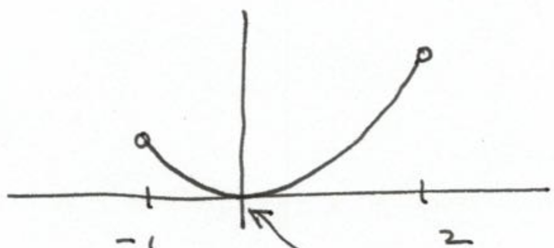
Thm (Fermat)

Assume f has a local max or local min at c, and that $f'(c)$ exists. ← implies continuous at c.

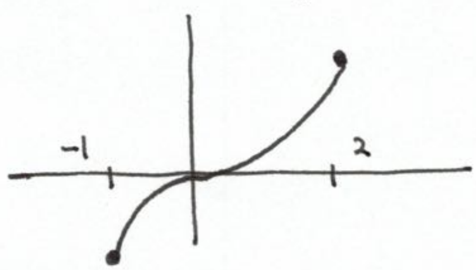
Then, $f'(c) = 0$

Ex $f(x) = x^2, \quad -1 < x < 2$.



$f'(0) = 2 \cdot 0 = 0$

Ex $f(x) = x^3, \quad -1 \leq x \leq 2$.



① local min: $x = -1$
right derivative $f'_+(-1) = 3(-1)^2 = 3$.

② local max: $x = 2$
left derivative $f'_-(2) = 3 \cdot 2^2 = 12$

③ $f'(x) = 0$ for some x in $(-1, 2)$?
$f'(x) = 3x^2 = 0 \Rightarrow x = 0$.

① Fermat's thm not applicable because we only have a right derivative at $x = -1$, but not a derivative.

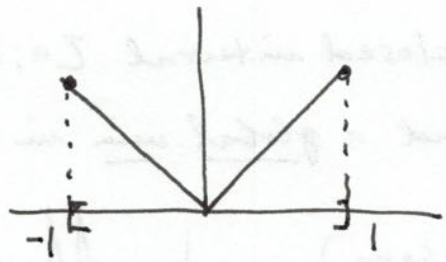
② Similar.

③ Fermat says: If c local min/max and f different. at c
$\Rightarrow f'(c) = 0$

but it does not say $f'(c) = 0 \Rightarrow$ local min/max.

not correct.

Ex $f(x) = |x|, \quad -1 \le x \le 1$.

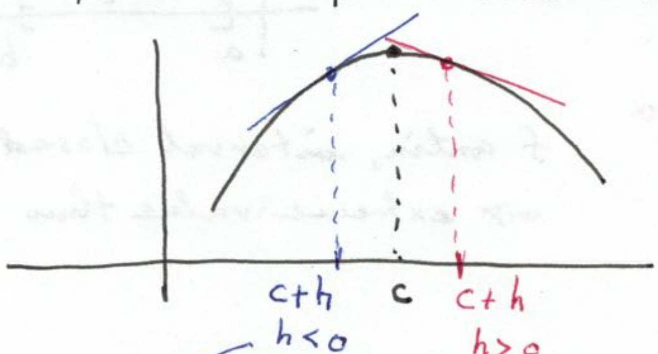


local max : $x = -1, 1$

local min : $x = 0$

but f not differentiable at 0 .

Explanation for Fermat's theorem .



local max at c ; by assumption $f'(c)$ exists .

$\Rightarrow f(c+h) - f(c) \le 0$

divide by h and let $h \to 0^+$

$$f'_+(c) = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \le 0 .$$

$h<0$: $f(c+h) - f(c) \le 0$

divide by $h < 0$ $\Rightarrow f'_-(c) = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \ge 0$ (ineq. flips because we divide by negative number h)

$$0 \le f'_-(c) = f'(c) = f'_+(c) \le 0$$

$$\Rightarrow f'(c) = 0 .$$

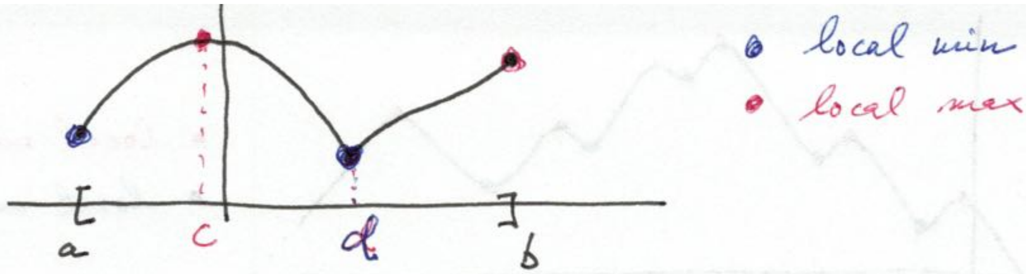
Def (critical number).

A point c in the domain of f is a critical number if:

- $f'(c) = 0$
- or $f'(c)$ does not exist.

Thm If f has a local min/max at c, then c is a critical number.

Ex.



critical points: a because derivative does not exist.

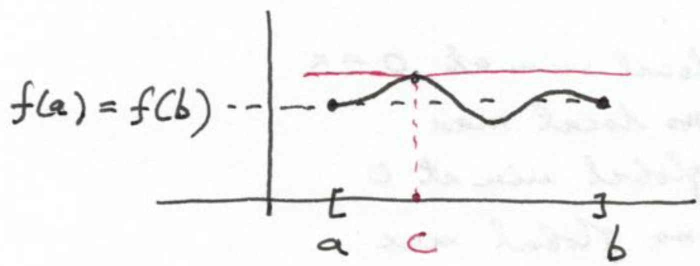
b " " " " "

c because $f'(c) = 0$.

d because derivative does not exist.

The mean value theorem.

Thm (Rolle).



Assume f is continuous on $[a,b]$, and differentiable in (a,b), and that $f(a) = f(b)$.

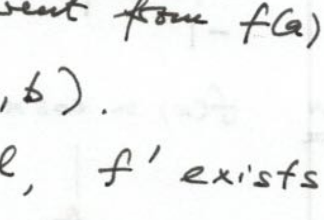
Then, there is a point c in (a,b) such that

$$f'(c) = 0 .$$

Because: Extreme value thm $\Rightarrow$ f has a global min/max in $[a,b]$

① If the global min/max both have the value $f(a) = f(b)$, then $f(x) = \text{const}$.

$\Rightarrow f'(x) = 0$ for any x in (a,b).



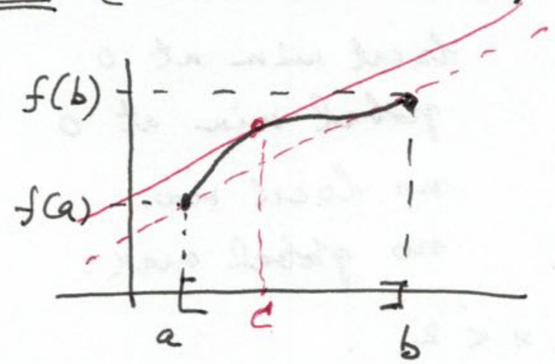
② f not constant $\Rightarrow$ either global min, or global max, or both are different from $f(a) = f(b)$

$\Rightarrow$ must be located at c in (a,b).

but in the interior of the interval, f' exists.

$\Rightarrow$ by Fermat, $f'(c) = 0$.

Thm (mean value thm).



Assume f continuous on $[a, b]$ and differentiable in (a, b). Then, there is a point c in (a, b) such that

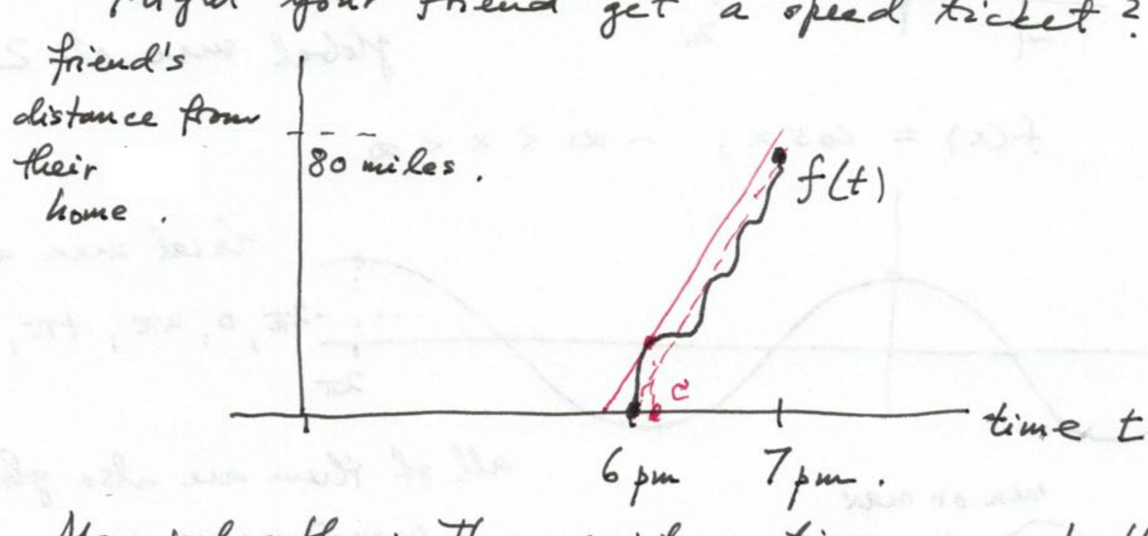
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

(the tangent line at $f(c)$ is parallel to the straight line connecting $f(a)$ and $f(b)$).

Note: Rolle's thm is the special case where $f(a) = f(b)$.

Ex Your friend lives 80 miles away from you, leaves home at 6 pm and arrives at your place at 7 pm. Max speed on highway is 65 mph.

Might your friend get a speed ticket?



Mean value thm: There exists a time c such that instantaneous velocity

$$f'(c) = \frac{80}{1} \text{ mph}$$