8/30/18 **Exponential functions.** (integer exponents).

$$a^1 = a \qquad a > 0$$

$$a^2 = a \cdot a$$

$$a^3 = a \cdot a \cdot a$$

$$\vdots$$

$$a^n = \underbrace{a \cdot a \cdots a}_{n \text{ times}}$$

"a to the power n"

a base, n exponent.

$$\boxed{a^m \cdot a^n = \underbrace{a \cdots a}_{m \text{ times}} \cdot \underbrace{a \cdots a}_{n \text{ times}} = a^{m+n}}$$

If $n > m$:

$$\frac{a^n}{a^m} = \frac{\overbrace{\cancel{a \cdots a}}^{m} \cdot \overbrace{a \cdots a}^{n-m}}{\underbrace{\cancel{a \cdots a}}_{m}} = a^{n-m} = a^{n+(-m)} = a^n \cdot \frac{1}{a^m}$$

$$\Rightarrow \boxed{a^{-m} = \frac{1}{a^m}}$$

$$a^0 = a^{n-n} = \frac{a^n}{a^n} = 1$$

$$0^n = 0$$

0^0 not defined (can't divide by zero).

$$(a^m)^n = \left.\begin{array}{l} \underbrace{a \cdots a}_{m} \\ \cdot \underbrace{a \cdots a}_{m} \\ \vdots \\ \cdot \underbrace{a \cdots a}_{m} \end{array}\right\} n \text{ times} = a^{m \cdot n} = (a^n)^m$$

Exponentials with fractional powers. $a>0$

$$\left(a^{\frac{5}{2}}\right)^2 = a^{\frac{5}{2}\cdot 2} = a^5 \Rightarrow a^{\frac{5}{2}} = \sqrt{a^5}$$

$p, q > 0$ integers.

$$\left(a^{\frac{p}{q}}\right)^q = a^{\frac{p}{q}\cdot q} = a^p \Rightarrow a^{\frac{p}{q}} = \sqrt[q]{a^p}$$

In general:

$$a^{\frac{1}{q}} = \sqrt[q]{a}$$

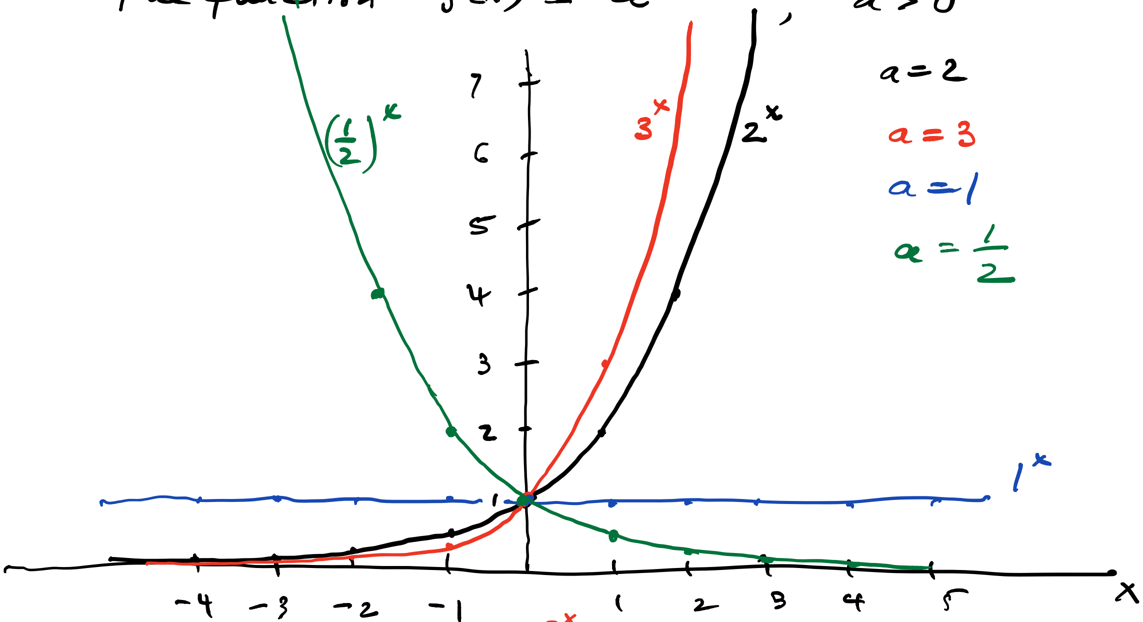
Ex $$2^{1.212} = 2^{\frac{1212}{1000}} = \sqrt[1000]{2^{1212}}$$

Ex a^{π} $\qquad \pi = 3.1415\ldots\ldots$

a^{π} is close to a^3
closer to $a^{3.1}$
even closer to $a^{3.14}$
$\vdots$

more and more precise approximations.
$\Rightarrow$ taking a limit.

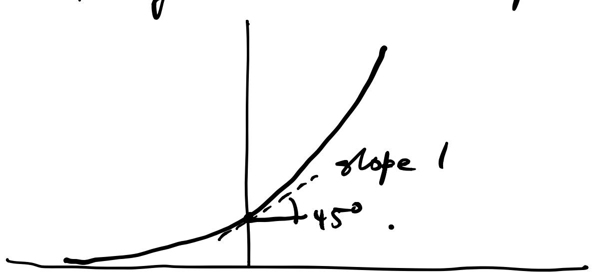
The function $f(x) = a^x$, $a > 0$



2^x and $\underbrace{(\frac{1}{2})^x}_{=2^{-x}}$ have graphs that are mirror images of one another (across y-axis)

Similarly true for a^x and a^{-x}

There is a particular number for a such that a^x crosses the y-axis with slope 1.



That value is $e = 2.71828\ldots$

Euler number.

Ex: Growth rates of bacteria.
Assume every second, a bacterium splits into two.

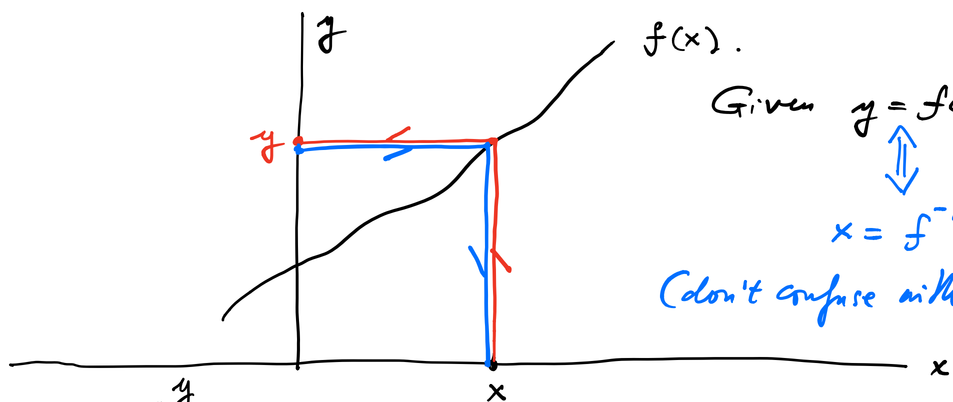
t time, in seconds.

$n(t)$ number of bacteria at time t.

$$n(0)$$
$$n(1) = 2\,n(0)$$
$$n(2) = 2\,n(1) = 2^2\,n(0)$$
$$n(3) = 2 \cdot n(2) = 2^3\,n(0)$$
$$\vdots$$
$$n(t) = 2^t\,n(0).$$

9/4/18

Inverse functions.

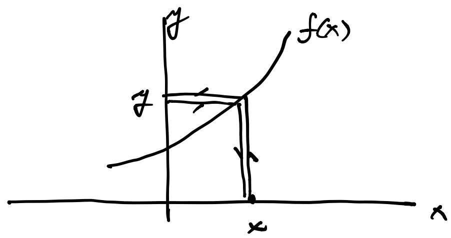


Given $y = f(x)$.
$$\Updownarrow$$
$$x = f^{-1}(y)$$
(don't confuse with $\frac{1}{f(y)}$!!).

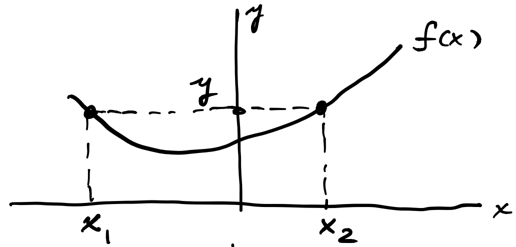
$$\left.\begin{aligned} f^{-1}(\overset{=y}{f(x)}) &= x \\ f(\underset{=x}{f^{-1}(y)}) &= y \end{aligned}\right\}$$

f and f^{-1} mutually undo what the other one does.

Ex

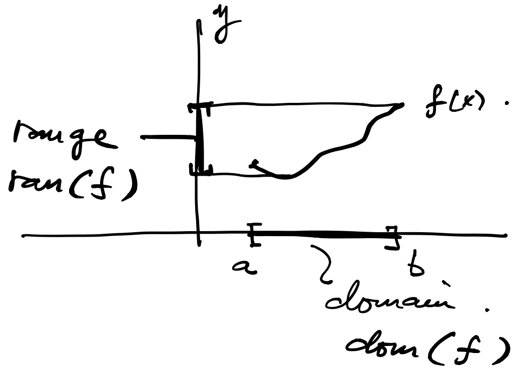


inverse fct well-defined



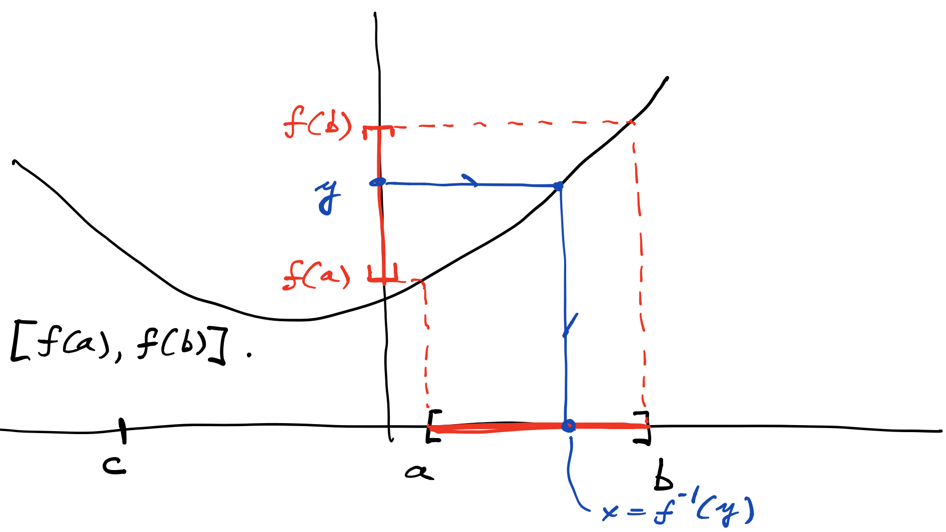
here it's not, because a fct can have at most one output per input.

Need to be careful about domain and range of a fct.



<u>Notation</u>:

$[a,b]$	interval with both a,b included.	
(a,b)	"	without a,b
$(a,b]$	"	with b included, a excluded.
$[a,b)$	"	with b excluded, a included.



$\operatorname{ran}(f) = [f(a), f(b)]$.

(

inverse fct well-defined for this domain, because here, there is one value of x for each value of y.

On $[a,b]$, f has precisely one y-value per x-value.

In this case, f is said to be <u>one-to-one</u> (1-to-1), or <u>injective</u>.

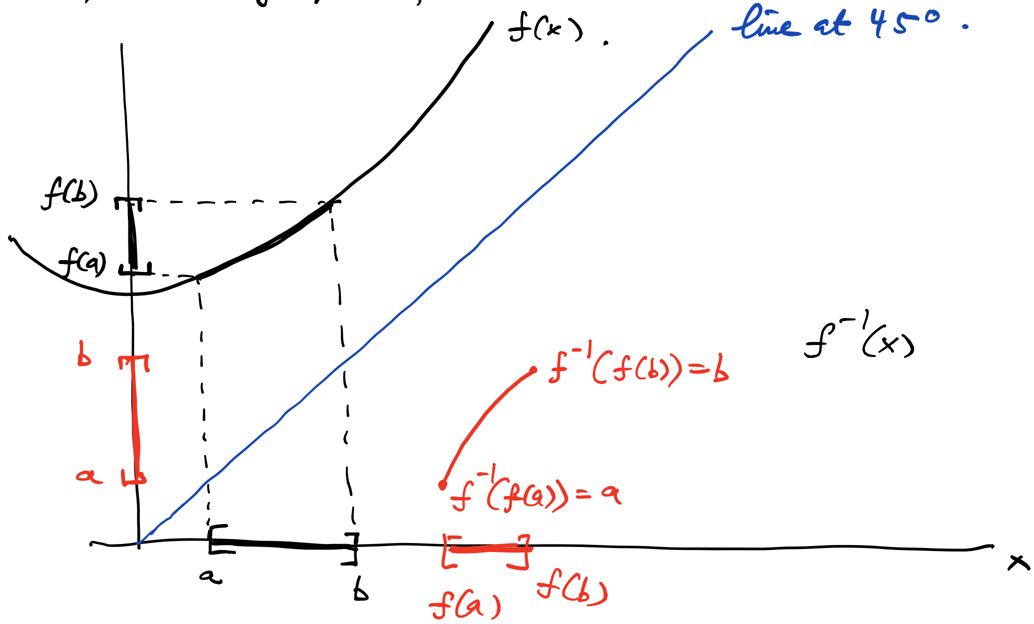
<u>Note:</u> f <u>not</u> injective on $[c,b]$.

The inverse function for f exists when f is injective on its domain.

Then, the domain of f^{-1} is the range of f

the range of f^{-1} is the domain of f.

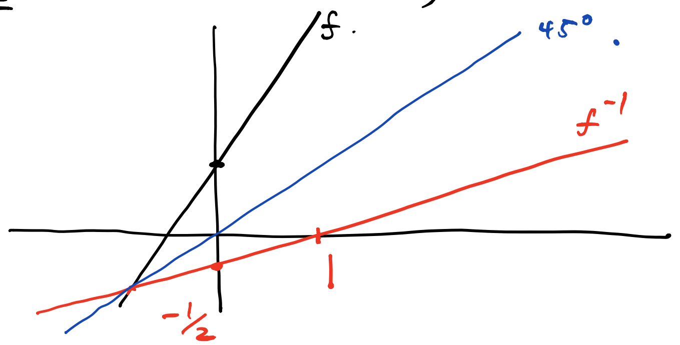
Next, draw graph of f^{-1}:



The graph of f^{-1} is obtained from the mirror image of the graph of f, across the diagonal at 45°.

"element of"

Ex: $f(x) = 2x + 1$, for $x \in (-\infty, \infty)$



Find inverse fct.

$y = f(x) = 2x + 1.$

Solve for x

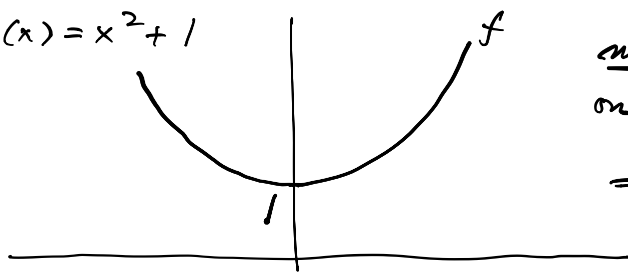
$\Rightarrow y - 1 = 2x$

$\Rightarrow x = \frac{y}{2} - \frac{1}{2} = f^{-1}(y)$

Switch $x \Leftrightarrow y$.

$y = \frac{x}{2} - \frac{1}{2} = f^{-1}(x)$

Ex. $f(x) = x^2 + 1$

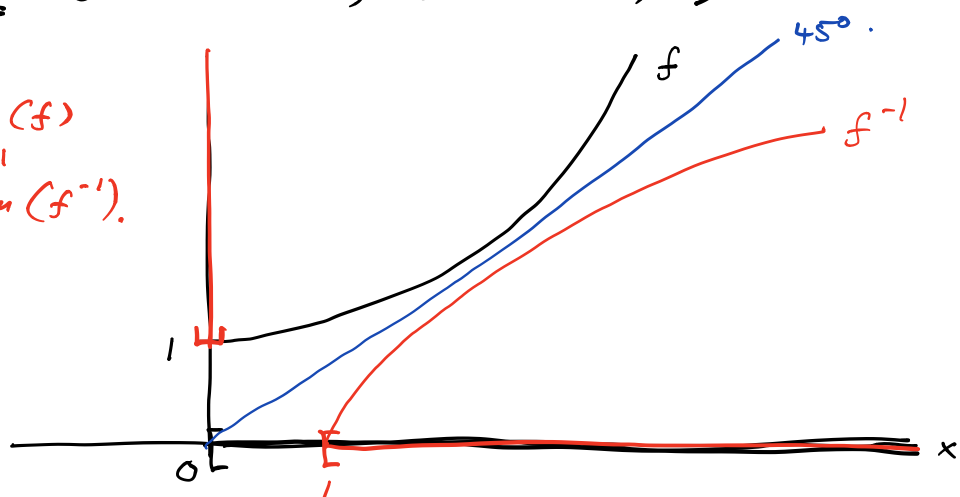


not injective on $(-\infty, \infty)$

$\Rightarrow f^{-1}$ not defined.

Ex $f(x) = x^2 + 1$, for $x \in [0, \infty)$

ran (f) $=$ dom (f^{-1}).



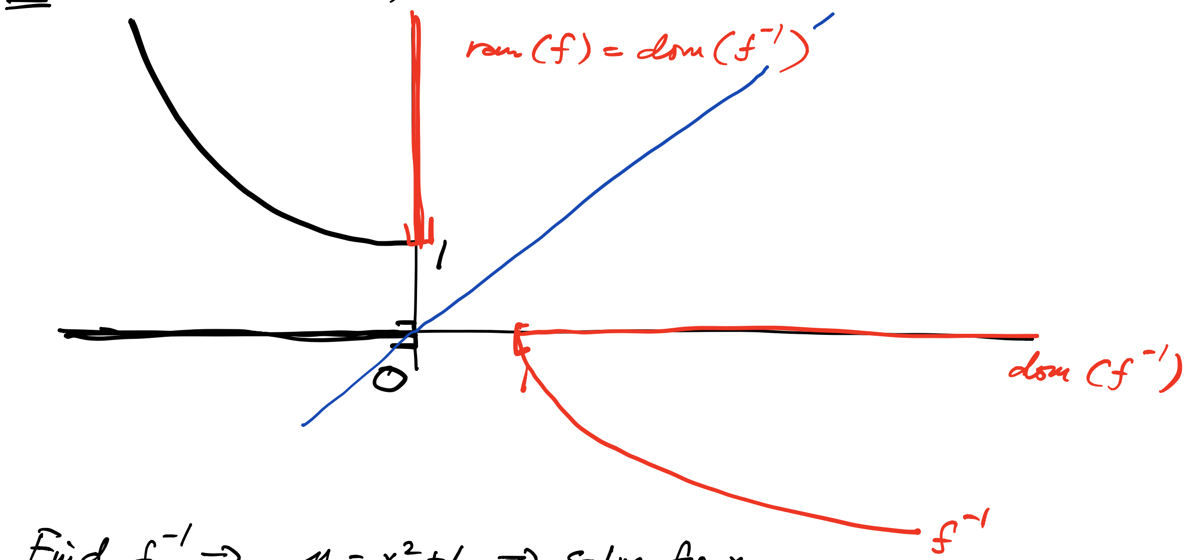
Find f^{-1}: $y = x^2 + 1 \Rightarrow$ solve for x.

$\Rightarrow y - 1 = x^2 \Rightarrow x = \sqrt{y - 1}$

positive because x positive on this domain.

Switch $x \leftrightarrow y \Rightarrow y = \sqrt{x-1} = f^{-1}(x)$

Ex $f(x) = x^2+1$, for $x \in (-\infty, 0]$



Find $f^{-1} \Rightarrow \quad y = x^2+1 \Rightarrow$ solve for x

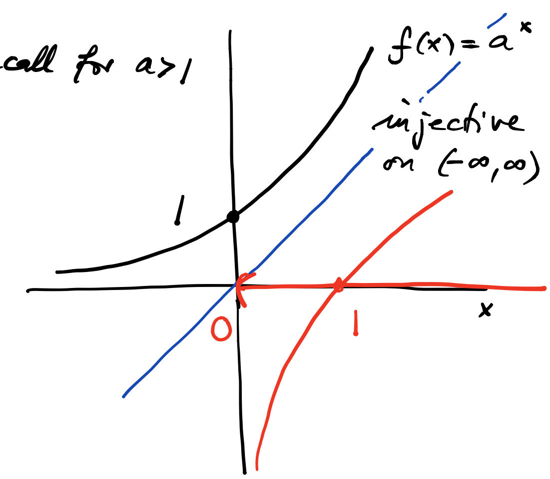
$\Rightarrow \quad y-1 = x^2 \quad \Rightarrow \quad x = -\sqrt{y-1}$

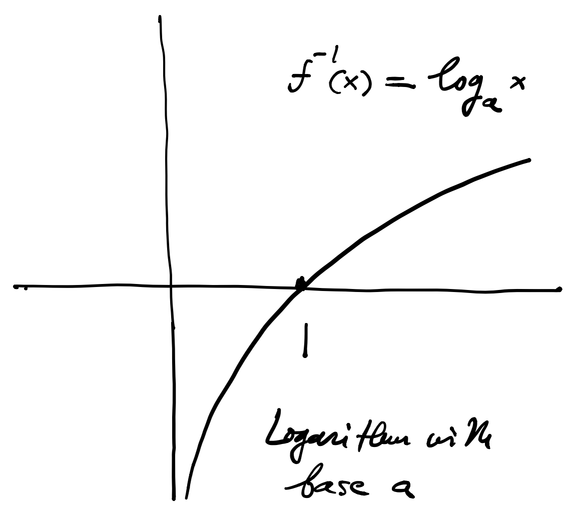
negative because x is negative on this domain.

Switch $x \leftrightarrow y \Rightarrow \quad y = -\sqrt{x-1} = f^{-1}(x)$.

9/6/18 **Logarithms as inverse fcts of exponential fcts.**

Recall for $a > 1$



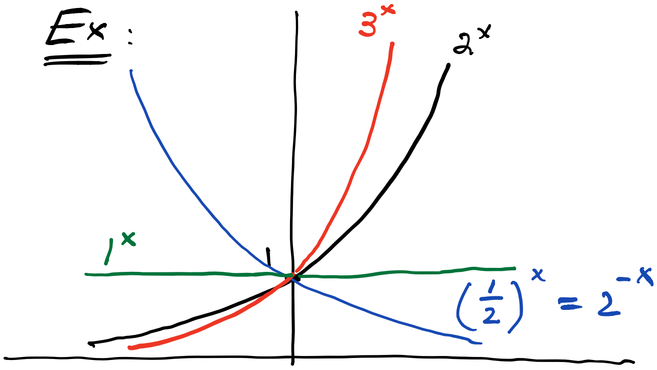


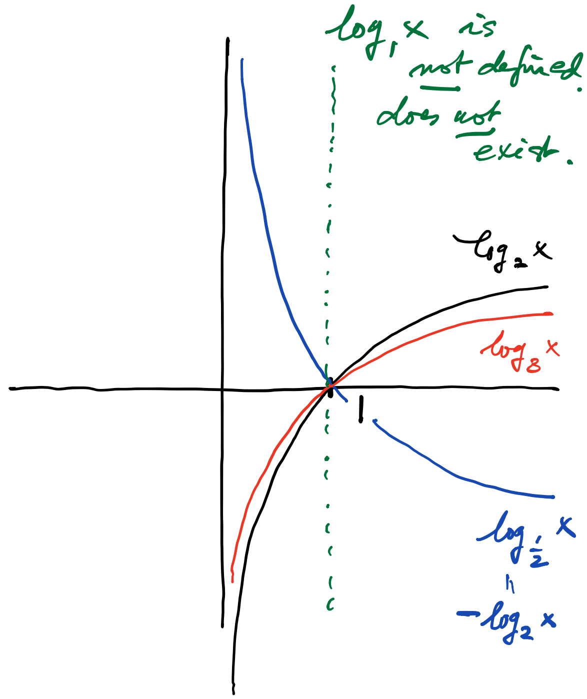
$\mathrm{dom}(f) = (-\infty, \infty)$

$\mathrm{ran}(f) = (0, \infty)$

$$\boxed{\begin{aligned} a^{\log_a x} &= x \\ \log_a(a^x) &= x. \end{aligned}}$$

Ex :





$$x \cdot y = \boxed{a^{\log_a(x \cdot y)}}$$

$$= \underbrace{a^{\log_a x}}_{x} \cdot a^{\log_a y} = \boxed{a^{\log_a x + \log_a y}}$$

$$\Rightarrow \log_a(x \cdot y) = \log_a x + \log_a y$$

$$\frac{x}{y} = \boxed{a^{\log_a\left(\frac{x}{y}\right)}}$$

$$= x \cdot \frac{1}{y} = \underbrace{a^{\log_a x}}_{x} \cdot \frac{1}{a^{\log_a y}} = a^{\log_a x} \cdot a^{-\log_a y}$$

$$= \boxed{a^{\log_a x - \log_a y}}$$

$$\boxed{\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y}$$

Special case: $x = 1$

$$\boxed{\log_a\left(\frac{1}{y}\right) = \underbrace{\log_a 1}_{=0} - \log_a y = -\log_a y}$$

Ex $\log_a(x^{10}) = \log_a(x \cdot x^9)$

$$= \log_a x + \log_a \underbrace{x^9}_{x \cdot x^8}$$

$$= \log_a x + \log_a x + \log_a x^8$$

$$= \underbrace{\log_a x + \dots + \log_a x}_{10 \text{ times}} = 10 \log_a x$$

More generally:

$$\boxed{\log_a x^r = r \log_a x, \quad \text{for any } r \in (-\infty, \infty).}$$

Ex $\log_a \sqrt{x} = \log_a x^{\frac{1}{2}} = \frac{1}{2} \log_a x.$

$\log_a \frac{1}{x} = \log_a x^{-1} = -\log_a x$

$\log_a 1 = \log_a x^0 = 0 \cdot \log_a x = 0$

$\log_a 0$ not defined.

Ex Check that

$$\log_b x = (\log_a x) \cdot (\log_b a)$$

$$x = b^{\log_b x} \stackrel{?}{=} b^{(\log_a x) \cdot (\log_b a)}$$

$$= \Big(\underbrace{b^{\log_b a}}_{a}\Big)^{\log_a x}$$

$$= a^{\log_a x} = \boxed{x}$$

Correct.

Ex Find

$$\underbrace{\log_2 4^{17}}_{} - \underbrace{3^{\log_9 81}}_{} = 34 - 9 = \underline{\underline{25}}.$$

$$17 \log_2 4 = 17 \log_2 2^2 = 34 \underbrace{\log_2 2}_{1}$$

$$(\log_a a = 1)$$

$$3^{\log_9 9^2} = 3^{2 \underbrace{\log_9 9}_{1}} = 3^2 = 9$$

Notation: e (Euler number) as base:

$$\boxed{\log_e x = \ln x}$$

"natural logarithm".

Ex: $x = 27$, $a = 3$, $b = 9$:

$$\log_9 27 = \Big(\overbrace{\log_3 27}^{3}\Big)\Big(\overbrace{\log_9 3}^{\log_9 9^{1/2} = \frac{1}{2}}\Big). = \frac{3}{2}$$

Ex $e^{8(\ln 2)(\log_2 x)} = (\underbrace{e^{\ln 2}}_{2})^{8\log_2 x}$

$$= (\underbrace{2^{\log_2 x}}_{x})^{8} = x^8$$

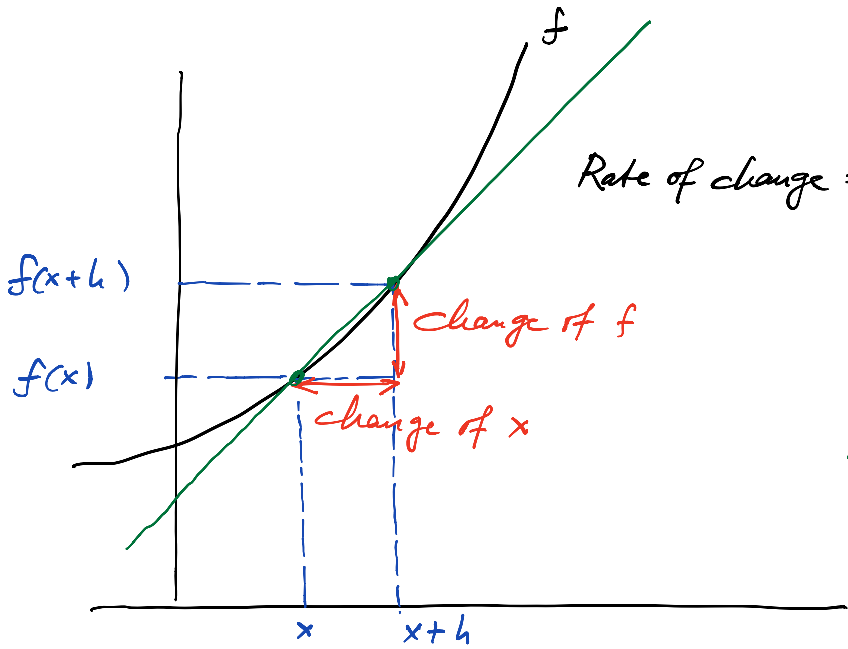
Limits

Calculus is very different from algebra in that we do not usually try to solve equations *exactly*, but we will try to find arbitrarily good approximations with precise error control.

Ex: $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = \dots$ gets arbitrarily close to 2.

$$\lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots + \frac{1}{2^n}\right) = 2$$

Rate of change of a function.



$$\text{Rate of change} = \frac{\text{Change of } f}{\text{Change of } x}$$

$$= \frac{f(x+h) - f(x)}{\cancel{x}+h - \cancel{x}}$$

$=$ slope of green line.

Def $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ derivative of f at x.

Ex: $f(x) = x^2$

$$(x^2)' = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}$$

$$= \lim_{h \to 0} (2x + h) = 2x$$

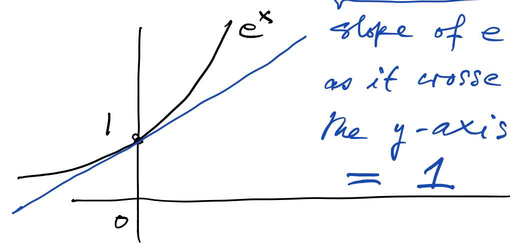
Ex: $(e^x)' = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$

$$= \lim_{h \to 0} \frac{e^x e^h - e^x}{h}$$

$$= e^x \lim_{h\to 0} \frac{e^h - 1}{h} = e^x \underbrace{\lim_{h\to 0} \frac{e^{0+h} - e^0}{h}}$$

slope of e^x as it crosses the y-axis. $= 1$

$$= e^x$$



Ex: $\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}}$

$$= \lim_{x\to 1} (x+1) = 2.$$

9/11/18

Class notes from 9/11/2018 are on a separate file in the same folder.

9/13/2018:

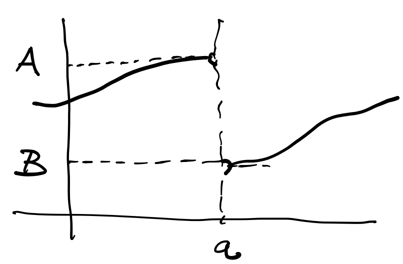
Left and right limits.

Def Left limit $\lim_{x\to a^-} f(x) = A$ — $f(x)$ approaches A when x approaches a from the left.

Right limit $\lim_{x\to a^+} f(x) = B$ — $f(x)$ approaches B when x approaches a from the right.

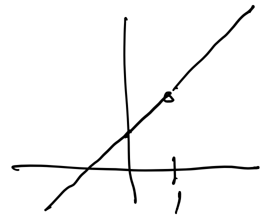
Ex: $\lim_{x\to 0^+} \frac{x}{|x|} = 1$

$\lim_{x\to 0^-} \frac{x}{|x|} = -1$

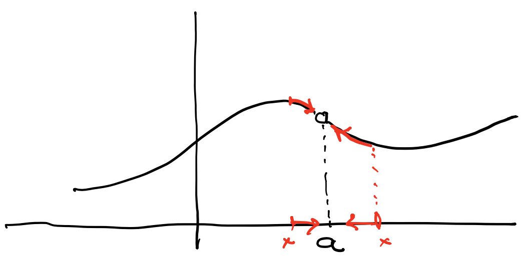


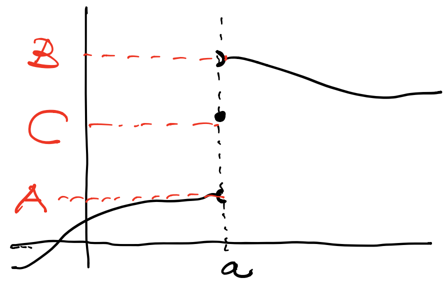
Ex: $\lim_{x\to 1^-} \frac{x^2-1}{x-1} = \lim_{x\to 1^-}(x+1) = 2$

$\lim_{x\to 1^+} \frac{x^2-1}{x-1} = \lim_{x\to 1^-}(x+1) = 2$



Thm $\lim_{x\to a} f(x)$ exists if and only if both $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist, and they have the same value.



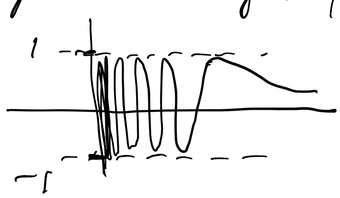


A: left limit $\lim_{x\to a^-} f(x)$

B: right limit

C: function value.

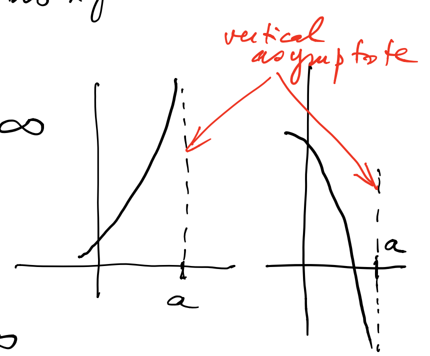
Ex A left or right limit might fail to exist:



$\sin\left(\frac{1}{x}\right)$, $x>0$

no right limit as $x\to 0^+$

Def $\lim_{x\to a^-} f(x) = +\infty$ or $-\infty$

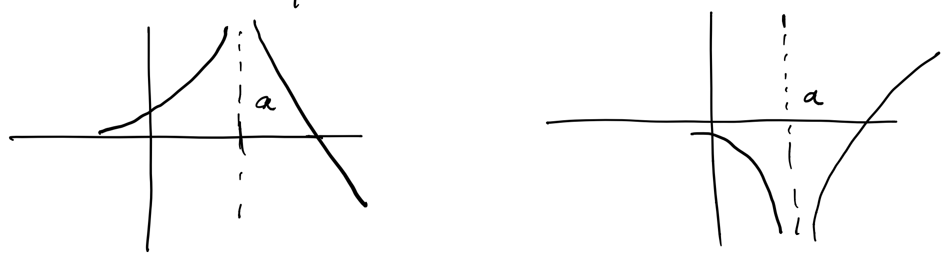


$\lim_{x\to a^+} f(x) = +\infty$ or $-\infty$

$\lim_{x\to a} f(x) = +\infty$ or $-\infty$

means that both $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$

are equal to $+\infty$ or $-\infty$.



Limit laws

Thm Assume that $\lim_{x \to a^{\pm}} f(x)$, $\lim_{x \to a^{\pm}} g(x)$ both exist and are finite.

Then: 1) $\lim_{x \to a^{\pm}} \left(\alpha f(x) \pm \beta g(x) \right) = \alpha \lim_{x \to a^{\pm}} f(x) \pm \beta \lim_{x \to a^{\pm}} g(x)$

for any numbers α, β.

2) $\lim_{x \to a^{\pm}} \left(f(x) \cdot g(x) \right) = \left(\lim_{x \to a^{\pm}} f(x) \right) \cdot \left(\lim_{x \to a^{\pm}} g(x) \right)$

3) assume that $\lim_{x \to a^{\pm}} g(x) \neq 0$, Then,

$$\lim_{x \to a^{\pm}} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a^{\pm}} f(x)}{\lim_{x \to a^{\pm}} g(x)}$$

Note: The statements are true for $\lim_{x \to a}$ everywhere, or $\lim_{x \to a^-}$, or $\lim_{x \to a^+}$ everywhere.

Ex $\lim_{x \to 0} (2x+3) \sin x \overset{2)}{=} \underbrace{\lim_{x \to 0} (2x+3)}_{3} \cdot \underbrace{\lim_{x \to 0} \sin x}_{0} = 0$

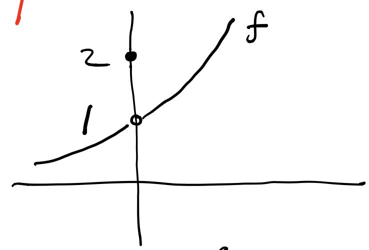
Ex $\displaystyle \lim_{x\to 0^+} \left(x\cdot \cos(2+x)\right)\cdot \frac{1}{|x|}$.

$$= \lim_{x\to 0^+} \cos(2+x)\cdot \frac{x}{|x|}$$

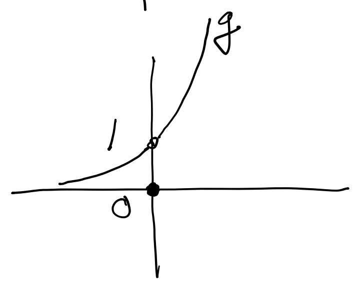
$$\overset{2)}{=} \underbrace{\left(\lim_{x\to 0^+} \cos(2+x)\right)}_{\cos 2}\cdot \underbrace{\left(\lim_{x\to 0^+} \frac{x}{|x|}\right)}_{1} = \cos 2 .$$

Ex

$$f(x) = \begin{cases} 2^x & \text{if } x\neq 0 \\ 2 & \text{if } x=0 \end{cases}$$



$$g(x) = \begin{cases} 3^x & \text{if } x\neq 0 \\ 0 & \text{if } x=0 . \end{cases}$$

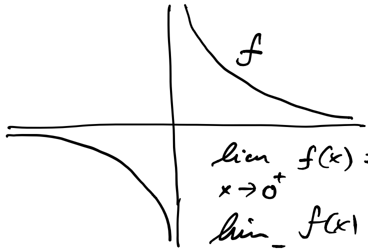


$$\lim_{x\to 0} \frac{f(x)}{g(x)} \overset{3)}{=} \frac{\lim_{x\to 0} f(x)}{\lim_{x\to 0} g(x)} = \frac{1}{1} = 1$$

we are allowed to use the limit rules because $\lim_{x\to 0} f(x) = 1$, and $\lim_{x\to 0} g(x) = 1 \neq 0$ (both finite)

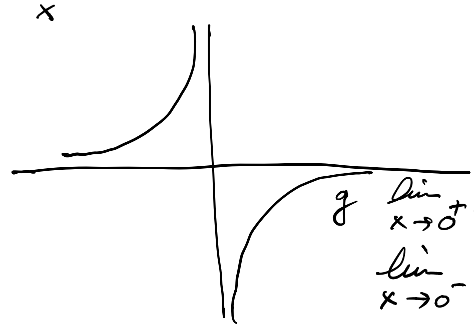
The fact that the fct value $g(0)=0$ does <u>not</u> matter.
Function value of $\frac{f(x)}{g(x)}$ at $x=0$ is <u>not</u> defined.

Ex $f(x) = \frac{1}{x}$, $g(x) = -\frac{2}{x}$



$\lim\limits_{x\to 0^+} f(x) = +\infty$

$\lim\limits_{x\to 0^-} f(x) = -\infty$.



$\lim\limits_{x\to 0^+} g(x) = -\infty$

$\lim\limits_{x\to 0^-} g(x) = +\infty$

no fct values, no limits at $x=0$.

$$\lim_{x\to 0^+}\left(f(x)+g(x)\right) = \lim_{x\to 0^+}\left(\frac{1}{x}-\frac{2}{x}\right) = \lim_{x\to 0^+}\left(-\frac{1}{x}\right) = -\infty$$

we did **not** use limit rules, they can't be used here.

Compare:

$$\lim_{x\to 0^+} f(x) + \lim_{x\to 0^+} g(x) = \underbrace{\lim_{x\to 0^+}\frac{1}{x}}_{+\infty} + \underbrace{\lim_{x\to 0^+}\left(-\frac{2}{x}\right)}_{-\infty}$$

$=$ **not** defined.

Ex $7+\infty = \infty$

Is $\infty - \infty = 0$ correct? **NO!!!**

If yes, then

$$7 = 7 + 0 = \underbrace{7+\infty}-\infty$$
$$= \infty - \infty = 0$$

Impossible $\Rightarrow$ $\infty - \infty$ **not** zero

it's **undefined**

Ex $3 \times \infty = \infty$

Is $\frac{\infty}{\infty} = 1$ correct? NO!!!

If yes, then

$$3 = 3 \times 1 = 3 \times \frac{\infty}{\infty}$$

$$= \frac{3 \times \infty}{\infty} = \frac{\infty}{\infty} = 1$$

Impossible $\Rightarrow$ $\frac{\infty}{\infty}$ is not 1

it's undefined

9/18/2018.

Comparison methods.

Thm If $f(x) \leq g(x)$ for all x near a (except possibly at a) and if $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ both exist (not necessarily finite)

Then:

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x).$$

This is also true if $\lim_{x \to a}$ is replaced by $\lim_{x \to a^-}$ or $\lim_{x \to a^+}$.

Ex: $\underbrace{\lim_{x \to 0^+} \frac{1+x^4}{x^3}}_{= \infty} \geq \lim_{x \to 0^+} \frac{1}{x^3} = \infty$ (because $1+x^4 \geq 1$)

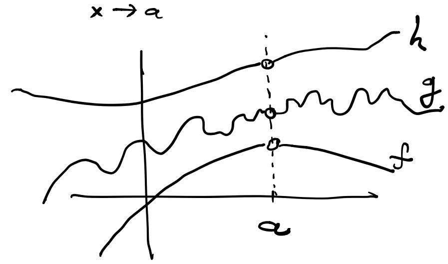
Thm (squeeze)

If $f(x) \le g(x) \le h(x)$ for all x near a (except possibly at a) and all have well-defined limits as $x \to a$ (not necessarily finite)

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x) \le \lim_{x\to a} h(x)$$

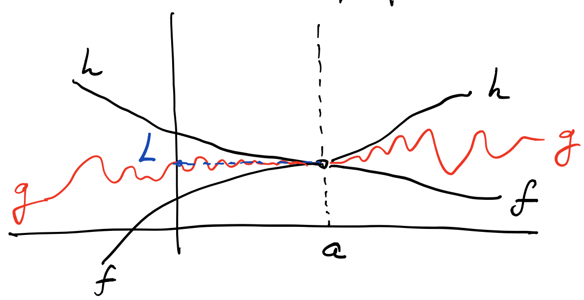
In particular, if

$$\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$$



then

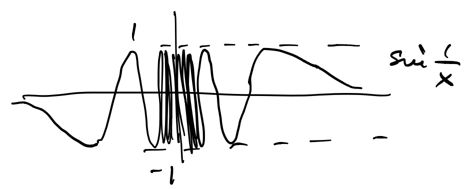
$$\lim_{x\to a} g(x) = L$$



This is also true with $\lim\limits_{x\to a}$ replaced by $\lim\limits_{x\to a^-}$ or $\lim\limits_{x\to a^+}$.

Ex $\quad f(x) = x^2 \sin\frac{1}{x}$.

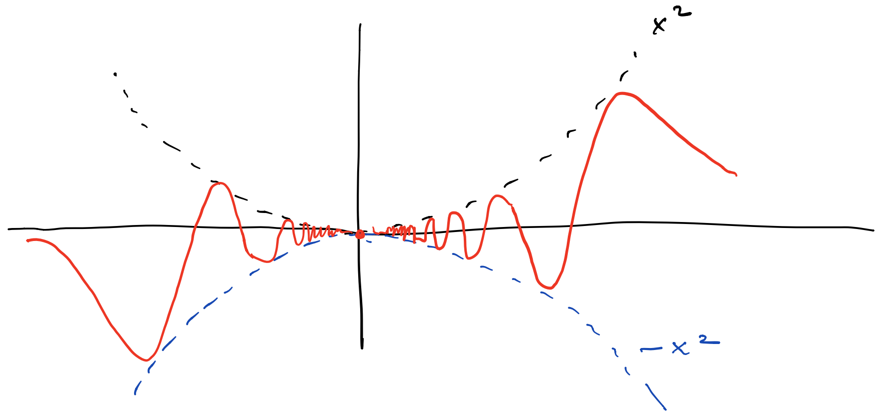
Find $\lim\limits_{x\to 0} x^2 \sin\frac{1}{x}$.



$$(-1)\cdot x^2 \le x^2 \cdot \underbrace{\sin\frac{1}{x}}_{-1 \le \sin\frac{1}{x} \le 1} \le 1\cdot x^2$$

$$\underbrace{\lim_{x\to 0}\left(-x^2\right)}_{=0} \le \lim_{x\to 0} x^2 \cdot \sin\frac{1}{x} \le \underbrace{\lim_{x\to 0} x^2}_{=0}$$

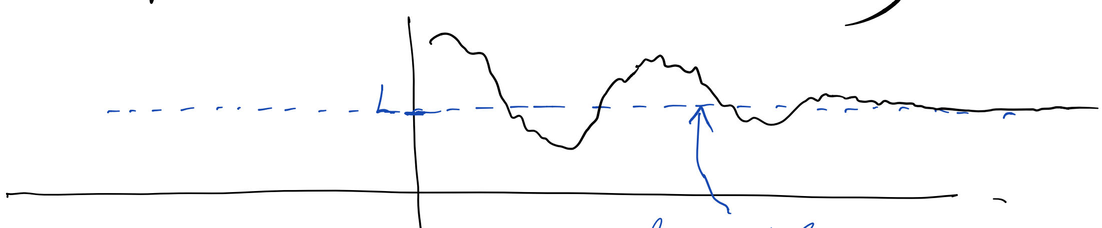
$$\Rightarrow \lim_{x\to 0} x^2 \cdot \sin\frac{1}{x} = 0.$$



Limits at infinity, horizontal asymptotes.

Def $\lim\limits_{x \to \pm\infty} f(x) = L$

"$f(x)$ approaches L with arbitrary precision as x grows to ∞ (respectively, to $-\infty$)"



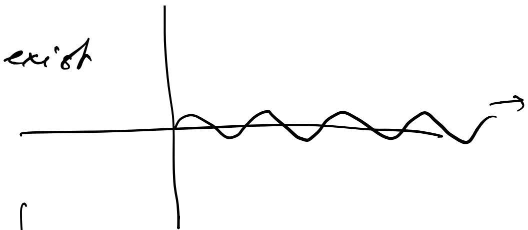
Ex $$f(x) = \frac{3x^3 + 2x^2 + 1}{4x^3 - x}$$

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{\cancel{x^3}\left(3 + \frac{2}{x} + \frac{1}{x^3}\right)}{\cancel{x^3}\left(4 - \frac{1}{x^2}\right)}$$

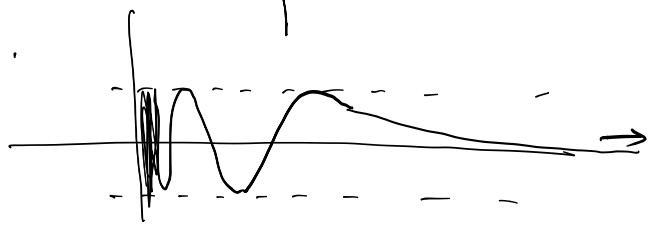
$$= \frac{\lim\limits_{x \to \infty}\left(3 + \frac{2}{x} + \frac{1}{x^3}\right)}{\lim\limits_{x \to \infty}\left(4 - \frac{1}{x^2}\right)} = \frac{3}{4}.$$

Limit laws apply when $x \to \infty$ or $x \to -\infty$, and $\lim\limits_{x \to \pm\infty} f(x)$ and $\lim\limits_{x \to \pm\infty} g(x)$ are finite

Ex $\lim_{x\to\infty} \sin x$ does not exist



$$\lim_{x\to\infty} \sin\frac{1}{x} = 0.$$

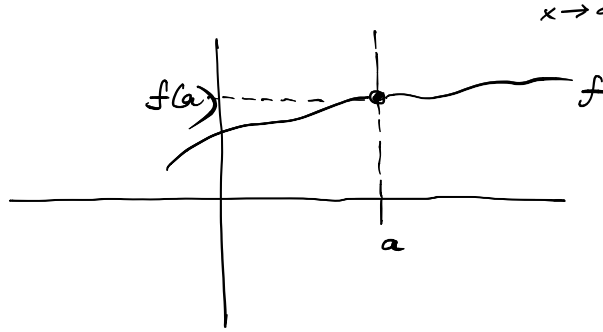


Ex
$$\lim_{x\to\infty} \frac{-x^5+x^4}{3+x^4} = \lim_{x\to\infty} \frac{x^{\cancel{5}}\left(-1+\frac{1}{x}\right)}{\cancel{x^4}\left(1+\frac{3}{x^4}\right)}$$
$$= \lim_{x\to\infty} x \cdot \underbrace{\frac{-1+\frac{1}{x}}{1+\frac{3}{x^4}}}_{\to -1} = -\infty$$

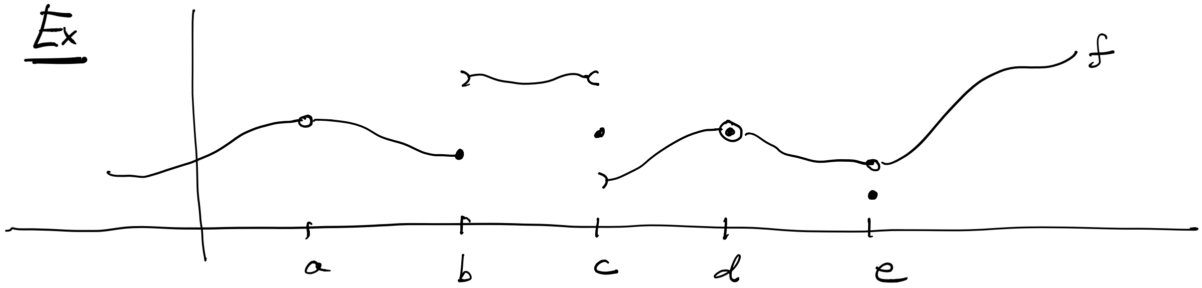
Continuity of functions.

Def: f is continuous at $x=a$ if:

1) f has a function value $f(a)$ at $x=a$.
2) $\lim_{x\to a} f(x)$ is well-defined.
3) Function value $=$ limit
$$f(a) = \lim_{x\to a} f(x).$$



Ex



	Left limit	Right limit	Limit	Fct value	Continuous.
$x=a$	Y	Y	Y	N	N
$x=b$	Y	Y	N	Y	N
$x=c$	Y	Y	N	Y	N
$x=d$	Y	Y	Y	Y	Y
$x=e$	Y	Y	Y	Y	N

Ex $\quad f(x) = x^2 \sin\frac{1}{x}$

$$\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$$

but: f has no function value at $x=0$.

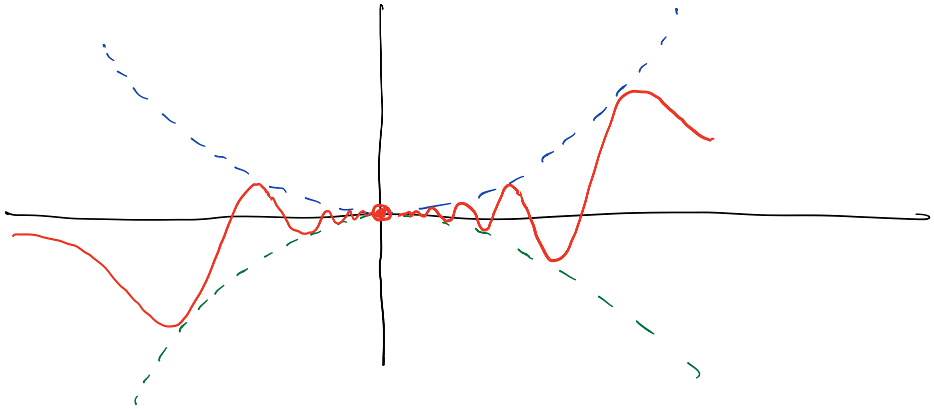
$\Rightarrow$ not continuous at $x=0$.

Ex:

$$g(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} x^2 \sin\frac{1}{x} = 0,$$

fct value: $g(0) = 0 = \lim\limits_{x\to 0} x^2 \sin\frac{1}{x} \Rightarrow g$ is continuous at $x=0$.



Left and right continuity

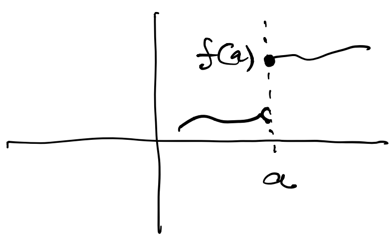
Def f is right continuous at $x=a$ if

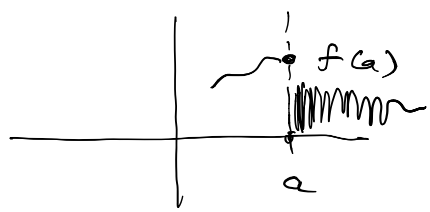
$$f(a) = \lim_{x\to a^+} f(x)$$

f is left continuous at $x=a$ if

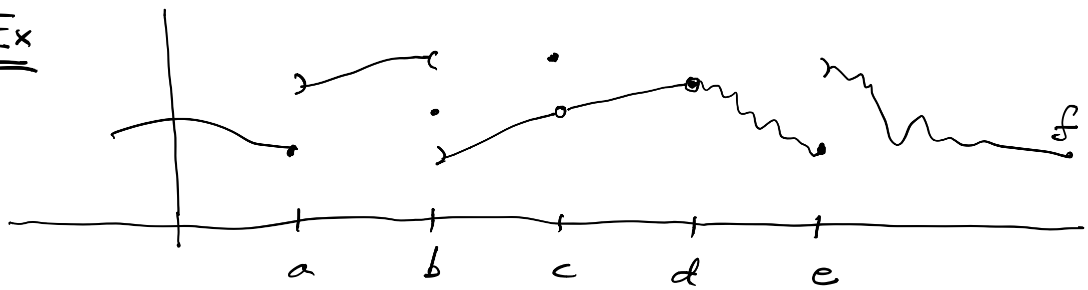
$$f(a) = \lim_{x\to a^-} f(x)$$

" function value = right / left limit ".





Ex

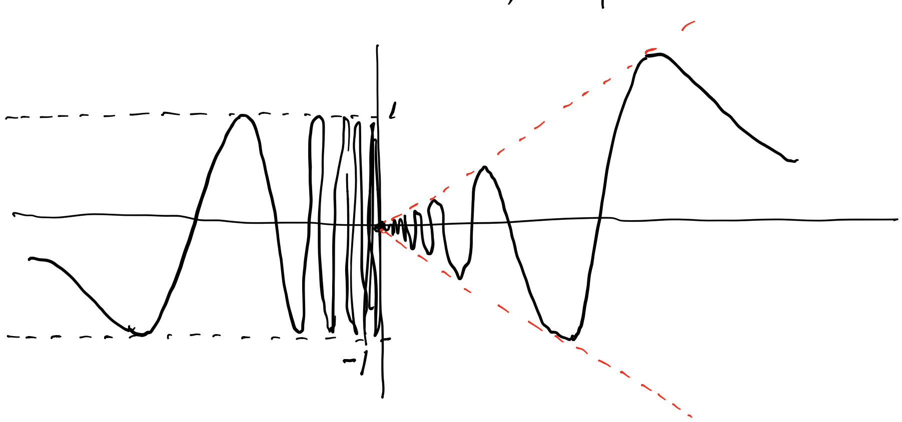


	left contin	right contin	continuous .
$x = a$	Y	N	N
$x = b$	N	N	N
$x = c$	N	N	N
$x = d$	Y	Y	Y
$x = e$	Y	N	N

Thm f is continuous at $x=a$ if and only if it is both left and right continuous at $x=a$.

Ex:

$$f(x) = \begin{cases} \sin\frac{1}{x}, & \text{if } x < 0 \\ x \sin\frac{1}{x}, & \text{if } x > 0 \\ 0, & \text{if } x = 0. \end{cases}$$



$\lim_{x\to 0^-} f(x)$ does not exist $\Rightarrow$ not left continuous at 0

$\lim_{x\to 0^+} f(x) = 0 = f(0) \Rightarrow$ right continuous at $x=0$.

Not continuous at $x=0$.

Thm (Continuity laws)

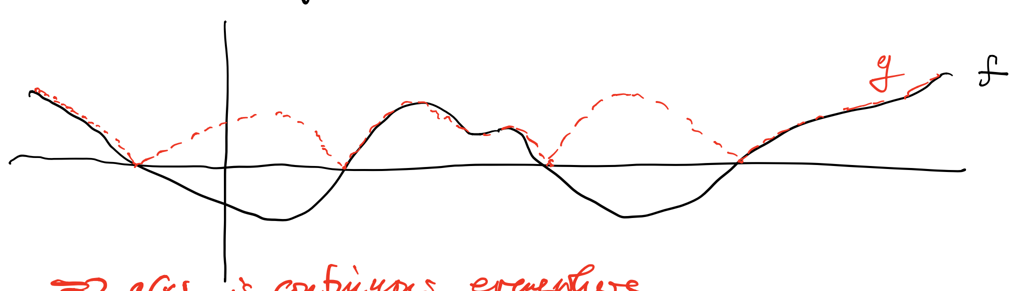
If f and g are continuous at $x=a$, then so are

$$\alpha f \pm \beta g \quad (\alpha, \beta \text{ some numbers})$$
$$f \cdot g$$
$$\frac{f}{g}, \text{ if } g(a) \neq 0.$$

Also true with left/right continuous instead of continuous.

Ex Assume $f(x)$ continuous everywhere.

Where is $g(x) = |f(x)|$ continuous?

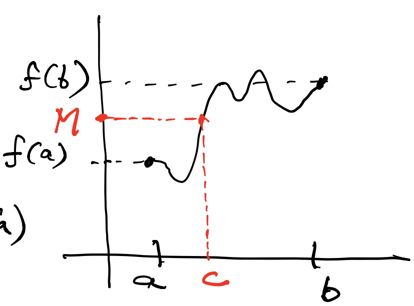


$\Rightarrow g(x)$ is continuous everywhere.

Thm (Intermediate value thm).

Assume f is continuous on $[a,b]$.

Then, for any number M between $f(a)$ and $f(b)$, there is a number



c in $[a,b]$ such that $f(c) = M$.

<u>Note</u>: This guarantees that there is <u>at least</u> one such c, but there could be more.

<u>Ex</u> Show that the equation

$$f(x) = 4x^3 - 6x^2 + 3x - 2 = 0$$

has a root between $x=1$ and $x=2$.

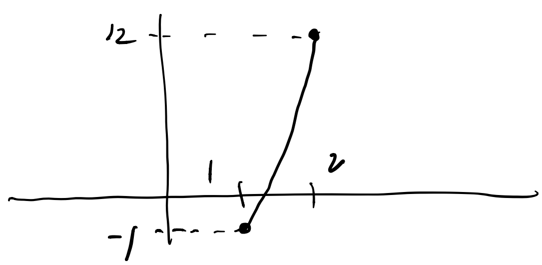
<u>Note</u>: f is continuous everywhere

$\Rightarrow$ IVT applicable on $[1,2]$.

$$f(1) = 4\cdot 1^3 - 6\cdot 1^2 + 3\cdot 1 - 2 = -1$$

$$f(2) = \underbrace{4\cdot 8}_{32} - \underbrace{6\cdot 4}_{24} + \underbrace{3\cdot 2}_{6} - 2 = 12$$

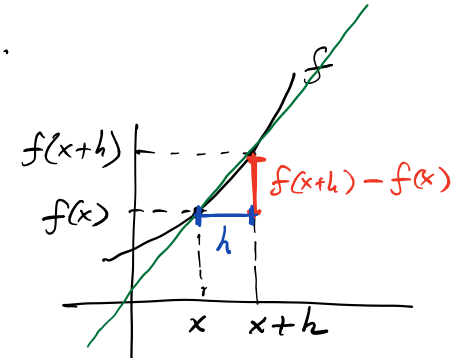
$\Rightarrow$ because 0 is between $f(1) = -1$ and $f(2) = 12$, there is a number c in $[1,2]$ such that $f(c) = 0$



Left and right derivatives.

<u>Recall</u>: Derivative of f at x

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$



Slope of tangent line at $f(x)$.

If $f'(x)$ exists, we say that f is differentiable at x.

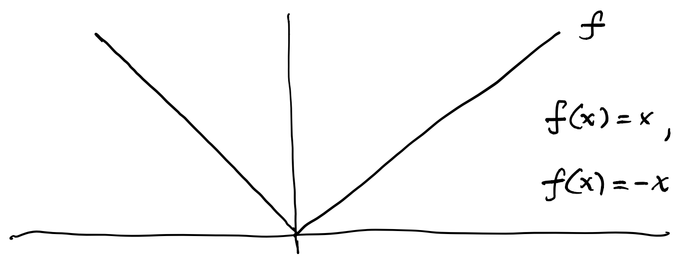
Def Left derivative of f at x:

$$f'_-(x) = \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h}$$

Right derivative of f at x:

$$f'_+(x) = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$$

Ex $f(x) = |x|$



$f(x) = x, \ x > 0$

$f(x) = -x, \ x < 0$

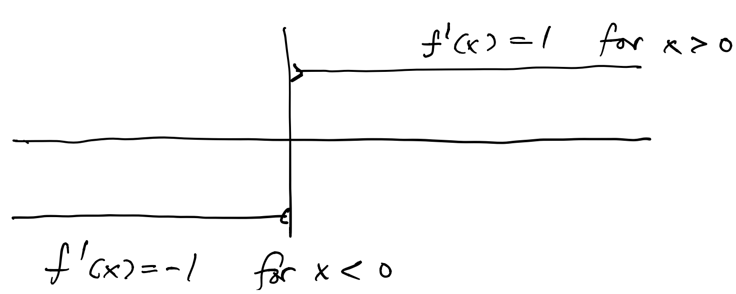
$x > 0 \Rightarrow f'(x) = 1$

$x < 0 \Rightarrow f'(x) = -1$

$x = 0 \Rightarrow f'$ does not exist.

$$f'_+(0) = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{\overbrace{h - 0}^{1}}{h} = 1$$

$$f'_-(0) = \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{-h - 0}{\underbrace{h}} = -1$$



There is no derivative at zero because

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

<u>cannot exist</u> : The left limit and the right limit are <u>not</u> the same.

<u>Thm</u> f has a derivative at x if and only if the left and right derivatives exist, and they are equal.

9/25/2018

<u>Thm</u>: If f is differentiable at $x = a$, then f is continuous at $x = a$.

<u>Ex</u>: Check this!

Need to check that $\lim_{x \to a} f(x) = f(a)$

$$\Updownarrow$$

$$\lim_{x \to a} f(x) - f(a) = 0$$

$x = a + h$ $\quad \Updownarrow$

$$\lim_{h \to 0} f(a+h) - f(a) = 0$$

$$\Updownarrow$$

$$\lim_{h \to 0} \underbrace{\frac{f(a+h) - f(a)}{h}} \cdot \underbrace{h} = 0$$

both terms have finite limits as $h \to 0$ $\Rightarrow$ use limit laws!

$$\Updownarrow$$

$$\underbrace{\left(\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\right)}_{f'(a)}\cdot\underbrace{\left(\lim_{h\to 0} h\right)}_{0} = 0.$$

Review for Midterm 1.

Ok: 1 page of handwritten notes. (no Quest examples).

Pencil, #2

Eraser.

No: Calculators, impersonators.

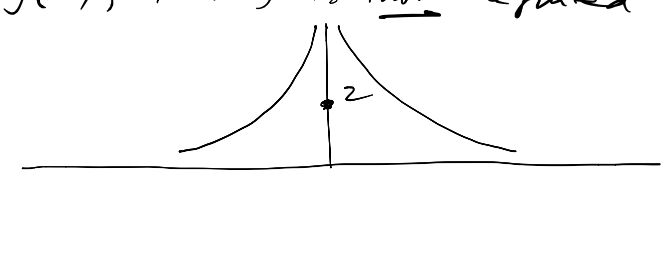
Ex True or false:

If $\lim_{x\to 6}(f(x)\cdot g(x))$ exists, then it equals $f(6)\cdot g(6)$

$\to$ F: Only true if both f, g are continuous at $x=6$.

Ex True or false:

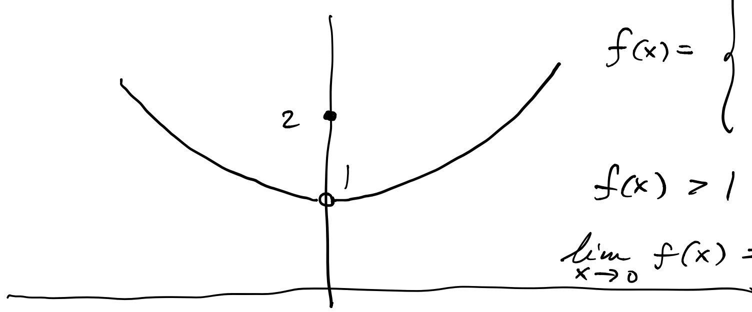
If the line $x=0$ is a vertical asymptote of $f(x)$, then f is not defined at $x=0$.



$$f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$$

Ex True or false:

If $f(x) > 1$ for all x, and $\lim_{x \to 0} f(x)$ exists, then $\lim_{x\to 0} f(x) > 1$.



$$f(x) = \begin{cases} 1 + x^2 & \text{if } x \neq 0 \\ 2 & \text{if } x = 0. \end{cases}$$

$f(x) > 1$

$\lim_{x\to 0} f(x) = 1.$

Ex Does the limit $\displaystyle\lim_{x\to 1} \frac{x^2-1}{|x-1|}$ exist? If yes, what is it?

Do left and right limit separately, and check if they are equal.

$$\lim_{x\to 1^+} \frac{x^2-1}{|x-1|} = \lim_{x\to 1^+} \frac{x^2-1}{x-1} = \lim_{x\to 1^+} \frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}}$$

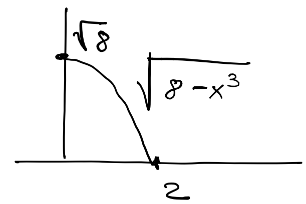
$$= \lim_{x\to 1^+} (x+1) = 2.$$

$$\lim_{x\to 1^-} \frac{x^2-1}{|x-1|} = \lim_{x\to 1^-} \frac{x^2-1}{-(x-1)} = -\lim_{x\to 1^-} \frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}}$$

$$= -\lim_{x\to 1^-} (x+1) = -2$$

$\Rightarrow$ The limit does <u>not</u> exist.

Ex $\lim_{x \to 2^-} \sqrt{8 - x^3}$



$$\lim_{\underbrace{x \to 2^-}} \sqrt{\underbrace{8 - x^3}_{>0}} = 0$$

$$x < 2 \Rightarrow x^3 < 8 \Rightarrow 8 - x^3 > 0$$

$$\lim_{x \to 2^+} \sqrt{\underbrace{8 - x^3}_{<0}} \quad \text{\underline{not defined}}.$$

Ex $\log_3\left(x - \sqrt{x^2 - 9}\right) + \log_3\left(x + \sqrt{x^2 - 9}\right)$

$$= \log_3\left(\left(x - \sqrt{x^2-9}\right) \cdot \left(x + \sqrt{x^2 - 9}\right)\right).$$

$$= \log_3\left(x^2 - (x^2 - 9)\right)$$

$$= \log_3 9 = 2$$

16/2/2018

Derivatives: Product rule, quotient rule, chain rule.

Thm Assume f, g differentiable at x. Then,

Product rule: $\left(f(x) \cdot g(x)\right)' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

Quotient rule: $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)}$

$g(x) \neq 0$.

Ex Check the product rule.

$$\boxed{\left(f(x)\cdot g(x)\right)'} = \lim_{h\to 0} \frac{f(x+h)\cdot g(x+h) - f(x)\cdot g(x)}{h}$$

$$= \lim_{h\to 0} \frac{\left(f(x+h) - f(x) + f(x)\right) g(x+h) - f(x)\cdot g(x)}{h}$$

$$= \lim_{h\to 0} \frac{\left(f(x+h) - f(x)\right) g(x+h) + f(x)\, g(x+h) - f(x)\cdot g(x)}{h}$$

$$= \lim_{h\to 0} \frac{\left(f(x+h) - f(x)\right) g(x+h) + f(x)\left(g(x+h) - g(x)\right)}{h}$$

$$= \lim_{h\to 0} \frac{\left(f(x+h) - f(x)\right)}{h}\cdot g(x+h) + \lim_{h\to 0} f(x)\cdot \frac{\left(g(x+h) - g(x)\right)}{h}$$

$$\boxed{= f'(x)\cdot g(x) + f(x)\cdot g'(x)}$$

$$\lim_{h\to 0} \frac{\left(f(x+h) - f(x)\right) g(x+h)}{h} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\cdot g(x+h)$$

$$\frac{A\cdot B}{C} = \frac{A}{C}\cdot B$$

Ex $\left(xe^x\right)' = \underbrace{1\cdot e^x}_{f'(x)\cdot g(x)} + \underbrace{x\cdot e^x}_{f(x)\cdot g'(x)} = (1+x)\cdot e^x$

$f(x) = x \qquad f'(x) = 1$

$g(x) = e^x \qquad g'(x) = e^x$

Ex: $\left(x^2\right)' = (x\cdot x)' = 1\cdot x + x\cdot 1 = 2\cdot x$

$f(x) = x \qquad f'(x) = 1$

$g(x) = x \qquad g'(x) = 1$

Ex: $(x^3)' = (x \cdot x^2)' = 1 \cdot x^2 + \underbrace{x \cdot (2x)}_{2x^2} = 3x^2$

$f(x) = x \quad f'(x) = 1$

$g(x) = x^2 \quad g'(x) = 2x$

Ex: $\left(\frac{1}{g(x)}\right)' = ?$

$$1 = g(x) \cdot \frac{1}{g(x)}$$

$$1' = 0 = \left(g(x) \cdot \frac{1}{g(x)}\right)'$$

$$= g'(x) \cdot \frac{1}{g(x)} + g(x) \cdot \left(\frac{1}{g(x)}\right)'$$

$$\Rightarrow \quad g(x) \cdot \left(\frac{1}{g(x)}\right)' = -\frac{g'(x)}{g(x)}$$

$$\Rightarrow \quad \left(\frac{1}{g(x)}\right)' = -\frac{g'(x)}{g^2(x)}$$

Ex Check quotient rule.

$$\boxed{\left(\frac{f(x)}{g(x)}\right)'} = \left(f(x) \cdot \frac{1}{g(x)}\right)'$$

$$= f'(x) \cdot \frac{1}{g(x)} + f(x) \cdot \left(\frac{1}{g(x)}\right)'$$

$$= f'(x) \cdot \frac{1}{g(x)} - f(x) \frac{g'(x)}{g^2(x)}$$

$$= \frac{f'(x)}{g(x)} - \frac{f(x)\,g'(x)}{g^2(x)}$$

$$\boxed{= \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{g^2(x)}}$$

Ex $$\left(\frac{e^x}{x}\right)' = \frac{e^x\cdot x - e^x\cdot 1}{x^2} = e^x\cdot\frac{x-1}{x^2}$$

Ex $$\left(\frac{x^2+1}{x-2}\right)' = \frac{\overbrace{2x}^{f'(x)}\cdot\overbrace{(x-2)}^{g(x)} - \overbrace{(x^2+1)}^{f(x)}\cdot\overbrace{1}^{g'(x)}}{\underbrace{(x-2)^2}_{g^2(x)}}$$

$f(x) = x^2+1$ $\quad f'(x) = 2x$

$g(x) = x-2$ $\quad g'(x) = 1$

$$= \frac{2x^2-4x-x^2-1}{(x-2)^2} = \frac{x^2-4x-1}{(x-2)^2}$$

Chain rule.

Thm Assume $g(x)$ is differentiable at x, and f is differentiable at $g(x)$. Then,

$$\big(f(g(x))\big)' = f'(g(x))\cdot g'(x)$$

Ex $f(x) = e^x$, $g(x) = x^2$ $\Big\}$ $f(g(x)) = e^{x^2}$ $\qquad f'(x) = e^x$, $g'(x) = 2x$.

$$\underset{\overset{\shortparallel}{f(g(x))}}{\left(e^{x^2}\right)'} = f'(g(x))\cdot g'(x)$$

$$= \underbrace{e^{x^2}}_{f'(g(x))}\cdot\underbrace{2x}_{g'(x)}$$

Ex $(\ln x)' = ?$

$$x = e^{\ln x}$$

$$x' = 1 = \left(e^{\ln x}\right)' = f'(g(x)) \cdot g'(x)$$

$f(x) = e^x \qquad f'(x) = e^x$

$g(x) = \ln x$

$$= \underbrace{e^{\ln x}}_{x} \cdot (\ln x)'$$

$$\Rightarrow 1 = x \cdot (\ln x)'$$

$$\Rightarrow (\ln x)' = \frac{1}{x}$$

Ex $(e^x)' = ?$

$$x = \ln(e^x)$$

$$x' = 1 = \left(\ln(e^x)\right)'$$

$f(x) = \ln x \qquad f'(x) = \frac{1}{x}$

$g(x) = e^x$

$$= f'(g(x)) \cdot g'(x)$$

$$= \frac{1}{e^x} \cdot (e^x)'$$

$$\Rightarrow \quad 1 = \frac{1}{e^x}\cdot(e^x)' \qquad \Rightarrow (e^x)' = e^x .$$

Ex $(x^x)' = ?$

$$x^x = (\underbrace{e^{\ln x}}_{x})^x = e^{x\cdot\ln x}$$

$$(x^x)' = (e^{x\cdot\ln x})'$$

$f(x) = e^x \qquad f'(x) = e^x$

$g(x) = x\cdot\ln x \qquad g'(x) = 1\cdot\ln x + x\cdot\frac{1}{x} = \ln x + 1$

$$= f'(g(x))\cdot g'(x) .$$

$$= \underbrace{e^{x\cdot\ln x}}_{x^x}\cdot(1+\ln x) .$$

$$= x^x\cdot(1+\ln x)$$

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Ex $(a^x)' = ?$ $\quad a > 0$

$$\boxed{(a^x)'} = \left((\underbrace{e^{\ln a}}_{a})^x\right)' = \left(e^{\overbrace{x\cdot\ln a}^{f(g(x))}}\right)'$$

$$(\underbrace{e^{\ln a}}_{a})^x = a^x$$

$f(x) = e^x \qquad f'(x) = e^x$

$g(x) = x\cdot\ln a \qquad g'(x) = \ln a$

chain rule

$$= \underbrace{e^{x\cdot\ln a}}_{f'(g(x))}\cdot\underbrace{\ln a}_{g'(x)}$$

$f'(g(x))$

$$\boxed{(a^x)' = a^x \cdot \ln a}$$

<u>Ex</u> $\left(a^{x^3}\right)' = ?$

$f(x) = a^x \qquad f'(x) = a^x \ln a$

$g(x) = x^3 \qquad g'(x) = 3x^2.$

$$= \underbrace{a^{x^3} \cdot \ln a}_{f'(g(x))} \cdot (3x^2).$$

<u>Ex</u> r real number.

$$(x^r)' = ?$$

$$\left(e^{\ln x}\right)^r = e^{r \ln x}.$$

$$\boxed{(x^r)'} = \left(e^{r \ln x}\right)'$$

$f(x) = e^x \qquad f'(x) = e^x$

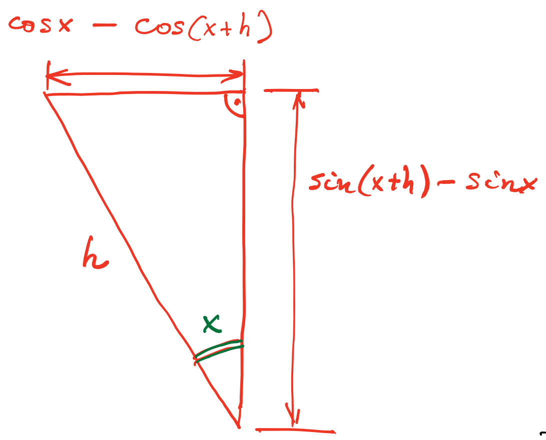
$g(x) = r \ln x \qquad g'(x) = r \cdot \frac{1}{x}$

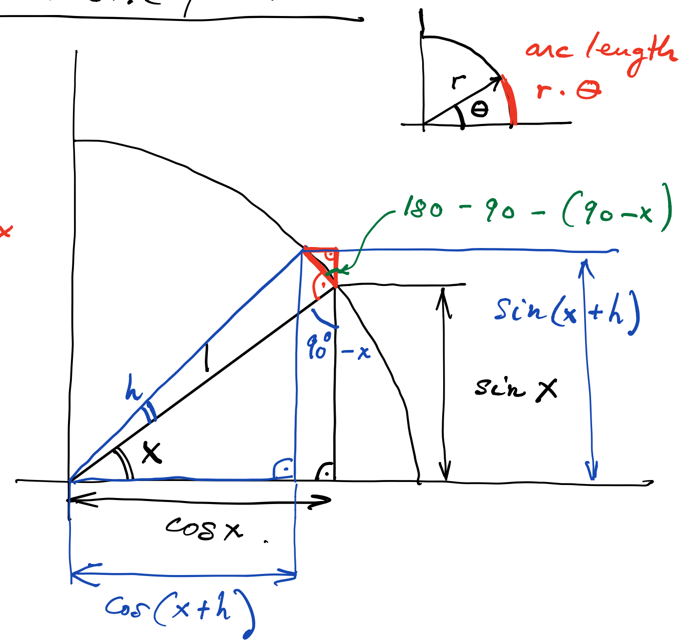
$$= \underbrace{e^{r \ln x}}_{f'(g(x))} \cdot \underbrace{r \cdot \frac{1}{x}}_{g'(x)}.$$

$$= x^r \cdot r \cdot \frac{1}{x}$$

$$\boxed{= r \cdot x^{r-1}}$$

Derivatives of trigonometric functions.





$$\cos x = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= (\sin x)'$$

$$\sin x = \lim_{h\to 0} \frac{\cos x - \cos(x+h)}{h} = -\lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} = -(\cos x)'$$

$$\Rightarrow (\cos x)' = -\sin x .$$

<u>Ex</u> Pythagoras $\cos^2 x + \sin^2 x = 1$

<u>Ex</u> $\tan x = \dfrac{\sin x}{\cos x}.$

$$(\tan x)' = \frac{(\sin x)' \cdot \cos x - \sin x \cdot (\cos x)'}{\cos^2 x}.$$

$$= \frac{\cos^2 x - (-\sin^2 x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}.$$

Ex $F(x) = e^{rx}$ $\qquad f(x) = e^x \quad f'(x) = e^x$

$\qquad g(x) = r\cdot x \quad g'(x) = r.$

$$F'(x) = \underbrace{e^{rx}}_{f'(g(x))} \cdot \underbrace{r}_{g'(x)} = r\cdot F(x)$$

$$F(0) = 1$$

Thm If $f(x)$ satisfies $f(0)=1$ and $f'(x) = r\,f(x)$ then $f(x) = e^{rx}$

Ex $i^2 = -1$, i imaginary unit.

Look at

$$f(x) = \cos x + i\sin x.$$

$$f'(x) = -\sin x + i\cos x$$

Compare this $i f(x) = i(\cos x + i \sin x) = i\cos x - \sin x$

$$\Rightarrow \quad f'(x) = i f(x)$$

also, $f(0) = \underbrace{\cos 0}_{1} + i\underbrace{\sin 0}_{0} = 1.$

$$\Rightarrow \quad f(x) = \boxed{e^{ix} = \cos x + i \sin x}$$ Euler's formula.

Ex $e^{i\alpha} \cdot e^{i\beta} = e^{i(\alpha+\beta)}$

$\qquad$ " $\qquad$ " $\qquad$ "

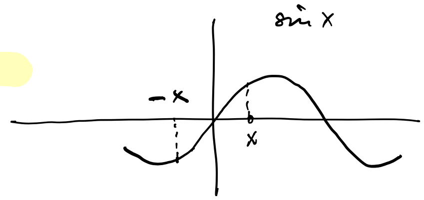
$(\cos\alpha + i\sin\alpha)\cdot(\cos\beta + i\sin\beta) = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$

$= \cos\alpha\cdot\cos\beta + i\sin\alpha\ i\sin\beta$

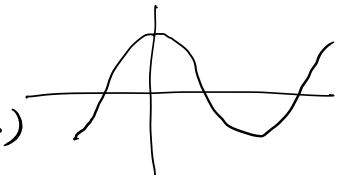
$\quad + i\sin\alpha\cos\beta + i\sin\beta\cos\alpha$

$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta .$$

$$+ i\left(\sin\alpha\cos\beta + \sin\beta\cos\alpha\right)$$



$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha .$$



Ex

$$\cos(\alpha-\beta) = \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta)$$
$$= \cos\alpha\cos\beta - \sin\alpha(-\sin\beta)$$
$$= \cos\alpha\cos\beta + \sin\alpha\sin\beta .$$

Ex

$\alpha = \beta$

$$\cos^2\alpha + \sin^2\alpha = 1 \Rightarrow \sin^2\alpha = 1 - \cos^2\alpha$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 \Rightarrow \cos^2\alpha = \frac{1+\cos 2\alpha}{2}$$

$$\sin 2\alpha = 2\cos\alpha\sin\alpha$$