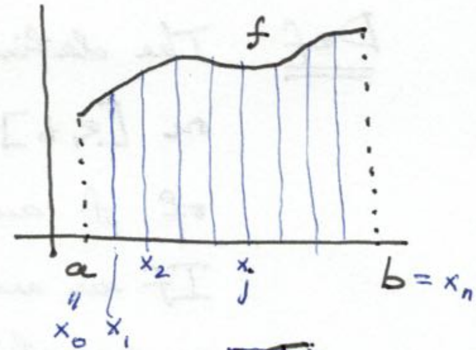
Def (definite integrals).

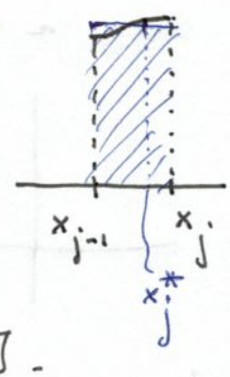
Assume f continuous fct on $[a,b]$.

Let $\Delta x = \frac{b-a}{n}$ and $x_j = a + j\cdot\Delta x$

Pick a sample point x_j^* in $[x_{j-1}, x_j]$



$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \Big(\underbrace{f(x_1^*)\Delta x + f(x_2^*)\Delta x + \dots + f(x_n^*)\Delta x}_{\text{Riemann sum}} \Big)$$

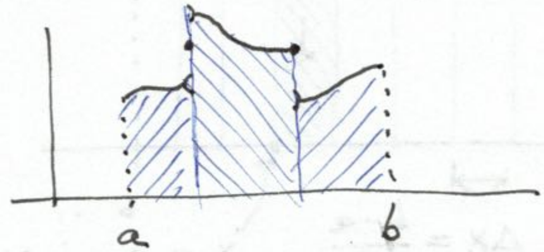


is the definite integral of f from a to b,
if this limit exists.

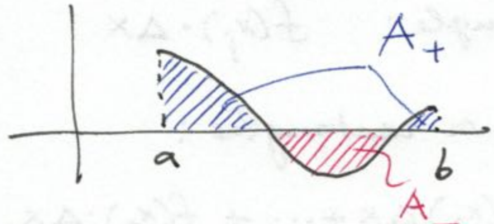
If it exists, we call f integrable on $[a,b]$.

"definite": boundary points a, b are given.

Thm If f is continuous on $[a,b]$, or if it has only finitely many jump discontinuities, then f is integrable.

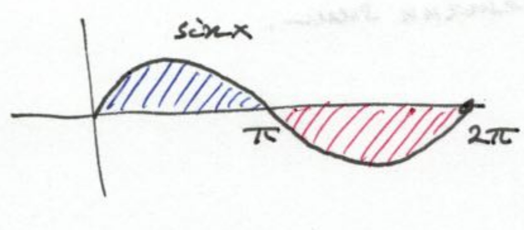


Positive & negative areas:



$$\int_a^b f(x)\,dx = A_+ - A_-$$

Ex $\int_0^{2\pi} \sin x\,dx = 0$



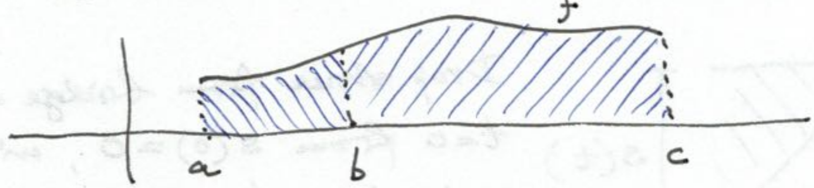
Properties of definite integrals.

Thm

$$\int_a^b c\, f(x)\,dx = c\int_a^b f(x)\,dx$$

$$\int_a^b \big(f(x) \pm g(x)\big)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$$

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$



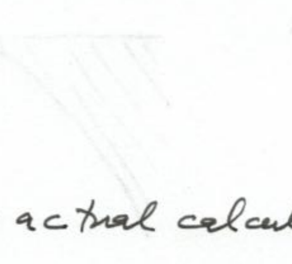
Note: Riemann sums are too cumbersome for actual calculations. They are extremely useful for numerical evaluation of integrals using a computer.

The fundamental theorem of calculus.

Bridge between differentiation and integration.
Discovered by Barrow in 17th century (advisor of Isaac Newton).

Consider: $A(x) = \int_a^x f(t)\,dt$

t integration variable
(indicates Riemann sum & limit)



Already checked: $A'(x) = f(x)$.

Thm (Fundamental theorem of calculus, Part I)

Assume f continuous on $[a,b]$. Then,

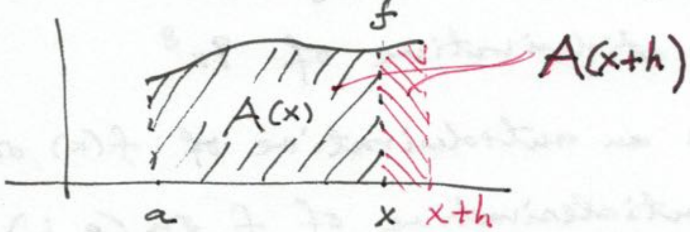
$$A(x) = \int_a^x f(t)\,dt$$

is continuous for x in $[a,b]$, and differentiable for x in (a,b)

$$A'(x) = f(x).$$

(differentiation undoes what integration does).

Recall:



$$\underbrace{A(x+h) - A(x)}_{////} = f(x)\cdot h$$

$$A'(x) = \lim_{h\to 0} \frac{A(x+h) - A(x)}{h} = f(x)$$

Ex: $A(x) = \int_1^x e^{t^2}\,dt.$

Find $A'(x) = e^{x^2}$

Ex: $A(x) = \int_2^x \left(t^{10} + \ln t - e^t\right)dt$

Find $A'(x) = x^{10} + \ln x - e^x$

Ex: $g(x) = \int_2^{x^2} \left(e^{t^2} + \ln t^3\right)dt = A(x^2)$

Find $g'(x) \underset{\substack{\uparrow \\ \text{chain rule}}}{=} A'(x^2)\cdot 2x = \left(e^{x^4} + \ln x^6\right)\cdot 2x$

Thm (Fundamental thm of calculus, Part II).

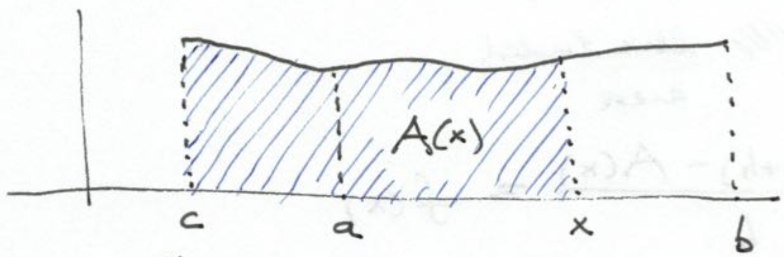
If f is continuous on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) =: F(x)\Big|_a^b$$

where F is <u>an</u> antiderivative of f.

Ex $\int_1^2 e^t\,dt = e^t\Big|_1^2 = e^2 - e^1$

Check:



$A(x) = \int_c^x f(t)\,dt$ is <u>an</u> antiderivative of f.

$$\Rightarrow \int_a^b f(t)\,dt = A(b) - A(a)$$

For any arbitrary antiderivative $F(x)$, we have

$$F(x) = A(x) + C \Rightarrow A(x) = F(x) - C$$

$$\begin{aligned} \Rightarrow \int_a^b f(t)\,dt &= A(b) - A(a) \\ &= (F(b) - C) - (F(a) - C) \\ &= F(b) - \cancel{C} - F(a) + \cancel{C} \\ &= F(b) - F(a) =: F(t)\Big|_a^b \end{aligned}$$

Ex $\int_2^{10} \frac{1}{x}\,dx = \ln x\Big|_2^{10} = \ln 10 - \ln 2 = \ln\frac{10}{2} = \ln 5$.

Ex $\int_1^2 t e^{t^2}\,dt = \frac{1}{2} e^{t^2}\Big|_1^2 = \frac{1}{2}(e^4 - e^1)$.

Ex $\int_1^x t^{10}\,dt = \frac{t^{11}}{11}\Big|_1^x = \frac{x^{11}}{11} - \frac{1}{11}$. (check: derivative in x is $x^{10} = \left(\int_1^x t^{10}\,dt\right)'$)