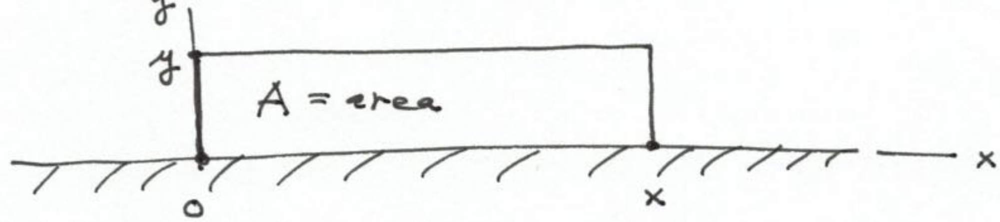
Optimization problems.

Goal: Find best possible outcome, under given constraints.

Ex: Find biggest house in best school district, costing least (given budget), and closest to work.

Ex: Build a fence at one wall of your house.



Make area A as large as possible, under the condition that the length of the fencing material is 100 m.

$A = x \cdot y$ area.

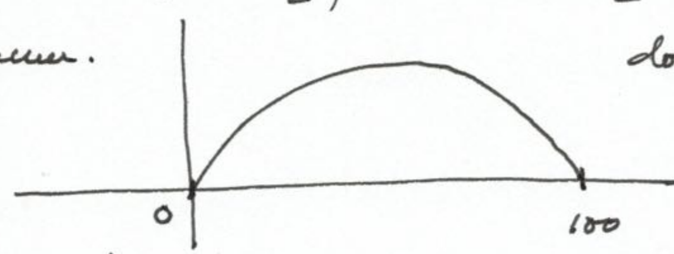
$2y + x = 100$ m length

Eliminate y: solve for y, using the equation that does not involve the quantity to be optimized.

$y = 50 - \frac{x}{2}$.

$\Rightarrow A = x \cdot \left(50 - \frac{x}{2}\right) = 50x - \frac{x^2}{2}$.

Find maximum. domain: $[0, 100]$.



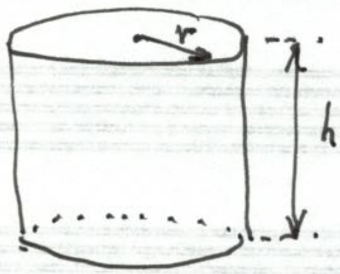
Crit. pts: $A'(x) = 50 - x = 0 \Rightarrow x = 50$.

$A''(x) = -1 \Rightarrow A''(50) = -1 < 0 \Rightarrow$ local max

crit pts at boundaries: $x = 0$, $x = 100 \Rightarrow$ local min.

$\Rightarrow$ for $\underline{x = 50}$, global max $\Rightarrow$ area is maximal, for given length 100 m.

Ex: Build a can (cylindrical) holding 1 l of food, with least material.



$$\text{Volume} \quad V = \underbrace{\pi r^2}_{\text{area of disc}} h = 1\,l$$

$$\text{Area} \quad A = 2\pi r^2 + \underbrace{2\pi r}_{\text{circumference}} h$$

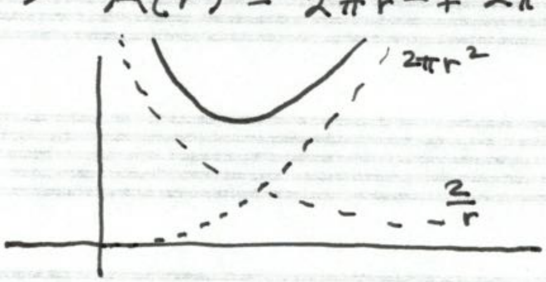
2 discs, top, bottom

Use eq for $\underline{\text{volume}}$ to eliminate one variable

because it is a $\underline{\text{known}}$ quantity.

$$h = \frac{1}{\pi r^2}$$

$$\Rightarrow A(r) = 2\pi r^2 + 2\pi r \frac{1}{\pi r^2} = 2\pi r^2 + \frac{2}{r}.$$



domain: $(0, \infty)$.

A is differentiable on this domain.

$\underline{\text{Crit points:}}$ $A'(r) = 0 = 4\pi r - \frac{2}{r^2}$. $\Rightarrow 4\pi r = \frac{2}{r^2}$.

$$\Rightarrow r^3 = \frac{1}{2\pi} \Rightarrow r = \sqrt[3]{\frac{1}{2\pi}}$$

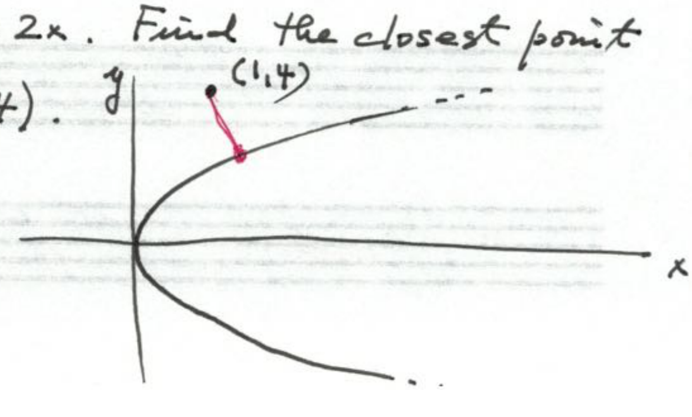
$$A''(r) = 4\pi + \frac{4}{r^3} \Rightarrow A''\left(\sqrt[3]{\frac{1}{2\pi}}\right) = 4\pi + \frac{4}{\left(\sqrt[3]{\frac{1}{2\pi}}\right)^3} > 0$$

$\Rightarrow$ local min.

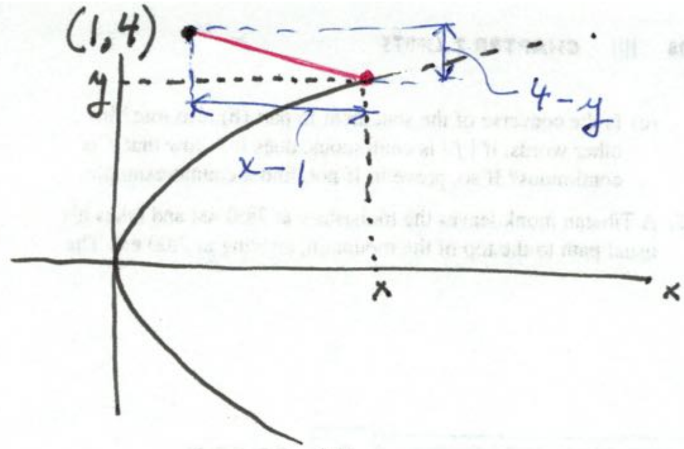
Towards $0, \infty$: $A(r) \to \infty$ as $r \to 0$, or $r \to \infty$

$\Rightarrow \sqrt[3]{\frac{1}{2\pi}}$ is global min.

Ex: Consider a parabola $y^2 = 2x$. Find the closest point on it to the point $(1, 4)$.



Distance from a point (x, y) on the curve to the point $(1,4)$.



$$d = \sqrt{(x-1)^2 + (y-4)^2}$$

(x,y) sits on the parabola:

$$y^2 = 2x.$$

solve for x: $x = \frac{y^2}{2}$.

$$d = \sqrt{\left(\frac{y^2}{2} - 1\right)^2 + (y-4)^2}$$ minimize in y.

Instead of minimizing d, we can minimize $d^2 \Rightarrow$ removes $\sqrt{\ }$.

$$d^2 = \left(\frac{y^2}{2} - 1\right)^2 + (y-4)^2$$

domain: $(-\infty, \infty)$.

d^2 differentiable everywhere

Crit pts: $(d^2)' = 2\left(\frac{y^2}{2} - 1\right) \cdot \frac{\cancel{2} y}{\cancel{2}} + 2(y-4)\cdot 1$

$$= y^3 - \cancel{2y} + \cancel{2y} - 8 = 0$$

$$\Rightarrow y^3 = 8 \Rightarrow y = 2$$

2nd derivative test: $(d^2)'' = 3y^2 \Rightarrow (d^2)''(2) = 3 \cdot 2^2 > 0$

$\Rightarrow$ local minimum.

As $y \to -\infty$, or $y \to \infty$, we have $d^2 \to \infty$.

at min: $x = \frac{y^2}{2} \Rightarrow x = \frac{2^2}{2} = 2 \Rightarrow (2,2)$ is the minimum location for $d^2 \Rightarrow$ also for d.

Antiderivatives

Def F is an antiderivative of f on an interval (a,b) if $F'(x) = f(x)$ for all x in (a,b).

Ex $\frac{x^2}{2}$ is an antiderivative of x

$\ln x$ is an antiderivative of $\frac{1}{x}$.