$\underline{\underline{\text{Ex}}}$ $f(x) = x^3 + 3x + 2$.

How many roots does $f(x) = 0$ have?

$$f(-1) = -1 - 3 + 2 < 0$$
$$f(1) = 1 + 3 + 2 > 0$$

f continuous $\Rightarrow$ intermediate value thm, $f(c) = 0$ for some c in $(-1, 1)$.

$f'(x) = 3x^2 + 3 > 0 \Rightarrow f$ is increasing everywhere
$\Rightarrow$ graph can cross x-axis only once.
$\Rightarrow$ there is only one root.

Indeterminate forms and de l'Hôpital's rule

$\lim_{x \to a} \frac{f(x)}{g(x)}$ in situations where $\left\{ \begin{array}{l} f \to 0 \\ g \to 0 \end{array} \right\}$ or $\left\{ \begin{array}{l} f \to \pm\infty \\ g \to \pm\infty \end{array} \right\}$.

$\underline{\underline{\text{Thm}}}$ (de l'Hôpital's rule).

Assume f, g differentiable near a, and

$$\lim_{x \to a} f(x) = 0, \quad \lim_{x \to a} g(x) = 0$$
$$\left(\text{or } \lim_{x \to a} f(x) = \pm\infty, \quad \lim_{x \to a} g(x) = \pm\infty \right)$$

Then,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

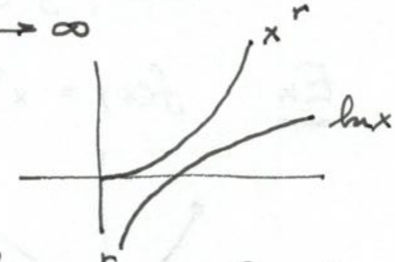
Remark: Similarly for left & right limits ($x \to a^{\pm}$).

$\underline{\underline{\text{Ex}}}$: $\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1$

Ex: $\lim\limits_{x\to 1} \dfrac{x^2-1}{x-1} = \lim\limits_{x\to 1} \dfrac{2x}{1} = 2.$

Ex: $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x^2} = \lim\limits_{x\to 0} \dfrac{\sin x}{2x} = \lim\limits_{x\to 0} \dfrac{\cos x}{2} = \dfrac{1}{2}.$

Ex: Compare x^r, $r>0$, with $\ln x$ as $x \longrightarrow \infty$



$$\lim_{x\to\infty} \frac{\ln x}{x^r} = \lim_{x\to\infty} \frac{\frac{1}{x}}{r x^{r-1}} = \lim_{x\to\infty} \frac{1}{r x^r}$$

$$\underset{\substack{\uparrow \\ r>0.}}{=} 0 \quad \text{for all } r>0.$$

$\lim\limits_{x\to\infty} x^r = \infty$, $\lim\limits_{x\to\infty} \ln x = \infty$

$\Rightarrow$ for example, $\lim\limits_{x\to\infty} \dfrac{\ln x}{x^{0.00000000000001}} = 0$

Ex: Let $n>0$ be an integer. Compare e^x with x^n, as $x\to\infty$

$\lim\limits_{x\to\infty} x^n = \infty$
$\lim\limits_{x\to\infty} e^x = \infty$

$$\lim_{x\to\infty} \frac{e^x}{x^n} = \lim_{x\to\infty} \frac{e^x}{n x^{n-1}}$$

$$= \lim_{x\to\infty} \frac{e^x}{n(n-1)x^{n-2}}$$

$$= \dots = \lim_{x\to\infty} \frac{e^x}{n(n-1)(n-2)\cdots 2\cdot 1\cdot x^0}$$

$$= \lim_{x\to\infty} \frac{e^x}{n!} = \infty$$

$\Rightarrow$ for example, e^x goes faster to ∞ than $x^{1,000,000,000}$.

Indeterminate powers.

$\lim\limits_{x\to a} (f(x))^{g(x)}$ where

1) $f\to 0^+$, $g\to 0$
2) $f\to\infty$, $g\to 0$
3) $f\to\infty$, $g\to\pm\infty$

Assume that $f(x) \geq 0$ when x is close enough to a

$$(f(x))^{g(x)} = \left(e^{\ln f(x)}\right)^{g(x)} = e^{g(x)\ln f(x)}$$

$$\Rightarrow \lim_{x\to a} (f(x))^{g(x)} = \lim_{x\to a} e^{g(x)\ln f(x)} \underset{\uparrow}{=} e^{\lim_{x\to a} g(x)\ln f(x)}$$

exponential fct is continuous everywhere.

__Ex__ Find $\lim_{x\to\infty} x^{1/x}$.

$$\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty}\left(e^{\ln x}\right)^{\frac{1}{x}} = \lim_{x\to\infty} e^{\frac{1}{x}\ln x} = e^{\lim_{x\to\infty}\frac{\ln x}{x}} = e^{0} = 1$$

$$\lim_{x\to\infty}\frac{\ln x}{x} \underset{\uparrow}{=} \lim_{x\to\infty}\frac{\frac{1}{x}}{1} = \lim_{x\to\infty}\frac{1}{x} = 0$$

de l'Hôpital.

Why is de l'Hôpital's rule correct?

Special case: f, g differentiable at a, and $f, g \to 0$ as $x \to a$.

f, g continuous at a $\Rightarrow f(a) = 0 = g(a)$

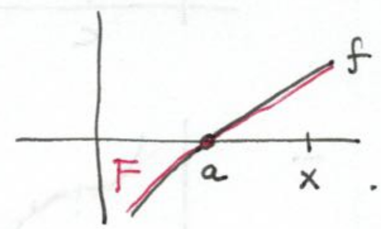
$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f(x) - \underbrace{f(a)}_{=0}}{g(x) - \underbrace{g(a)}_{=0}}$$

$$= \lim_{x\to a}\frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} = \frac{f'(a)}{g'(a)}$$

In the general case: f, g differentiable near a, and $f, g \to 0$ as $x \to a$.

Define: $F(x) := \begin{cases} f(x) & \text{if } x \neq a \\ 0 & \text{if } x = a \end{cases}$

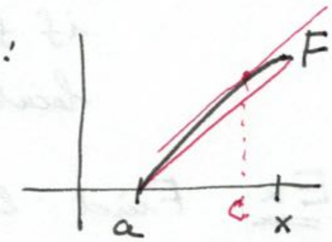
$G(x) := \begin{cases} g(x) & \text{if } x \neq a \\ 0 & \text{if } x = a . \end{cases}$



$\Rightarrow F, G$ are continuous in $[a, x]$, differentiable in (a, x)

Mean value thm: There are points c, d in (a, x):

$$F'(c) = \frac{F(x) - F(a)}{x - a} .$$



$$\Rightarrow F(x) = \underbrace{F(a)}_{=0} + F'(c)(x-a)$$

$$G(x) = \underbrace{G(a)}_{=0} + G'(d)(x-a) .$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{F(x)}{G(x)} = \lim_{x \to a} \frac{F'(c)\cancel{(x-a)}}{G'(d)\cancel{(x-a)}} = \lim_{x \to a} \frac{F'(x)}{G'(x)}$$

$$= \lim_{x \to a} \frac{f'(x)}{g'(x)} \quad \text{(since } c, d \to a \text{ as } x \to a \text{)}$$

Remark: $f, g \to \infty$ as $x \to \infty$:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{\frac{1}{g(x)}}{\frac{1}{f(x)}} = \lim_{x \to a} \frac{\frac{-g'(x)}{g^2(x)}}{\frac{-f'(x)}{f^2(x)}} = \lim_{x \to a} \frac{g'(x)}{f'(x)} \left(\frac{f(x)}{g(x)} \right)^2$$

$$= \left(\lim_{x \to a} \frac{g'(x)}{f'(x)} \right) \cdot \left(\lim_{x \to a} \frac{f(x)}{g(x)} \right)^2$$

cancel one power $\Rightarrow 1 = \left(\lim_{x \to a} \frac{g'(x)}{f'(x)} \right) \left(\lim_{x \to a} \frac{f(x)}{g(x)} \right) .$

$$\Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$