10/9/2018

Why is the chain rule correct? Recall $(f(g(x)))' = f'(g(x))\,g'(x)$

Check:

$$(f(g(x)))' = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

$$= \lim_{h \to 0} \frac{f\left(g(x) + \frac{g(x+h) - g(x)}{h}\, h\right) - f(g(x))}{h}$$

$$= \lim_{h \to 0} \frac{f(g(x) + \overbrace{g'(x) \cdot h}^{H}) - f(g(x))}{\underbrace{g'(x) \cdot h}_{H}} \cdot g'(x)$$

$$= \lim_{H \to 0} \frac{f(g(x) + H) - f(g(x))}{H} \cdot g'(x)$$

$$= f'(g(x)) \cdot g'(x)$$

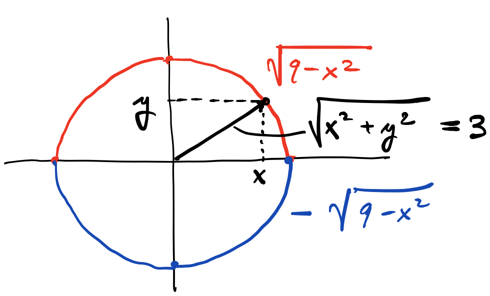
Implicit differentiation.

Sometimes, "solving y for x" is difficult.

Ex $\quad x^2 + y^2 = 9 \qquad$ circle of radius 3.

$$\Rightarrow y^2 = 9 - x^2$$

$$\Rightarrow y = \pm\sqrt{9 - x^2}$$



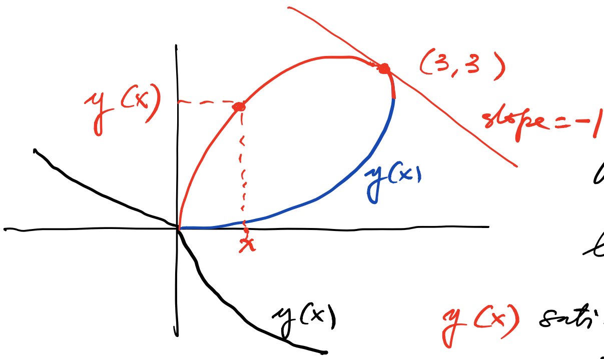
The circle is the combination of the graphs of 2 functions.

"Piecewise representable as a graph".

$\underline{\underline{Ex}}$ $\quad \underbrace{x^3 + y^3 = 6\,x \cdot y}_{\text{solutions describe a curve.}}$

For instance, $(3,3)$ lies on this curve:

$$\underbrace{3^3 + 3^3}_{2 \cdot 3^3} \overset{?}{=} \underbrace{6 \cdot 3 \cdot 3}_{2 \cdot 3 \cdot 3 = 2 \cdot 3^3}.$$



What is the slope of the tangent line at $(3,3)$? $\Rightarrow y'(3) = ?$

$y(x)$ satisfies

$$x^3 + y^3(x) = 6 \cdot x \cdot y(x)$$

($y(x)$, $y(x)$, $y(x)$ all satisfy the same equation)

Differentiate in x on both sides of equality sign.

$$\Rightarrow \quad 3x^2 + 3y^2(x) \cdot y'(x) = 6y(x) + 6x \cdot y'(x).$$

Solve for $y'(x)$:

$$3y^2(x) \cdot y'(x) - 6x \cdot y'(x) = 6y(x) - 3x^2$$

$$(3y^2(x) - 6x) \cdot y'(x) = 6y(x) - 3x^2.$$

$$\Longrightarrow \quad y'(x) = \frac{6y(x) - 3x^2}{3y^2(x) - 6x} = \frac{2y(x) - x^2}{y^2 - 2x}.$$

at $(3,3)$ $\Rightarrow \underline{y'(3)} = \dfrac{2 \cdot 3 - 3^2}{3^2 - 2 \cdot 3} = \underline{\underline{-1}}$ $\qquad$ slope $= -1$.

$x = 3$, $y(3) = 3$

$\underline{\underline{Ex}}$: Find $y'(x)$ if

$$\sin(x^2 + y^2) = y \cdot \cos x$$

Differentiate in x: (here, y is the same as $y(x)$)

$f(g(x))$ where $f(x) = \sin x$ $\Rightarrow$ $f'(x) = \cos x$

$g(x) = x^2 + y^2(x)$ $\qquad g'(x) = 2x + 2y \cdot y'$

$$\underbrace{\cos(x^2+y^2)}_{f(g(x))} \cdot \underbrace{(2x + 2y\,y')}_{g'(x)} = y'\cos x - y\sin x$$

$$\cos(x^2+y^2)\cdot 2x + \cos(x^2+y^2)\cdot 2y\cdot y' = y'\cos x - y\sin x$$

Solve for y':

$$\cos(x^2+y^2)\cdot 2y\cdot y' - y'\cos x = -\cos(x^2+y^2)\cdot 2x - y\cdot\sin x$$

$$\left(\cos(x^2+y^2)\cdot 2y - \cos x\right) y' = -\cos(x^2+y^2)\cdot 2x - y\cdot\sin x.$$

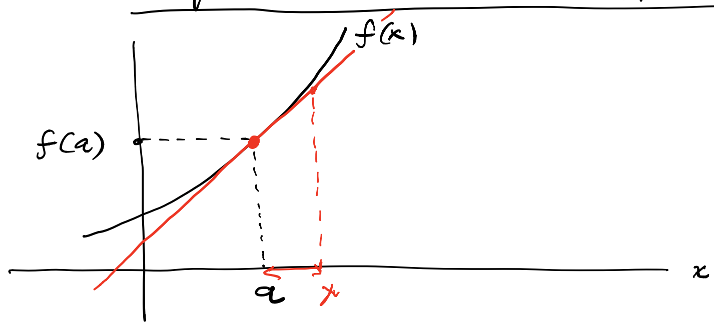
$$y' = \frac{-\cos(x^2+y^2)\cdot 2x - y\cdot\sin x}{\cos(x^2+y^2)\cdot 2y - \cos x}$$

Observe: $(0,0)$ lies on the curve:

$$\underbrace{\sin(\underbrace{0^2+0^2}_{0})}_{0} = \underbrace{\overset{\substack{y\\ \|}}{0}\cdot\underbrace{\cos 0}_{1}}_{0}$$

at $(0,0)$: $$y'(0) = \frac{-\cos(0+0)\cdot 2\cdot 0 - 0\cdot\sin 0}{\underbrace{\cos(0+0)\cdot 2\cdot 0}_{0} - \underbrace{\cos 0}_{1}} = \frac{0}{-1} = 0$$

Tangent line & linear approximation.



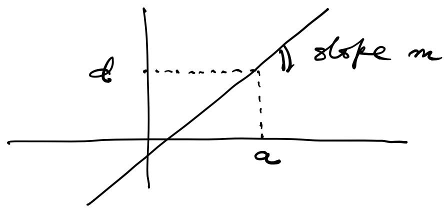
Formula for the tangent line: $y = f(a) + f'(a)\cdot(x-a)$

Check: linear fct in x

$y(a) = f(a) + f'(a)\cdot(a-a) = f(a)$

$y'(x) = f'(a)$ is constant $\Rightarrow$ graph is indeed a line.

Recall

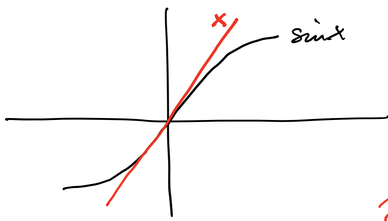


$y = d + m(x-a)$

Def: The function $y(x) = f(a) + f'(a)(x-a)$ is called the <u>linear approximation</u> (= <u>linearization</u>) of f at a.

For x close to a, the linearization is a good approximation of $f(x)$.

Ex Find linearization of $\sin x$ at 0: $f(x) = \sin x$, $a = 0$.



$\sin 0 = 0$

$\sin' 0 = \cos 0 = 1.$

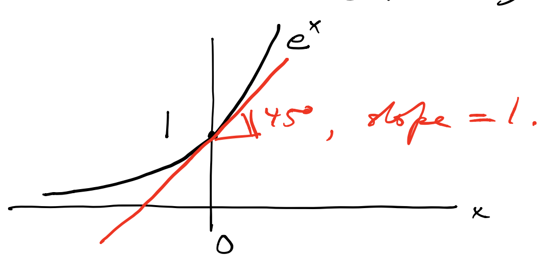
$y = \underbrace{\sin 0} + \sin' 0 \cdot (x - 0).$

$= 0 + 1 \cdot x = x$

Ex Find linearization of e^x at 0: $e^0 = 1$

$(e^x)'(x=0) = e^0 = 1$

$y = \underset{e^0}{\underset{\shortparallel}{1}} + \underset{(e^x)'(x=0) = e^0}{\underset{\shortparallel}{1}} \cdot (x - 0) = 1 + x$



Ex Calculate $\sqrt{4.02}$ without a calculator.

$$f(x) = \sqrt{x}. \quad (= x^{1/2})$$

$$x = 4.02$$

$$a = 4$$

pick a $\begin{cases} \text{close to } x \\ f(a) \text{ easy to calculate} \\ f'(a) \text{ "} \end{cases}$

$$f'(x) = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$\Rightarrow f(4) = \sqrt{4} = 2$$

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

$\Rightarrow$ Linearization
$$y = \underset{f(a)}{\underset{\shortparallel}{2}} + \underset{f'(a)}{\underset{\shortparallel}{\frac{1}{4}}}(\underbrace{4.02}_{x} - \underbrace{4}_{a})$$

$$= 2 + \underbrace{\frac{1}{4} \cdot 0.02}_{0.005} = 2.005$$

Ex Calculate $\ln 3$ without a calculator.

$$(e \approx 2.72)$$

$$f(x) = \ln x$$

$$f'(x) = \frac{1}{x}$$

$$a = e \approx 2.72$$

$$f(a) = \ln e = 1$$

$$f'(a) = \frac{1}{e} \approx \frac{1}{2.72}$$

Linearization:
$$y = \underset{f(a)}{\underset{\shortparallel}{1}} + \underset{\frac{1}{e} = f'(a)}{\underset{\approx}{\frac{1}{2.72}}}(\underset{x}{\underset{\shortparallel}{3}} - \underset{e = a}{\underset{\approx}{2.72}})$$

$$= 1 + \frac{1}{2.72} \cdot 0.28 = 1 + \underbrace{\frac{28}{272}} \approx 1.1$$

≈ 0.1

<u>Ex</u> Find $3^{3.01}$ in linear approximation.

$$f(x) = 3^x$$

$$f'(x) = 3^x \cdot \ln 3$$

$$x = 3.01$$

$$a = 3$$

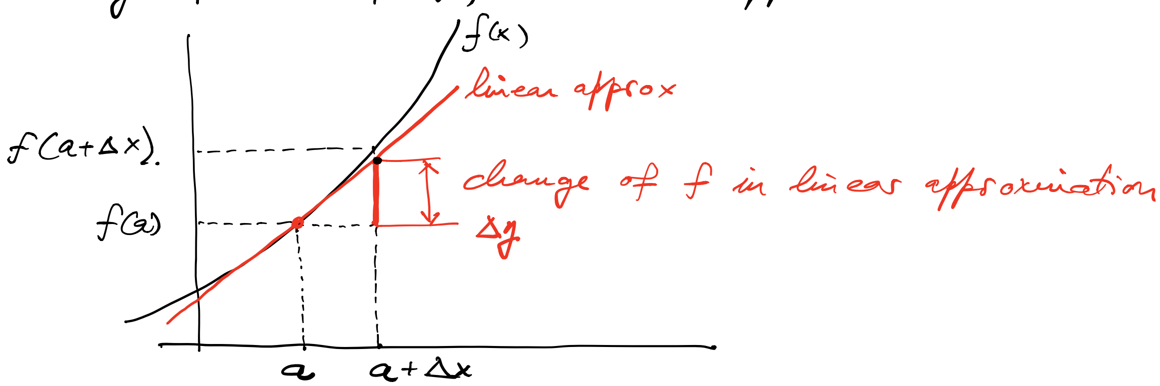
Linear approximation:

$$y = \underbrace{3^3}_{= f(a)} + \underbrace{3^3 \cdot \ln 3}_{= f'(a)} \cdot (\underbrace{3.01}_{= x} - \underbrace{3}_{= a})$$

$$\approx 27 + \underbrace{\underbrace{27 \cdot 1.1}_{29.7} \cdot 0.01}_{0.297} = 27.297$$

<u>Differentials</u>.

= Change of value of f, in linear approximation.



$$\boxed{\Delta y} = \underbrace{\cancel{f(a)} + f'(a)(\underbrace{\cancel{a}+\Delta x}_{x} - \cancel{a})}_{f(a+\Delta x) \text{ in linear approx.}} - \cancel{f(a)}$$

$$\boxed{= f'(a) \cdot \Delta x}$$

Ex Find $\ln(1.02)$ in linear approx, and determine the differential for $a=1$, $x=1.02$

$$f(x) = \ln x$$
$$f'(x) = \frac{1}{x}.$$

Linearization

$$y = \underbrace{\ln 1}_{f(a)} + \underbrace{\frac{1}{1}}_{f'(a)} \cdot (\underset{x}{1.02} - \underset{a}{1})$$
$$= 0 + 1 \cdot 0.02 = 0.02.$$

coincidentally equal because $f(a) = \ln 1 = 0$

Differential

$$\Delta y = f'(a) \cdot \Delta x = \frac{1}{1} \cdot 0.02 = 0.02$$

Note: Notations for derivatives: $f'(x)$, $\dfrac{df}{dx}$

Ex $\ln 8$ in linear approx, and differential.

$$f(x) = \ln x$$
$$f'(x) = \frac{1}{x}$$
$$x = 8, \quad a = e^2 \approx 2.72^2$$

$$f(a) = \ln e^2 = 2$$
$$f'(a) = \frac{1}{e^2} \approx \frac{1}{(2.72)^2}$$

$$e^2 \approx (2.72)^2 \approx \overbrace{3^2}^{g(a)} + \overbrace{2 \cdot 3}^{g'(a)} \cdot (2.72 - 3) = 9 + \underbrace{6 \cdot (-0.28)}_{-1.68}.$$

$g(x) = x^2$ $\quad x = 2.72$

$g'(x) = 2x$ $\quad a = 3$

$$= 7.32$$

Linearization

$$y = \underset{\underset{f(a)}{\shortparallel}}{2} + \underset{\underset{\frac{1}{e^2}}{\approx}}{\frac{1}{7.32}} \cdot (\underset{\underset{x}{\shortparallel}}{8} - \underset{\underset{a=e^2}{\approx}}{7.32}).$$

$$= 2 + \frac{1}{7.32} \cdot 0.68 = 2 + \frac{68}{\underbrace{732}} \approx 2.095$$

$$\approx \frac{70}{735} \approx \frac{1}{10.5}$$

$$\frac{1}{10.5} \approx \underbrace{\frac{1}{10}}_{k(a)} - \underbrace{\frac{1}{10^2}}_{k'(a)}(\underset{\underset{x}{\shortparallel}}{10.5} - \underset{\underset{a}{\shortparallel}}{10})$$

$k(x) = \frac{1}{x}$ $\quad a = 10$ $\qquad \shortparallel \; 0.1 - \underbrace{0.01 \cdot 0.5}_{0.005}$

$k'(x) = -\frac{1}{x^2}$ $\quad x = 10.5$ $\qquad = 0.095$

Related rates

Q: How does one quantity change when another quantity that it depends on changes?

Ex: Ball-shaped balloon of volume $V = \frac{4\pi}{3} r^3$, r = radius when air is pumped so that

$$\frac{dV}{dt} = V'(t) = 100 \text{ cm}^3/\text{sec}$$

then, what is the rate of change per sec of the radius, once $r = 25$ cm?

Want to know $\frac{dr}{dt} = ?$

chain rule

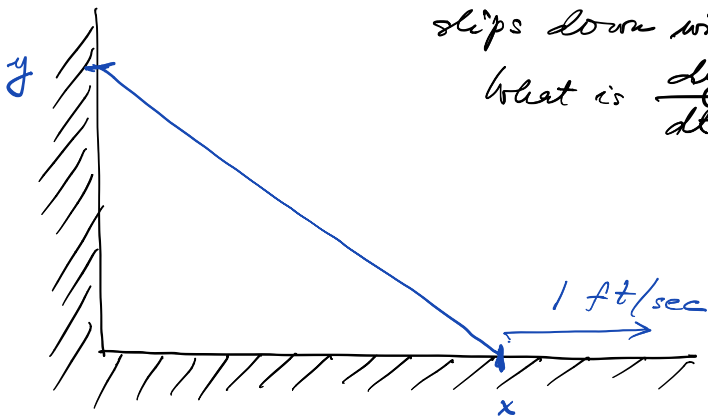
$\Rightarrow$ differentiate volume formula $\Rightarrow \frac{dV}{dt} = \frac{4\pi}{3} 3r^2 \cdot \frac{dr}{dt}$

$$\Rightarrow \; 100 = \frac{4\pi}{\cancel{3}} \cdot \cancel{3} \cdot 25^2 \cdot \frac{dr}{dt}.$$

$$\Rightarrow \; \frac{dr}{dt} = \frac{100}{25^2} \; \frac{1}{4\pi} = \frac{1}{25\pi} \; \text{cm/sec}.$$

10/16/2018

Ex A ladder of length 10 ft standing against a wall slips down with $\frac{dx}{dt} = 1$ ft/sec. What is $\frac{dy}{dt}$ when $x = 6$ ft?



Pythagoras:

$$x^2 + y^2 = 10^2 = 100.$$

Take derivative on both sides in t:

$$\cancel{2}x \cdot \frac{dx}{dt} + \cancel{2}y \cdot \frac{dy}{dt} = 0$$

$$\Rightarrow \; \frac{dy}{dt} = -\frac{x}{y}\,\frac{dx}{dt}.$$

when $x = 6$ ft

$$\Rightarrow y = \sqrt{\underbrace{100-36}_{64}} = 8 \text{ ft}.$$

$$= -\frac{6}{8} \cdot 1 \text{ ft/sec}$$

$$= -\frac{3}{4} \text{ ft/sec}.$$

Hyperbolic functions.

Def:

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

Similarity to trig. fcts:

Euler's formula, $i^2 = -1$

$$e^{ix} = \cos x + i \sin x$$

$$e^{-ix} = \cos(-x) + i \sin(-x)$$

$= \cos x - i\sin x$

$\cosh^2 x - \sinh^2 x = 1$

$\cosh(-x) = \cosh x$

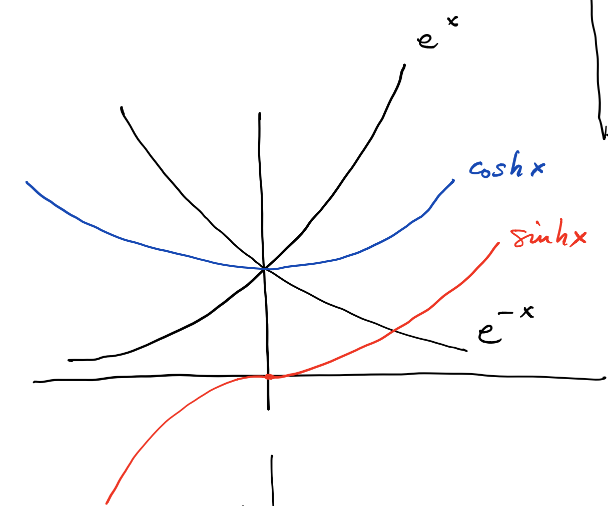
$\sinh(-x) = -\sinh x$

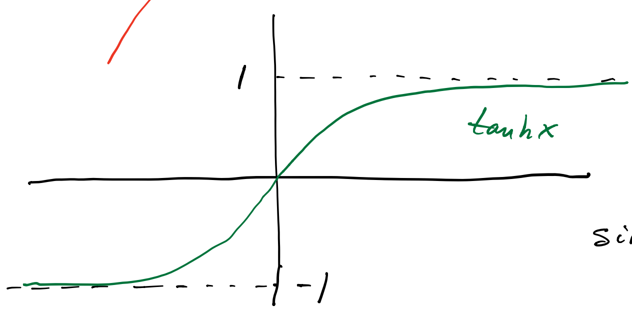
$\Rightarrow e^{ix} + e^{-ix} = 2\cos x$

$\Rightarrow \cos x = \dfrac{e^{ix}+e^{-ix}}{2} = \cosh(ix)$

$e^{ix} - e^{-ix} = 2i\sin x$

$\Rightarrow \sin x = \dfrac{e^{ix}-e^{-ix}}{2i} = \dfrac{\sinh(ix)}{i}$





$\sinh' x = \left(\dfrac{e^x - e^{-x}}{2}\right)'$

$= \dfrac{e^x + e^{-x}}{2} = \cosh x$

$\cosh' x = \left(\dfrac{e^x + e^{-x}}{2}\right)'$

$= \dfrac{e^x - e^{-x}}{2} = \sinh x$

$\sinh(-x) = \dfrac{e^{-x} - e^{-(-x)}}{2} = \dfrac{e^{-x} - e^{x}}{2}$

$= -\dfrac{e^x - e^{-x}}{2} = -\sinh x$

<u>Thm</u>

$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$

$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right)$

$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right).$

$\sinh^{-1}(\sinh x) = x$

$\sinh(\sinh^{-1} x) = x.$

<u>Check:</u> $\sinh y = \dfrac{e^y - e^{-y}}{2} = x$ solve y for x to get $y = \sinh^{-1} x$.

$\Rightarrow e^y - e^{-y} = 2x$

$\Rightarrow e^y - 2x - e^{-y} = 0.$

multiply by e^y

$\Rightarrow (e^y)^2 - 2x \cdot e^y - 1 = 0$

Quadratic equation for $Y := e^y$.

$$Y^2 - 2x \cdot Y - 1 = 0.$$

$$X^2 + bX + c = 0.$$

$$\Rightarrow X = \frac{-b}{2} \pm \frac{1}{2}\sqrt{b^2 - 4c}$$

$$0 < e^y = Y = -\frac{-2x}{2} \pm \frac{1}{2}\sqrt{(2x)^2 - 4(-1)}$$

$$= x \pm \frac{1}{2}\sqrt{4x^2+4}$$

$$= x \pm \sqrt{x^2+1}$$

$$\Rightarrow e^y = x + \underbrace{\sqrt{x^2+1}}_{>|x|} > 0$$

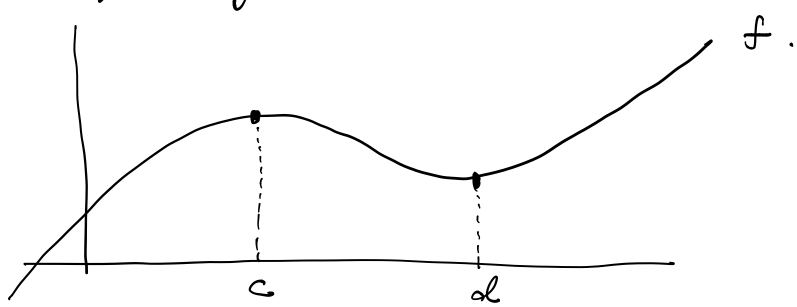
$$\Rightarrow \boxed{y = \ln\left(x + \sqrt{x^2+1}\right)}$$

$$\boxed{\sinh^{-1} x}$$

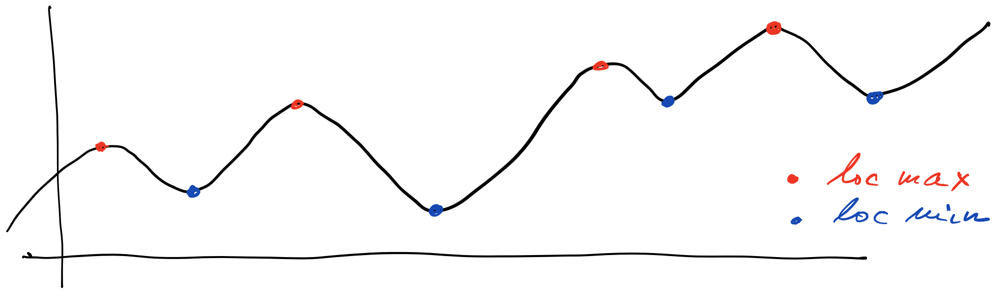
Applications of differentiation.

Maximum and minimum values.

Def f has a local max at c if $f(c) \geq f(x)$ for all x sufficiently close to c.
f has a local min at d if $f(d) \leq f(x)$ for all x sufficiently close to d.



Ex

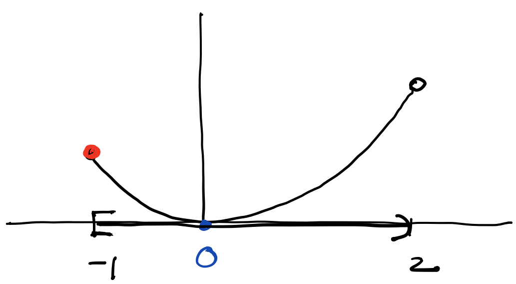


Def A point c in the domain D of f is a <u>global max</u> if $f(c) \geq f(x)$ for all x in D.

A point d in D is a <u>global min</u> if $f(d) \leq f(x)$ for all x in D.

(global min/max = absolute min/max).

Ex $f(x) = x^2$, for $-1 \leq x < 2$

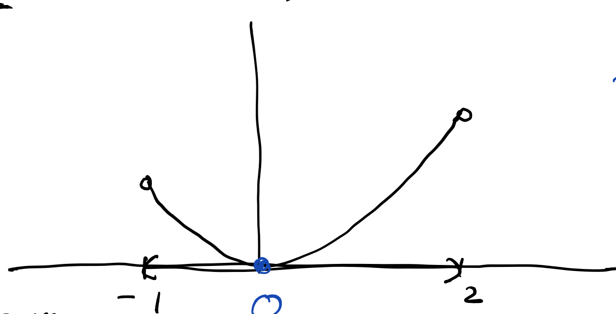


local, global min at 0

local max at -1.

no global max.

Ex $f(x) = x^2$, for $-1 < x < 2$



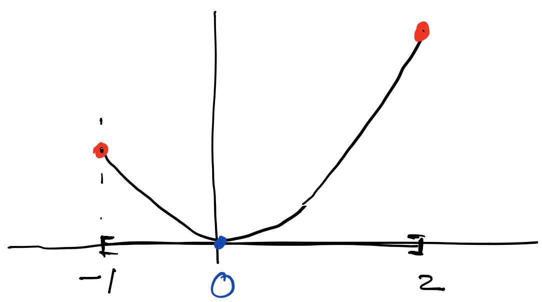
local, global min at 0

no local max

no global max.

10/18/2018

Ex $f(x) = x^2$, for $-1 \leq x \leq 2$

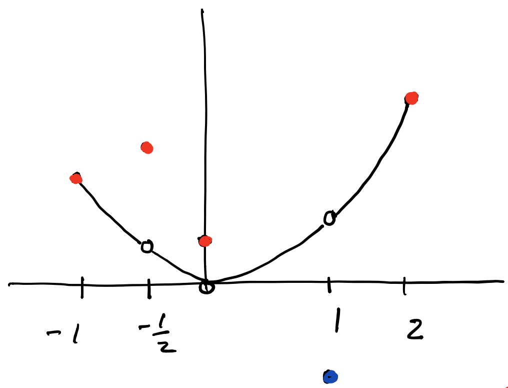


local, global min at 0.

local max at −1

local, global max at 2.

Ex



$-1 \leq x \leq 2$.

$$f(x) = \begin{cases} x^2 & \text{if } x \neq -\frac{1}{2},\ x \neq 0,\ x \neq 1 \\ \frac{3}{2} & \text{if } x = -\frac{1}{2} \\ \frac{1}{2} & \text{if } x = 0 \\ -1 & \text{if } x = 1 \end{cases}$$

local, global min at 1.

local, global max at 2.

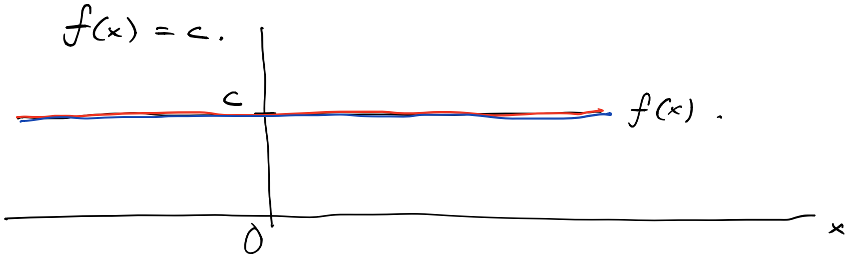
local max at −1

local max at $-\frac{1}{2}$

local max at 0

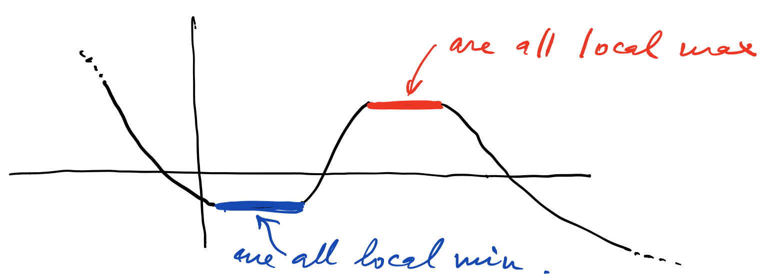
Every global min/max is also a local min/max.

Ex $f(x) = c$.

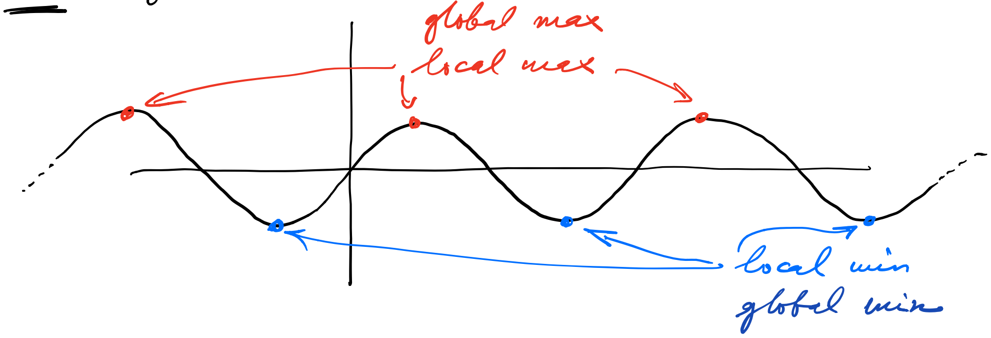


every point x is a local/global min/max.

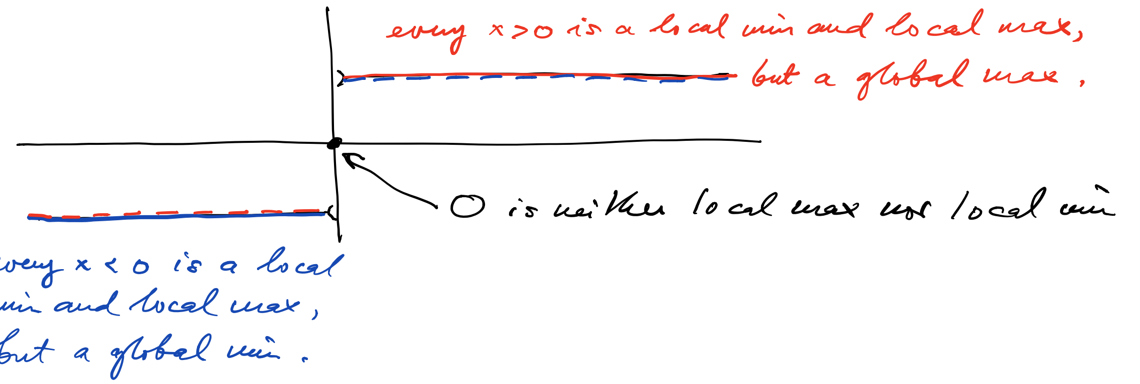
Ex



Ex $f(x) = \sin x$.



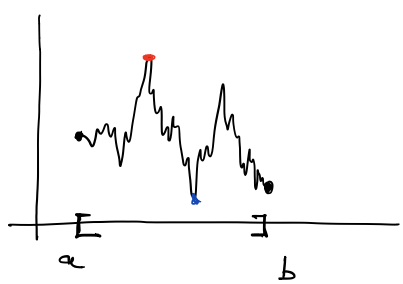
Ex $f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 0, & x = 0. \end{cases}$



Thm (Extreme value theorem).

Assume f is continuous on closed interval $[a,b]$.

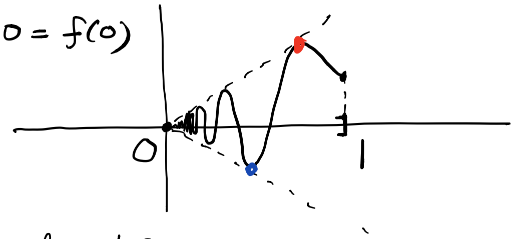
Then, f has a global max and a global min in $[a,b]$.



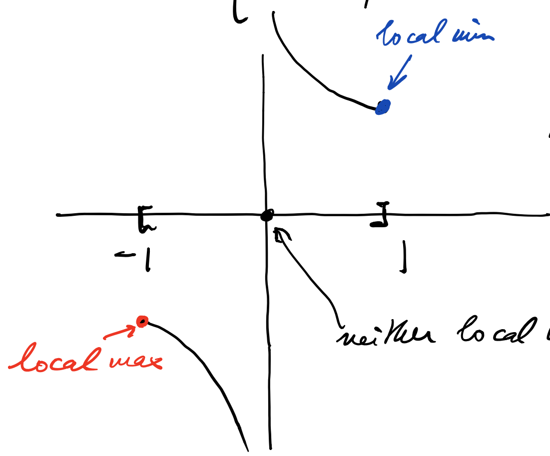
Ex $f(x) = \begin{cases} x \sin \frac{1}{x}, & 0 < x \leq 1 \\ 0, & x = 0. \end{cases}$ Domain $[0,1]$

Because $\lim_{x \to 0^+} f(x) = 0 = f(0)$

$\Rightarrow f$ is continuous on $[0, 1]$



Ex $f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ for $-1 \leq x \leq 1$.

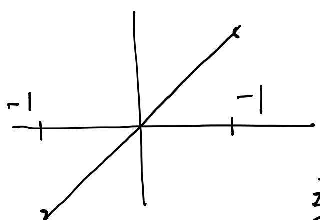


neither local min nor local max.

no global max nor global min

Not continuous $\Rightarrow$ Extreme value theorem is not applicable.

Ex $f(x) = x$ for $-1 < x < 1$



f has neither a global min nor a global max on $(-1, 1)$, but f is continuous.

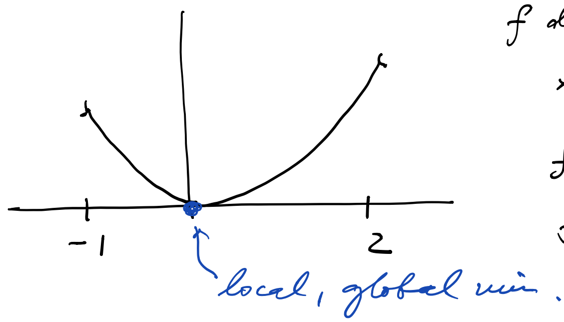
Because the domain is not a closed interval (does not contain both endpoints)

$\Rightarrow$ Extreme value thm is not applicable.

Thm (Fermat)

Assume that f has a local max or local min at c, and that $f'(c)$ exists. $\Rightarrow$ Then, $f'(c) = 0$.

Ex $f(x) = x^2$, $-1 < x < 2$

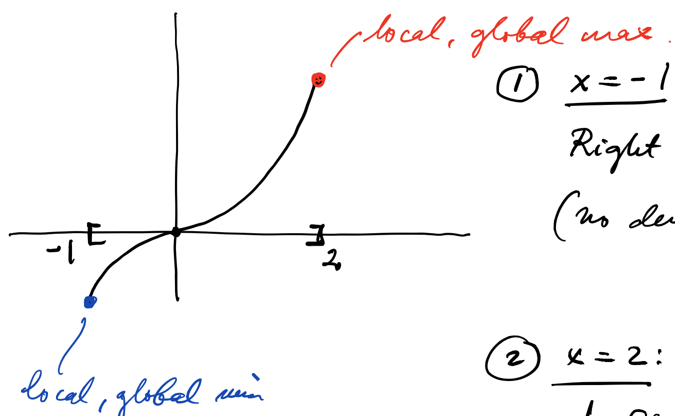


f differentiable for every x in $(-1, 2)$

$f'(x) = 2x$.

$f'(0) = 0$

Ex $f(x) = x^3$, $-1 \le x \le 2$



① $x = -1$: local and global min.

Right derivative : $f'_+(-1) = 3(-1)^2 = 3$

(no derivative here $\Rightarrow$ ~~Fermat~~).

② $x = 2$: local and global max

Left derivative : $f'_-(2) = 3 \cdot 2^2 = 12$

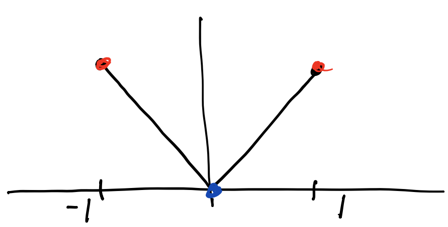
(no derivative here $\Rightarrow$ ~~Fermat~~).

③ $f'(0) = 3 \cdot 0^2 = 0$ neither local min nor local max

(careful : Fermat implies local min/max and differentiable at $c \Rightarrow f'(c) = 0$

But not the other way around !)

Ex $f(x) = |x|$ $-1 \le x \le 1$



local, global max at $x = -1$, $x = 1$.

local, global min at $x = 0$

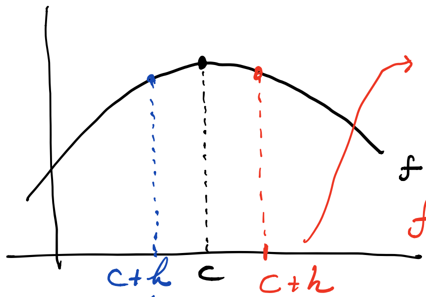
But : f not differentiable at $x = 0$.

$\Rightarrow$ Fermat not applicable here.

10/23/2018

Check: Explanation of Fermat's theorem.

local max at c, and $f'(c)$ exists.



$h>0$: $\quad f(c+h)-f(c) \le 0$

divide by h, and let $h \to 0^+$

$$f_+'(c) = \lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h} \le 0$$

$h<0$: $\quad f(c+h)-f(c) \le 0$

divide by h, and let $h \to 0^-$

$$f_-'(c) = \lim_{h\to 0^-} \frac{\overbrace{f(c+h)-f(c)}^{\le 0}}{\underbrace{h}_{<0}} \ge 0$$

$h<0$ negative number $\Rightarrow$ dividing by h flips the inequality sign.

$$0 \le f_-'(c) = f'(c) = f_+'(c) \le 0$$

f differentiable

$\Rightarrow f'(c) = 0$. This verifies Fermat's thm.

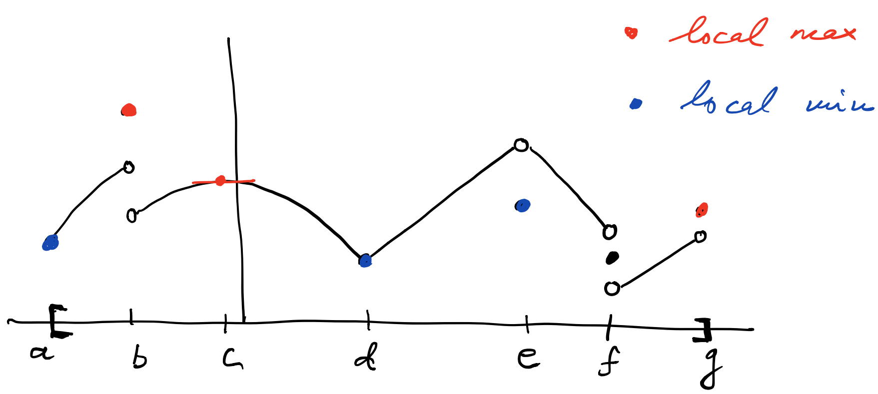
Def (critical number)

A point c in the domain of f is a critical number if: either $f'(c)=0$

or $f'(c)$ does not exist.

Thm If f has a local min/max at c, then c is a critical number.

Ex



Critical value

a because derivative does not exist, local min

b ——— " ———, local max

c local max with derivative zero

d local min, derivative does not exist.

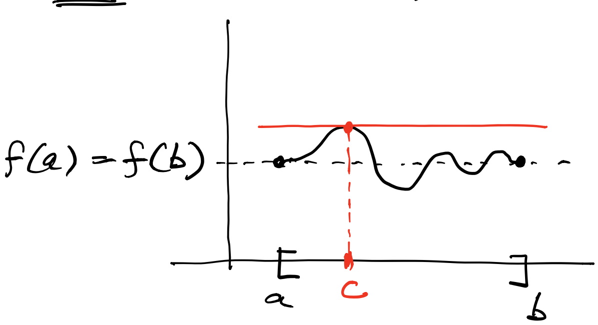
e local min, ——— " ———

f neither local min nor max, deriv. does not exist

g local max, deriv. does not exist.

The mean value theorem

Thm (Rolle's thm).



Assume f is continuous on $[a,b]$ with $f(a) = f(b)$, and that f is differentiable in (a,b).

Then, there is a point c in (a,b) such that $f'(c) = 0$.

Check: Extreme value thm: f continuous on $[a,b]$

$\Rightarrow$ there is a global min, global max

① $f(x) = f(a) = f(b)$ constant.

$\Rightarrow f'(x) = 0$ for all x in (a,b) CORRECT.

(2) f not constant.

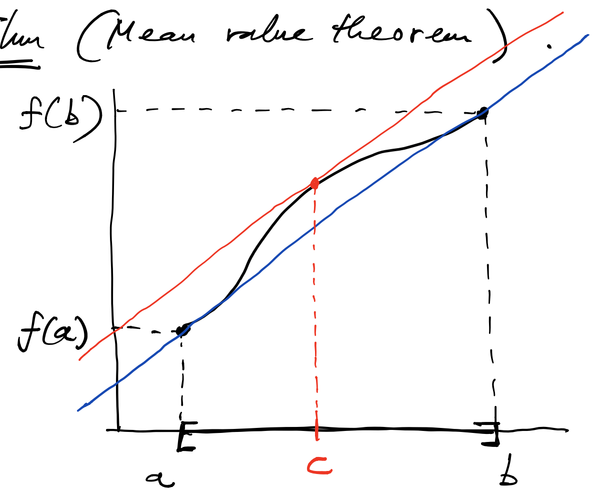
$\Rightarrow$ either global min or global max or both are not equal to $f(a)=f(b)$.

$\Rightarrow$ must be located at c in (a,b)

But: f differentiable in (a,b)

Therefore, Fermat says that $f'(c)=0$.]

Thm (Mean value theorem).



Assume f continuous on $[a,b]$ and differentiable in (a,b).

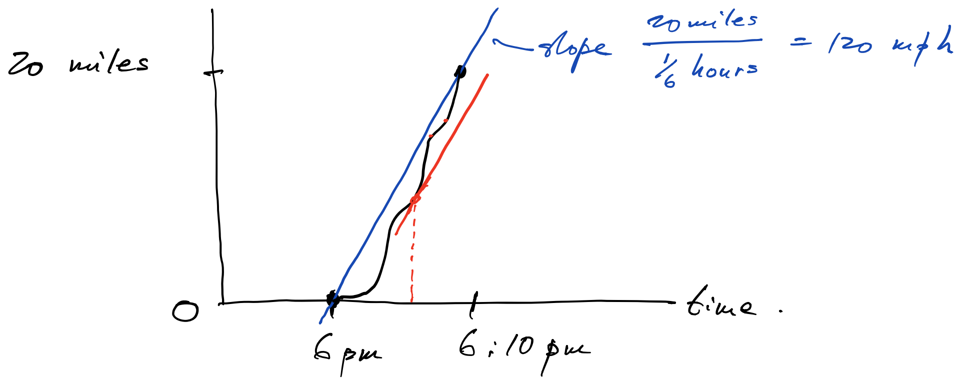
Then, there is a point c in (a,b) where

$$\underbrace{f'(c)}_{\text{slope of red tangent line.}} = \underbrace{\frac{f(b)-f(a)}{b-a}}_{\text{slope of blue line}}$$

$\Rightarrow$ "There is a point c between a and b where the tangent to the graph of c is parallel to the blue line (connecting the fct values at the endpoints a,b)".

Ex The local police station orders pizza at 6pm from a store 20 miles away. At 6:10 pm, the delivery person arrives and is arrested. Why?

(speed limit is 45 mph)

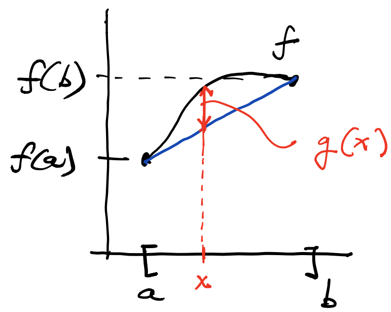


Legal reasoning: The delivery person must have driven at 120 mph at at least one moment between 6 pm and 6:10 pm.

Check (mean value theorem).

Equation for blue line:

$$y = f(a) + \underbrace{\frac{f(b)-f(a)}{b-a}}_{\text{slope}}(x-a)$$



$$g(x) = f(x) - \left(f(a) + \frac{f(b)-f(a)}{b-a}(x-a)\right).$$

$g(x)$ is continuous on $[a,b]$, differentiable on (a,b)

$g(a) = 0 = g(b)$

Rolle: There is c in (a,b) where $g'(c) = 0$.

$$g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow g'(c) = 0 = f'(c) - \frac{f(b)-f(a)}{b-a} \Rightarrow \boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}$$

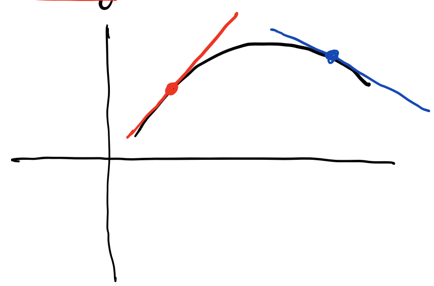
10/25/2018

Derivatives and shape of graphs.

What do we learn from f' about f?

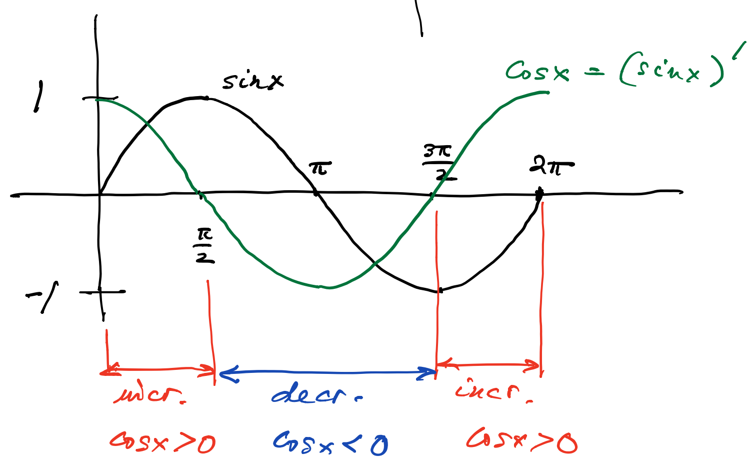
Increasing / decreasing.

- if $f'(x) > 0$, then f is increasing at x
- if $f'(x) < 0$, then f is decreasing at x

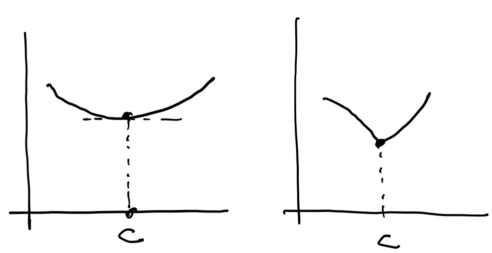


Ex $f(x) = \sin x$

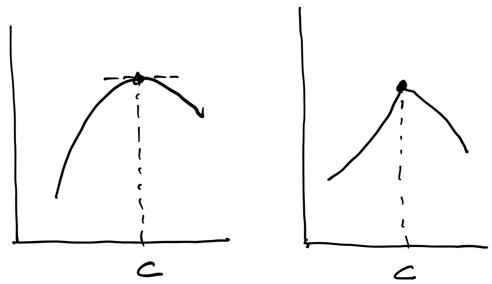
$0 \le x \le 2\pi$



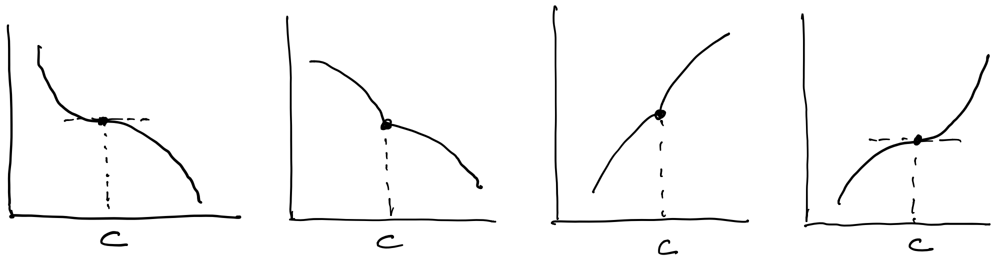
If f is continuous, and c is a critical number ($f'(c) = 0$, or $f'(c)$ does not exist).



if f changes from decreasing ($f' < 0$) to increasing ($f' > 0$), at c, then f has a local min at the critical number c.

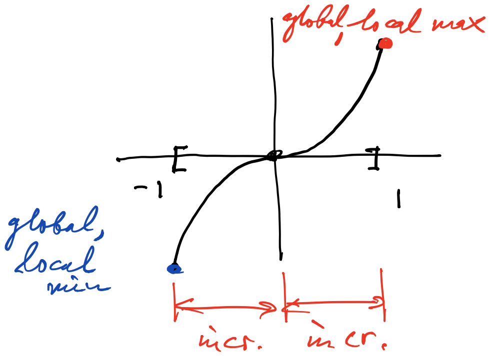


if f changes from increasing ($f'>0$) to decreasing ($f'<0$) at c, then f has a <u>local max</u> at the critical number c.



if f does not change from increasing ($f'>0$) to decreasing ($f'<0$) or vice versa at c, then the critical point c is neither a local min, nor a local max.

<u>Ex</u> $f(x) = x^3$, for $-1 \le x \le 1$.



Critical values: $-1, 0, 1$

no derivative

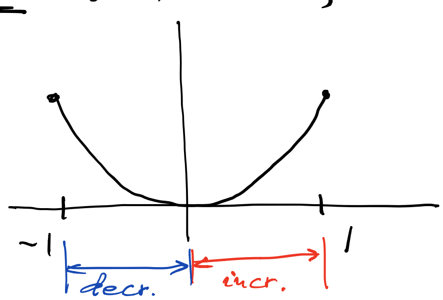
derivative $= 0$.

Increasing $f'(x) = 3x^2 > 0 \quad -1 \le x < 0$

Increasing. $f'(x) = 3x^2 > 0 \quad 0 < x \le 1$

neither local min nor local max at $x=0$.

<u>Ex</u> $f(x) = x^4$, $-1 \le x \le 1$.



f' does not exist

Critical values: $-1, 0, 1$

$f'(0) = 0$.

$f'(x) = 4x^3 < 0$, $-1 < x < 0$

<u>decreasing</u>

$f'(x) = 4x^3 > 0$, $0 < x < 1$.

<u>increasing</u>.

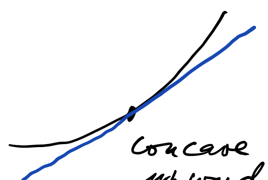
change from decr. to incr. at $x=0 \implies x=0$ is a <u>local</u> <u>min</u>

Second derivative.

f' ~ slope of the tangent line.

f'' ~ rate of change of the steepness (slope) of tangent line.

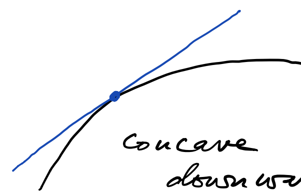
Def



concave upwards

(~ convex)

graph of f is <u>above</u> the tangent



concave downward

(~ concave)

graph of f is <u>below</u> the tangent.

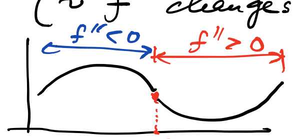
Thm (Concavity test)

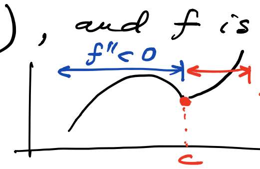
Assume f is <u>twice</u> differentiable in (a,b)

Then 1) if $\underbrace{f''(x) > 0}_{\text{steepness of tangent line increases.}}$ for x in $(a,b) \Rightarrow f$ is <u>concave up</u> in (a,b)

2) if $\underbrace{f''(x) < 0}_{\text{steepness of tangent line decreases.}}$ for x in $(a,b) \Rightarrow f$ is <u>concave down</u> in (a,b)

Def: A point c is an inflection point if f changes concavity at c (~ f'' changes sign), and f is continuous at c.

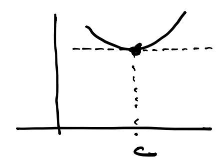




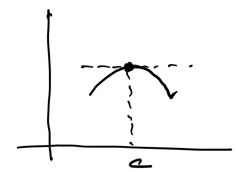
Thm (2nd derivative test)

Assume f'' is continuous near c.

1) if $f'(c) = 0$ and $f''(c) > 0$
then f has a local min at c.

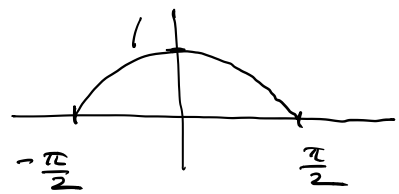


2) if $f'(c) = 0$ and $f''(c) < 0$
then f has a local max at c.



Remark: If $f'(c)$ does not exist, then $f''(c)$ also doesn't exist $\Rightarrow$ can't use 2nd derivative test.

Ex $f(x) = \cos x$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

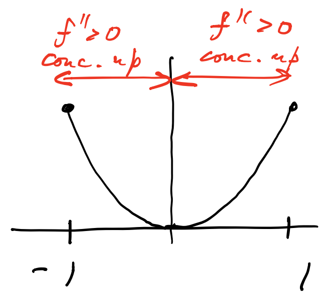


$f'(x) = 0$: $-\sin x = 0$
$\Rightarrow x = 0$

$f''(x) = -\cos x$: $f''(0) = -\cos 0 = -1 < 0$

$\Rightarrow$ by 2nd derivative test, $x = 0$ is a local max

Ex $f(x) = x^4$, $-1 \le x \le 1$.



Crit values $-1, 0, 1$.

At $x = 0$:

$f'(x) = 4x^3 \Rightarrow f'(0) = 0$

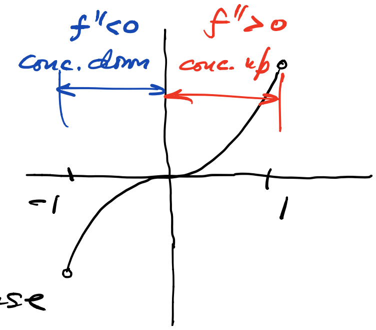
$f''(x) = 12x^2 \Rightarrow f''(0) = 0$. $\Rightarrow$ 2nd derivative test does not apply (because neither $f''(0) > 0$ nor $f''(0) < 0$)

Have to use 1st derivative; verify change from decreasing ($f' < 0$) to increasing ($f' > 0$).
Also, the concavity of f doesn't change at $0 \Rightarrow 0$ not an inflection pt.

Ex $f(x) = x^3$, $-1 < x < 1$.

$f'(x) = 3x^2$

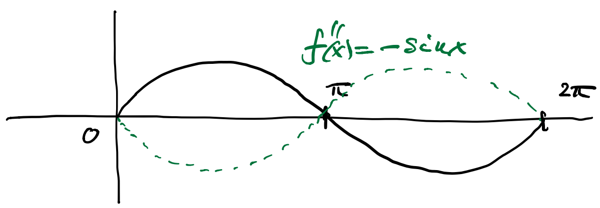
$f''(x) = 6x$. $\begin{cases} < 0, & -1 < x < 0 \\ > 0, & 0 < x < 1 \end{cases}$



$\Rightarrow$ $x=0$ is an inflection pt because the concavity of f changes.

$x=0$ is also a critical value because $f'(0)=0$.

Ex $f(x) = \sin x$, $0 \le x \le 2\pi$.



$f'(x) = \cos x$

$f''(x) = -\sin x$

$\Rightarrow \begin{cases} f'' > 0, & \pi < x < 2\pi \\ f'' < 0, & 0 < x < \pi. \end{cases}$

$\rightarrow$ $x = \pi$ is an inflection point

But: $f'(\pi) = \cos\pi = -1 \Longrightarrow x=\pi$ is not a critical value.

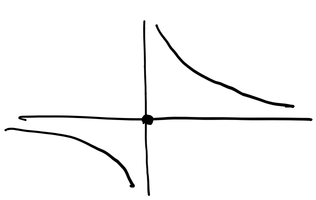
Remark: An inflection point does not need to be a critical value.

10/30/2018 2nd midterm on Nov 15, 2018

Final exam on Dec 15, 2018. Saturday,

RLM 4.102 7-10 PM

Ex $f(x) = \begin{cases} \frac{1}{x} & x \ne 0 \\ 0 & x = 0 \end{cases}$



$f'(x) = -\frac{1}{x^2}$

$f''(x) = \frac{2}{x^3}$ $\begin{cases} f'' < 0, & x < 0 \\ f'' > 0, & x > 0 \end{cases}$

Is $x=0$ an inflection point?

NO: Because f is not continuous at $x=0$.

Ex $f(x) = x^4 - 4x^3$, $-\infty < x < \infty$

Draw the graph (min, max, concavity, inflection pts, ...).

1) Zeros of $f(x) = x^4 - 4x^3 = 0 \Rightarrow \underline{x=0}, \underline{x=4}$.

2) Derivative $f'(x) = 4x^3 - 12x^2$

Critical points: $4x^3 - 12x^2 = 0 \Rightarrow \underline{x=0}, \underline{x=3}$

f decreasing/increasing: neither local max nor min; local min

$(-\infty, 0), (0, 3), (3, \infty)$

$\underbrace{f'<0 \quad f'<0}_{\text{decreasing}} \quad \underbrace{f'>0}_{\text{increasing}}$.

3) Second derivative. $f''(x) = 12x^2 - 24x$.

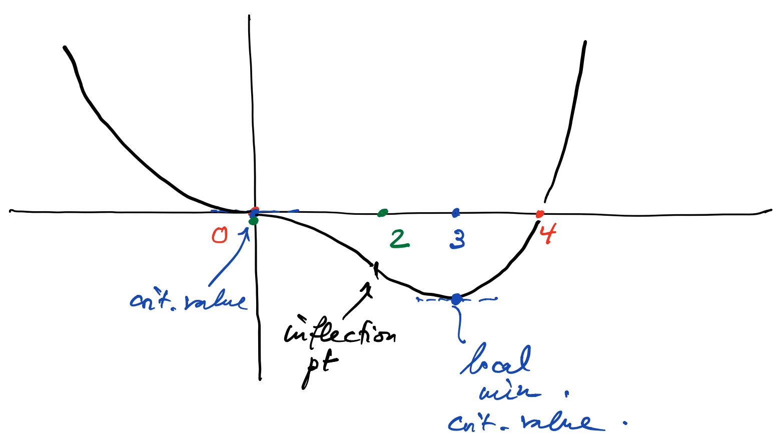
Candidates for inflection pts: $f''(x) = 12x^2 - 24x = 0$.

$\Rightarrow \underline{x=0}, \underline{x=2}$.

$(-\infty, 0) (0, 2) (2, \infty)$

$f''>0 \quad f''<0 \quad f''>0$

$\Rightarrow$ Inflection pts: $0, 2$.



Ex $f(x) = x^3 + 3x + 2$, $-\infty < x < \infty$.

How many roots $f(x) = 0$ have?

$\Rightarrow$ How many times does the graph of f intersect the x-axis?

Intermediate value thm:

$$f(-1) = -1 - 3 \cdot 1 + 2 = -2.$$

$$f(1) = 1 + 3 \cdot 1 + 2 = 6$$

$\Rightarrow$ there must be a root in $(-1, 1)$.

Now consider $f'(x) = 3x^2 + 3 > 0$

$\Rightarrow$ f is everywhere increasing.

$\Rightarrow$ can cross x-axis only once.

$\Rightarrow$ there is exactly one root of $f(x) = 0$.

Indeterminate forms and de l'Hôpital's rule.

Thm (de l'Hôpital's rule).

Assume f, g differentiable near $x = a$, and

$$\lim_{x \to a} f(x) = 0, \quad \lim_{x \to a} g(x) = 0$$

$$\left(\text{or } \lim_{x \to a} f(x) = \pm\infty, \quad \lim_{x \to a} g(x) = \pm\infty \right)$$

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Remark: Similarly for left/right limits and left/right derivatives of f, g.

Ex $\lim_{x\to 0} \frac{\sin x}{x} \underset{\text{de l'H}}{=} \lim_{x\to 0} \frac{\cos x}{1} = 1.$

Ex $\lim_{x\to 1} \frac{x^2-1}{x-1} \underset{\text{de l'H}}{=} \lim_{x\to 1} \frac{2x}{1} = 2.$

Ex $\lim_{x\to 0} \frac{1-\cos x}{x^2} \underset{\text{de l'H}}{=} \lim_{x\to 0} \frac{\sin x}{2x} \underset{\text{de l'H}}{=} \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}.$

Ex Compare x^r, $r>0$, with $\ln x$, as $x \to \infty$

$$\lim_{x\to\infty} x^r = \infty, \quad \lim_{x\to\infty} \ln x = \infty$$

$$\lim_{x\to\infty} \frac{\ln x}{x^r} \underset{\text{de l'H}}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{r x^{r-1}}$$

$$= \lim_{x\to\infty} \frac{1}{r\cdot x\cdot x^{r-1}} = \lim_{x\to\infty} \frac{1}{r x^r} = 0$$

$\Rightarrow$ x^r tends to ∞ faster than $\ln x$, for any $r>0$.

$\Rightarrow$ $x^{0.000000000000000000000000000000001}$

tends to ∞ faster than $\ln x$

Calculation: $r = 0.01$, $x = 10^{10,000}$

$$\Rightarrow x^r = \left(10^{10,000}\right)^{0.01} = 10^{100}$$

$\log_{10} x = \log_{10} 10^{10,000} = 10,000 = 10^4$ much smaller than 10^{100}

Ex Let $n>0$ be an integer.

Compare e^x with x^n as $x \to \infty$ $\left(\lim_{x\to\infty} e^x = \infty, \ \lim_{x\to\infty} x^n = \infty\right)$

$$\lim_{x\to\infty} \frac{e^x}{x^n} \underset{\text{de l'H}}{=} \lim_{x\to\infty} \frac{e^x}{n x^{n-1}}$$

$$\underset{\text{de l'H}}{=} \lim_{x\to\infty} \frac{e^x}{n(n-1)x^{n-2}} \underset{\text{de l'H}}{=} \cdots = \lim_{x\to\infty} \frac{e^x}{n(n-1)(n-2)\cdots\cdots 2x}$$

$$\underset{\text{de l'H}}{=} \lim_{x\to\infty} \frac{e^x}{\underbrace{n(n-1)(n-2)\cdots\cdot 3\cdot 2\cdot 1}_{n!}} = \infty$$

$\Rightarrow$ e^x tends to ∞ faster than x^n, for any $n > 0$.

Calculate: $x = 10{,}000$, $n = 100$

$$e^{10{,}000} \approx 3^{10{,}000} \approx (3^2)^{5000} \approx \underbrace{10^{5000}}_{\text{much bigger than}}$$

$$x^n = 10{,}000^{100} = (10^4)^{100} = 10^{400}$$