ft.

Differentiate (implicit!) in t:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\Rightarrow \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = -\frac{6}{8}\cdot 1 = -\frac{3}{4} \text{ ft/sec.}$$

negative because it slips down.

Hyperbolic functions.

Def:

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

Similar to trigon. fcts because:

$$e^{ix} = \cos x + i\sin x$$

$$e^{-ix} = \cos x - i\sin x$$

$$\Rightarrow \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh(-x) = -\sinh x$$

$$\cosh(-x) = \cosh x$$

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y.$$