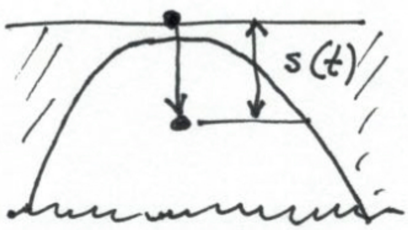
Another example of rate of change.

Change of position of object in time.

Consider a stone, falling from a bridge, and measure distance from location where it's dropped.



$$s(t) = 4.9\, t^2 \text{ meters}$$

t: time in seconds.

$\frac{9.81}{2}$ m/sec².

How fast after 5 sec?

One way to put it:

$$\bar{v} = \frac{s(5.1)-s(5)}{5.1-5} = \frac{4.9\,(5.1)^2 - 4.9 \cdot 5^2}{0.1} = 49.49 \text{ m/sec}.$$

estimate, using <u>average</u> <u>velocity</u> for the time between 5 and 5.1 seconds.

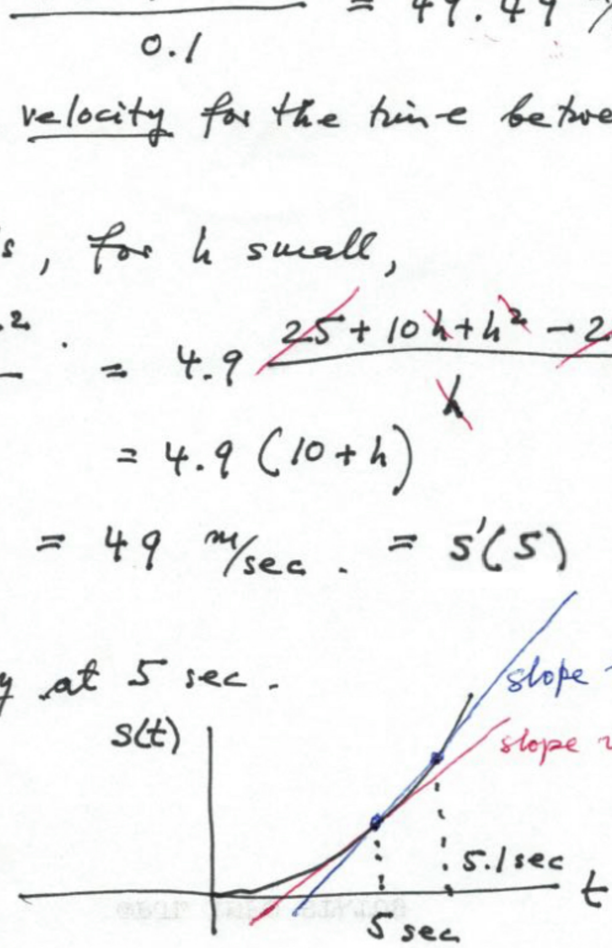
For a time $5+h$ seconds, for h small,

$$\frac{4.9\,(5+h)^2 - 4.9 \cdot 5^2}{h} = 4.9\, \frac{\cancel{25} + 10h + h^2 - \cancel{25}}{\cancel{h}}$$

$$= 4.9\,(10+h)$$

$$v = \lim_{h \to 0} 4.9\,(10+h) = 49 \text{ m/sec}. = s'(5)$$

instantaneous velocity at 5 sec.
= derivative of $s(t)$ at 5 sec



The limit of a function.

Recall the function $f(x) = \frac{x^2-1}{x-1}$.

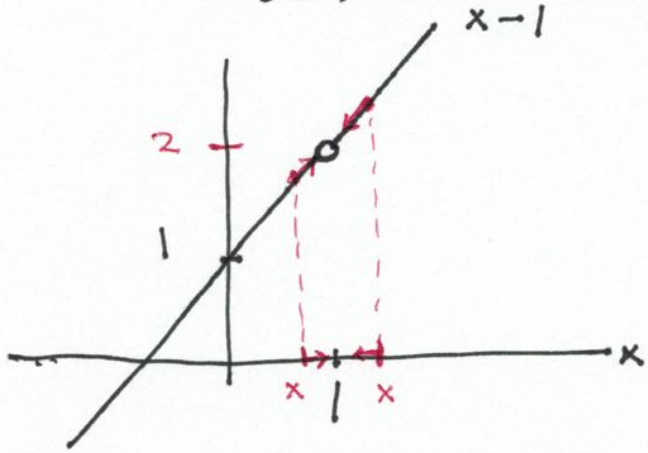
Def We say that f has a function value at x if $f(x)$ can be properly computed.

Ex $f(x) = \frac{x^2-1}{x-1}$ has a function value for all x not equal to 1.

But for $x = 1$, it has no well-defined function value.

When $x < 1$, or $x > 1$ (not equal to 1), then

$$f(x) = \frac{x^2-1}{x-1} = \frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}} = x+1$$



at $x=1$, f has no fct value $\Rightarrow$ hole in the graph

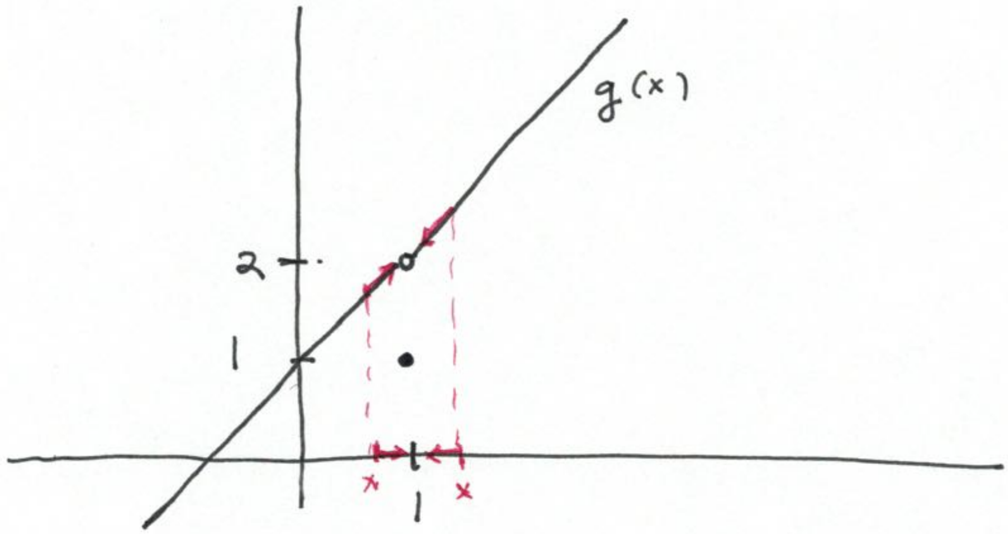
But: As x approaches 1, $f(x)$ approaches 2.

$$\Rightarrow \lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^2-1}{x-1} = 2.$$

Observe: f doesn't have a fct value at $x=1$, but it has a limit!

We could also cook up a function

$$g(x) = \begin{cases} \dfrac{x^2-1}{x-1} & \text{when } x \neq 1 \\ 1 & \text{when } x = 1 \end{cases}$$



$$\left.\begin{aligned} \lim_{x\to 1} g(x) &= 2 . \\ g(1) &= 1 \end{aligned}\right\}$$

g has a fct value at 1, and a limit at 1, but they are not the same.

Def (Limit of a function).

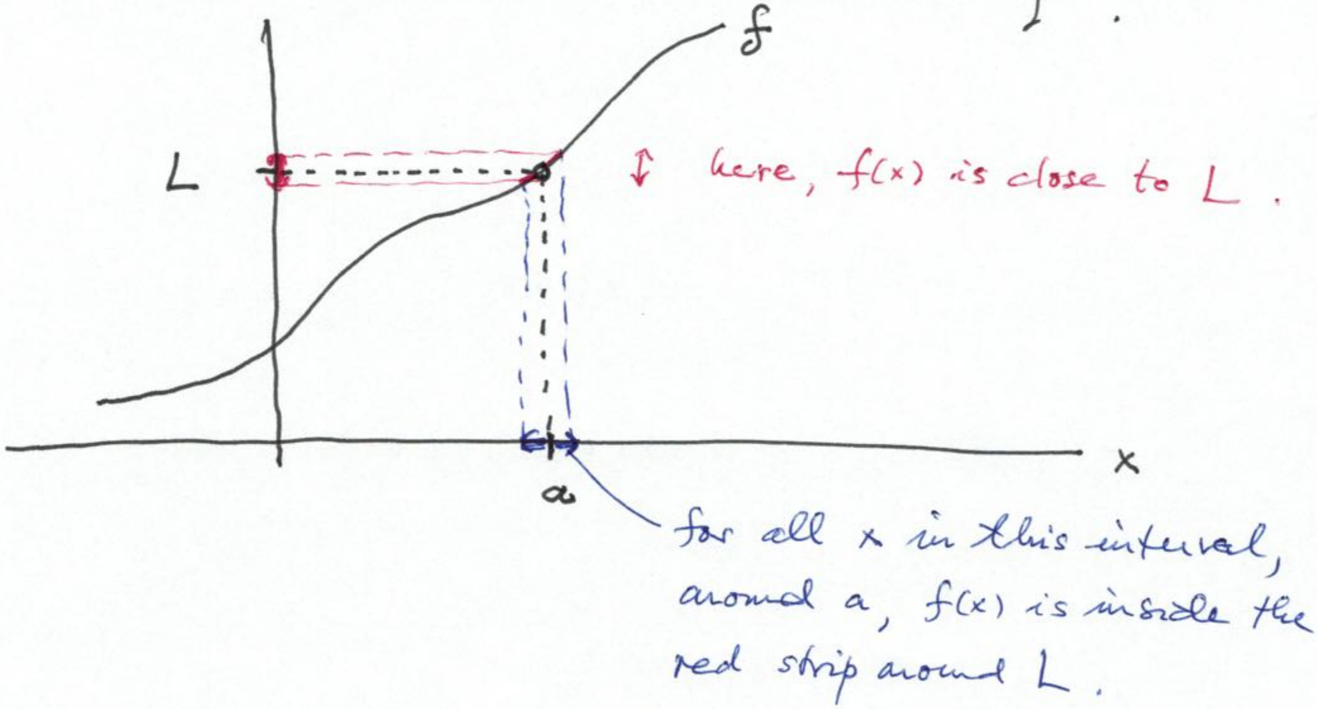
Assume that near $x = a$, $f(x)$ has well-defined function values.

Then, we write

$$\lim_{x \to a} f(x) = L$$

if we can bring $f(x)$ arbitrarily close to L by taking x sufficiently close to a.

(**Note**: f might or might not have a fct value at a, and even if it does, it doesn't need to match the limit).



Ex $\lim_{t \to 0} \frac{\sqrt{t^2+9} - 3}{t^2} \underset{\substack{\uparrow \\ \underline{\text{trick!}}}}{=} \lim_{t \to 0} \frac{\sqrt{t^2+9} - 3}{t^2} \cdot \frac{\sqrt{t^2+9} + 3}{\sqrt{t^2+9} + 3}$

$$= \lim_{t \to 0} \frac{\cancel{t^2} + \cancel{9} - \cancel{9}}{\cancel{t^2}\left(\sqrt{t^2+9} + 3\right)} = \lim_{t \to 0} \frac{1}{\sqrt{t^2+9} + 3}.$$

$$= \frac{1}{6}$$

we used $\left(\sqrt{t^2+9} - 3\right)\left(\sqrt{t^2+9} + 3\right) = \left(\sqrt{t^2+9}\right)^2 - 9$

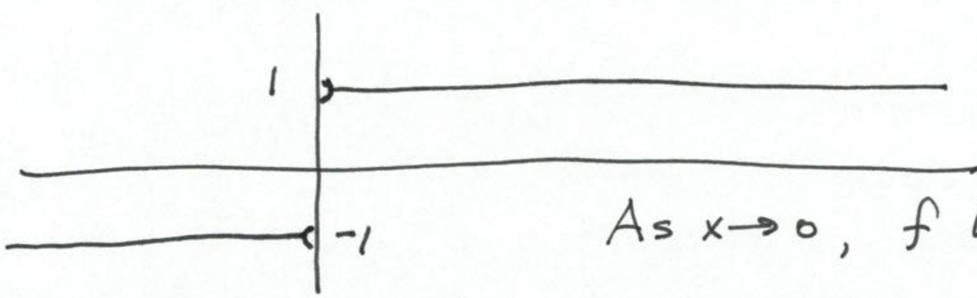
$$= t^2 + 9 - 9$$

recall: $(a+b)(a-b) = a^2 - b^2$

$$(a+b)^2 = a^2 + 2ab + b^2$$

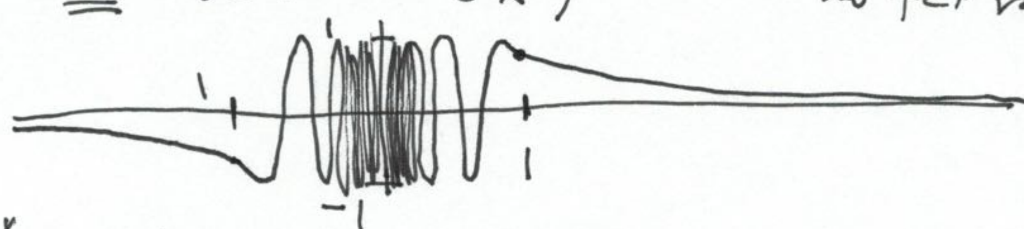
$$(a-b)^2 = a^2 - 2ab + b^2$$

Ex $f(x) = \frac{x}{|x|}$ no fct value at 0.



As $x \to 0$, f has no limit.

Ex $f(x) = \sin\left(\frac{1}{x}\right)$ no fct value value at $x=0$



As $x \to 0$,
$f(x)$ oscillates crazier
and crazier, but
does not have a limit.