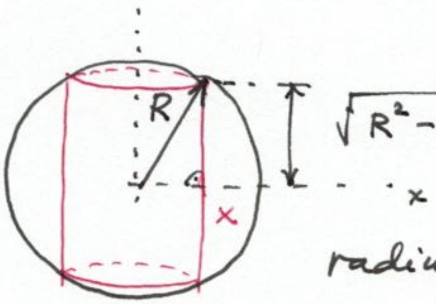
Ex Ball of radius R.



$\sqrt{R^2-x^2}$ half the height.

radius of cylinder: x

circumference: $2\pi x$

height: $2\sqrt{R^2-x^2}$

area: $A(x) = 2\pi x \, 2\sqrt{R^2-x^2}$

$$\text{Volume} = \int_0^R A(x)\,dx = 4\pi \int_0^R x\sqrt{R^2-x^2}\,dx.$$

$$u = R^2 - x^2 \Rightarrow du = -2x\,dx \Rightarrow x\,dx = -\tfrac{1}{2}du.$$

$$u(0) = R^2, \quad u(R) = 0$$

$$= 4\pi \int_{R^2}^{0} \sqrt{u}\left(-\tfrac{1}{2}\right)du$$

$$= 4\pi(-1)\int_0^{R^2} \sqrt{u}\left(-\tfrac{1}{2}\right)du$$

$$= 2\pi \int_0^{R^2} \sqrt{u}\,du.$$

$$= 2\pi \frac{2}{3} u^{3/2}\Big|_0^{R^2}$$

$$= 2\pi \frac{2}{3}\left((R^2)^{3/2} - 0^{3/2}\right).$$

$$= \underline{\underline{\frac{4\pi}{3} R^3}}$$

Ex $$f(x) = \int_2^x \underbrace{\left(\int_{-t}^{t^2} \cos(s^2)\, ds \right)}_{g(t)} dt = \int_2^x g(t)\, dt$$

$$f''(x) = ?$$

$$f'(x) = g(x) = \int_{-x}^{x^2} \cos(s^2)\, ds .$$

$$f''(x) = g'(x) = \cos(x^4)\cdot 2x - \cos(x^2)\cdot(-1) .$$

FToC: $$f(x) = \int_{h(x)}^{\ell(x)} g(t)\, dt$$

$$\Rightarrow f'(x) = g(\ell(x))\cdot \ell'(x) - g(h(x))\cdot h'(x) .$$

Ex $$\lim_{x\to\infty} \left(\sqrt{x^2+1} - \sqrt{x^2+2} \right) = ?$$

$$= \lim_{x\to\infty} \left(\sqrt{x^2+1} - \sqrt{x^2+2} \right) \frac{\sqrt{x^2+1} + \sqrt{x^2+2}}{\sqrt{x^2+1} + \sqrt{x^2+2}} .$$

$$= \lim_{x\to\infty} \frac{\cancel{x^2}+1 - (\cancel{x^2}+2)}{\sqrt{x^2+1} + \sqrt{x^2+2}} .$$

$$= \lim_{x\to\infty} \frac{-1}{\sqrt{x^2+1} + \sqrt{x^2+2}} = 0$$

Ex $$\lim_{x\to\infty} x \left(\sqrt{x^2+1} - \sqrt{x^2+2} \right) .$$

Try de l'Hôpital.

$$\lim_{x\to\infty} \frac{\sqrt{x^2+1} - \sqrt{x^2+2}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{1}{\cancel{2}\sqrt{x^2+1}}\cdot \cancel{2}x - \frac{1}{\cancel{2}\sqrt{x^2+2}}\cdot \cancel{2}x}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \left(\frac{x}{\sqrt{x^2+1}} - \frac{x}{\sqrt{x^2+2}} \right)(-x^2) .$$

Looks more complicated than before.

$$\lim_{x\to\infty} x\left(\sqrt{x^2+1}-\sqrt{x^2+2}\right)\frac{\sqrt{x^2+1}+\sqrt{x^2+2}}{\sqrt{x^2+1}+\sqrt{x^2+2}}$$

$$=\lim_{x\to\infty} x\,\frac{\cancel{x^2}+1-(\cancel{x^2}+2)}{\sqrt{x^2+1}+\sqrt{x^2+2}}$$

$$=\lim_{x\to\infty}\frac{-x}{\sqrt{x^2+1}+\sqrt{x^2+2}}$$

$$\sqrt{x^2+1}=\sqrt{x^2\left(1+\frac{1}{x^2}\right)}$$

$$=x\cdot\sqrt{1+\frac{1}{x^2}}$$

$$=\lim_{x\to\infty}\frac{-\cancel{x}}{\cancel{x}\sqrt{1+\frac{1}{x^2}}+\cancel{x}\sqrt{1+\frac{2}{x^2}}}$$

$$=\lim_{x\to\infty}\frac{-1}{\sqrt{1+\frac{1}{x^2}}+\sqrt{1+\frac{2}{x^2}}}=-\frac{1}{2}.$$

$$=\frac{-1}{\lim_{x\to\infty}\left(\sqrt{1+\underbrace{\frac{1}{x^2}}_{\to 0}}+\sqrt{1+\underbrace{\frac{2}{x^2}}_{\to 0}}\right)}=-\frac{1}{2}.$$

How to make de l'Hôpital work:

$$\lim_{x\to\infty} x\left(\sqrt{x^2+1}-\sqrt{x^2+2}\right)=\lim_{x\to\infty} x^2\left(\sqrt{1+\frac{1}{x^2}}-\sqrt{1+\frac{2}{x^2}}\right).$$

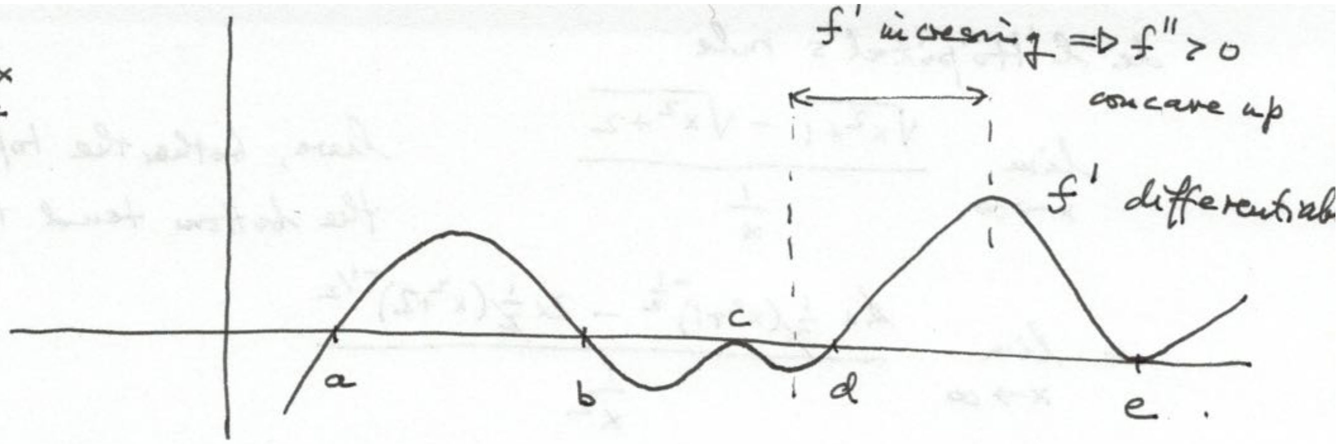
$$=\lim_{x\to\infty}\frac{\sqrt{1+\frac{1}{x^2}}-\sqrt{1+\frac{2}{x^2}}}{\frac{1}{x^2}}$$

$h=\frac{1}{x^2}$

$$=\lim_{h\to 0}\frac{\sqrt{1+h}-\sqrt{1+2h}}{h}\underset{\text{de l'H.}}{=}\lim_{h\to 0}\frac{\frac{1}{2\sqrt{1+h}}-\frac{2}{2\sqrt{1+2h}}}{1}=\frac{1}{2}-1$$

$$=-\frac{1}{2}.$$

Ex



f has the following properties:

$\underline{x=a}$: Crit pt. local min. (from − to +)

$\underline{x=b}$: Crit pt local max (from + to −).

$\underline{x=c}$: Crit pt, but neither min nor max.

$f''(c) = 0$ and $f''(x)$ changes sign at c ⟹ inflection pt.

$\underline{x=d}$: Crit pt, local min.

$\underline{x=e}$: Crit pt, neither min nor max

inflection pt.

Ex $f(x) = x^{x^2}$ Find $f'(x)$

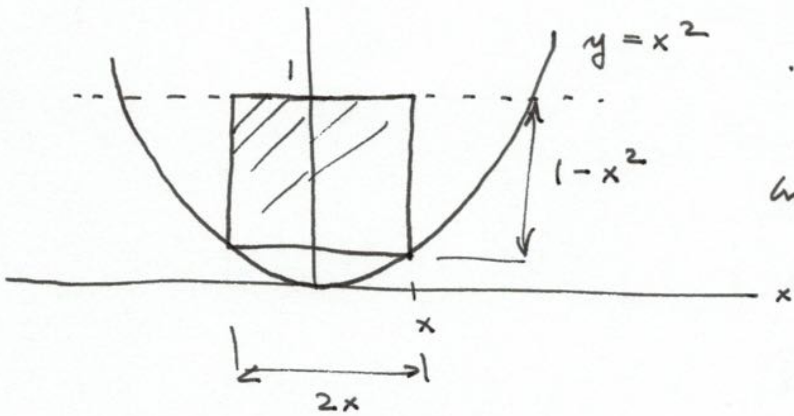
$x > 0$.

$$f(x) = \left(e^{\ln x}\right)^{x^2} = e^{x^2 \ln x}$$

$$f'(x) = \underbrace{e^{x^2 \ln x}}_{x^{x^2}} \cdot \underbrace{(x^2 \ln x)'}_{x^2 \cdot \frac{1}{x} + 2x \cdot \ln x}.$$

$$= x^{x^2} \cdot (x + 2x \cdot \ln x).$$

Ex



Find rectangle with largest area.

$$A(x) = 2x \cdot (1 - x^2).$$

$$A'(x) = 2(1 - x^2) + 2x(-2x).$$

$$= 2 - 2x^2 - 4x^2 = 2 - 6x^2 = 0$$

$$\Rightarrow x^2 = \frac{1}{3}. \Rightarrow \underline{x = \pm \frac{1}{\sqrt{3}}}.$$

$$A\left(\frac{1}{\sqrt{3}}\right) = \frac{2}{\sqrt{3}}\Big(\underbrace{1 - \frac{1}{3}}_{\frac{2}{3}}\Big) = \frac{4}{3\sqrt{3}}.$$