Antiderivatives.

$$F' = f$$

f is the derivative of F.

F is an antiderivative of f.

Ex x^9 is an antiderivative of $9x^8$

x^9+10 is an antiderivative of $9x^8$

Thm If $F(x)$ is an antiderivative of $f(x)$ on (a,b), then any other antiderivative of f on (a,b) has the form

$$F(x) + \text{constant}.$$

Therefore, the most general antiderivative of f is

$$F(x) + C$$

arbitrary constant.

Ex Find general antiderivative of $f(x)$:

$$f(x) = e^x \Longrightarrow F(x) = e^x + C$$

$$f(x) = \frac{1}{x^2} \Longrightarrow F(x) = -\frac{1}{x} + C$$

$$f(x) = x^r \Longrightarrow F(x) = \frac{x^{r+1}}{r+1} + C$$

$$f(x) = \cos x \Longrightarrow F(x) = \sin x + C.$$

$$f(x) = \sin x \longrightarrow F(x) = -\cos x + C.$$

$$f(x) = \frac{1}{x} \Longrightarrow F(x) = \ln x + C.$$

$$f(x) = 2^x \Longrightarrow F(x) = \frac{2^x}{\ln 2} + C$$

Ex $f(x) = \ln x$, $x > 0 \Rightarrow F(x) = ?$

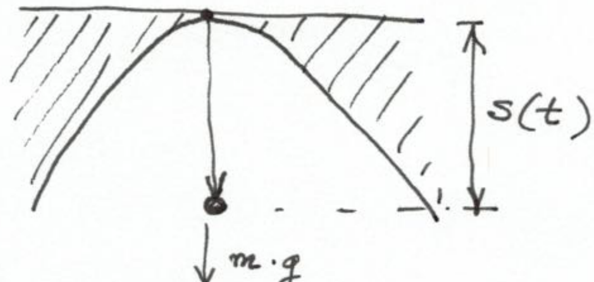
Look at $(x \cdot \ln x)' = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1$.

$$\Rightarrow \ln x = (x \cdot \ln x)' - 1$$

$$= (x \cdot \ln x - x)'$$

$\Rightarrow F(x) = x \cdot \ln x - x + C$ general antiderivative of $\ln x$

Ex



Drop stone from bridge at time $t = 0$ from $s(0) = 0$, with velocity $s'(0) = 1\ ^{m}/_{sec}$.
mass of stone: $m = 1$ kg.

Newton's equation: mass · acceleration = force.

$$\cancel{m} \cdot s''(t) = \cancel{m} \cdot g \quad \leftarrow 9.81\ ^{m}/_{sec^2}$$

$$\Rightarrow s''(t) = 9.81$$

$$\Rightarrow s'(t) = 9.81 \cdot t + C \Rightarrow s'(0) = 1 = 9.81 \cdot 0 + C.$$

$$\Rightarrow \underline{C = 1}.$$

$$\Rightarrow s'(t) = 9.81 \cdot t + 1$$

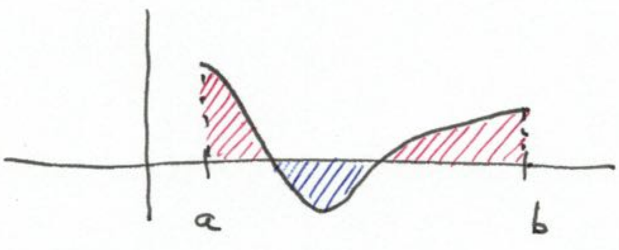
$$\Rightarrow s(t) = 9.81 \cdot \frac{t^2}{2} + t + D \Rightarrow s(0) = 0 = \underbrace{9.81 \cdot \frac{0^2}{2} + 0}_{0} + D$$

$$\Rightarrow \underline{D = 0}.$$

$$\Rightarrow s(t) = 9.81 \frac{t^2}{2} + t$$

Integrals.

Def The definite integral of a continuous function f on $[a, b]$ is the area enclosed between the graph of f and the x-axis.
If an area lies below the x-axis, it is counted with a negative sign.

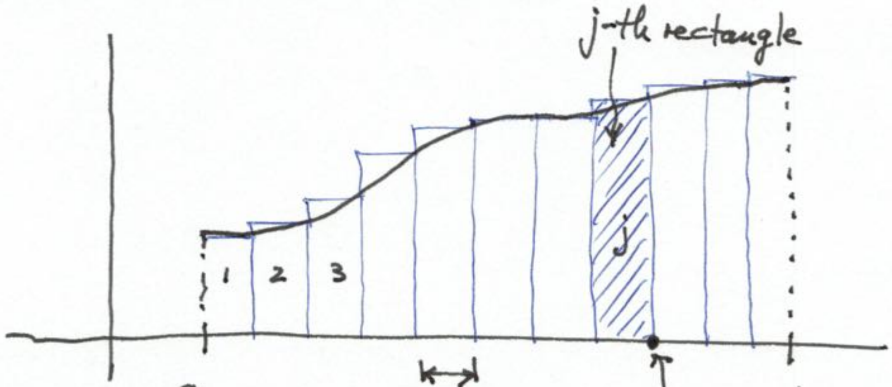


//// positive area $\Rightarrow$ add.

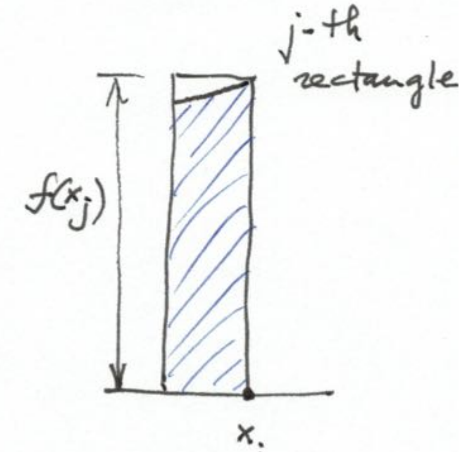
//// negative area $\Rightarrow$ subtract.

How to calculate this area?

j-th rectangle   n rectangles.



$$\Delta x = \frac{b-a}{n} \qquad x_j = a + j \cdot \Delta x$$



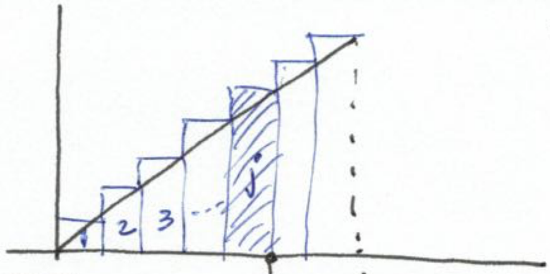
Area of j-th rectangle: $f(x_j) \cdot \Delta x$

Sum of areas of all n rectangles:

$$\underbrace{f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\cdot\Delta x}_{\text{Riemann sum.}} \xrightarrow{n\to\infty} \text{area under } f(x).$$

~ integral of f over $[a, b]$

Ex



$f(x) = x$

$a = 0$ $\qquad$ b

$\Delta x = \frac{b}{n}$. $\qquad$ $x_j = j\Delta x = j \cdot \frac{b}{n}$

height of j-th rectangle

$$f(x_j) = x_j$$

area of j-th rectangle

$$x_j \cdot \Delta x = \left(j \cdot \frac{b}{n}\right) \cdot \frac{b}{n} = j \cdot \frac{b^2}{n^2}$$

Sum of areas of all n rectangles:

$$\underbrace{1 \cdot \frac{b^2}{n^2}}_{\text{1st rectangle}} + 2 \cdot \frac{b^2}{n^2} + 3\frac{b^2}{n^2} + \dots + \underbrace{n\frac{b^2}{n^2}}_{n\text{-th rectangle.}}$$

$$= \underbrace{(1+2+3+\dots+n)}_{(1+n)\cdot\frac{n}{2}} \cdot \frac{b^2}{n^2} = \left(n^2+n\right)\frac{b^2}{2n^2}$$

$$= \frac{b^2}{2}\left(1 + \frac{1}{n}\right)$$ approximation of area with n rectangles

$n \to \infty$: $\quad \longrightarrow \frac{b^2}{2} = \lim_{n\to\infty} \frac{b^2}{2}\left(1+\frac{1}{n}\right)$

Ex: $1 + 2 + 3 + \dots + 7 + 8 + \dots + 13 + 14 = 15 \cdot 7 = (14+1) \cdot \underbrace{\frac{14}{2}}_{\text{number of pairs}}$

15 15 15

15

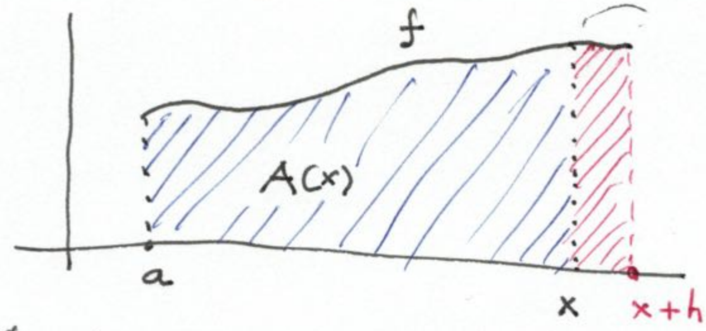
$1+2+3+\dots+7+8+9+\dots+13+14+15 = 16 \cdot 7 + 8$

16 16 16

$= 16 \cdot \left(7 + \frac{1}{2}\right)$

16

A more elegant way to calculate an integral:

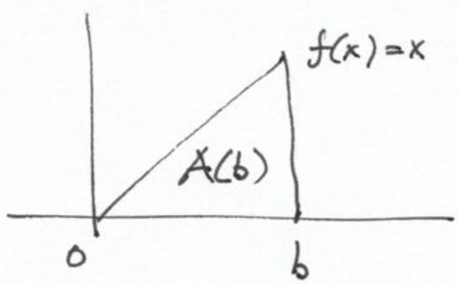


Q: By how much does $A(x)$ change when $x \to x+h$

$$\underbrace{A(x+h) - A(x)}_{\text{red area}} = f(x) \cdot h$$

$$\Rightarrow \underbrace{\lim_{h \to 0} \frac{A(x+h) - A(x)}{h}}_{A'(x)} = f(x)$$

Ex



$$A(x) = \frac{x^2}{2} + C.$$

$$A(0) = 0 \Rightarrow C = 0.$$

$$\Rightarrow A(b) = \frac{b^2}{2}$$