11/1/2018

# Indeterminate powers.

Assume $f(x) > 0$ for $x$ near $a$.

$$\lim_{x\to a} \left(f(x)\right)^{g(x)} \quad \text{where} \quad \begin{array}{l} 1)\ f\to 0^+,\ g\to 0 \\ 2)\ f\to\infty,\ g\to 0 \\ 3)\ f\to\infty,\ g\to\pm\infty \end{array}$$

If $f(x) > 0$ for $x$ near $a$

$$\left(f(x)\right)^{g(x)} = \left(e^{\ln f(x)}\right)^{g(x)} = e^{g(x)\cdot \ln f(x)}$$

$$\Rightarrow \boxed{\lim_{x\to a}\left(f(x)\right)^{g(x)}} = \lim_{x\to a} e^{g(x)\cdot\ln f(x)}$$

$$= \boxed{e^{\lim_{x\to a} g(x)\cdot \ln f(x)}}$$

if exponent has a finite limit.

Ex Find $\lim_{x\to\infty} x^{\frac{1}{x}}$ $\Rightarrow f(x) = x$, $g(x) = \frac{1}{x}$.

$$\lim_{x\to\infty} x^{\frac{1}{x}} = e^{\lim_{x\to\infty} \frac{1}{x}\ln x}$$

$$\Rightarrow \lim_{x\to\infty} \frac{\ln x}{x} \underset{\text{de l'H}}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

$$\Rightarrow \lim_{x\to\infty} x^{\frac{1}{x}} = e^0 = 1$$

Ex Find $\lim_{x\to 0^+} x^x$ $\Rightarrow f(x) = x$, $g(x) = x$

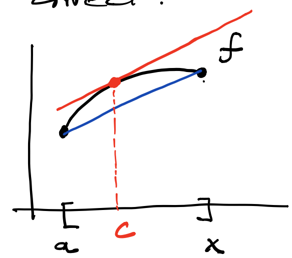
$$\lim_{x\to 0^+} x^x = e^{\lim_{x\to 0^+} x\ln x} = e^0 = 1,$$

because: $\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \underset{\text{de l'H}}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$

$$= \lim_{x \to 0^+} (-x) = 0$$

Check that de l'Hôpital's rule is correct.

Remember: Mean value thm.




$$f'(c) = \frac{f(x) - f(a)}{x - a}.$$

$$\Rightarrow f(x) - f(a) = f'(c) \cdot (x-a)$$

$$\Rightarrow \quad f(x) = f(a) + f'(c) \cdot (x-a) \qquad c \text{ in } (a,x)$$

$$g(x) = g(a) + g'(d) \cdot (x-a) \qquad d \text{ in } (a,x)$$

Assume $\lim_{x \to a} f(x) = 0 \Rightarrow f(a) = 0$

$\lim_{x \to a} g(x) = 0 \Rightarrow g(a) = 0$

$$\Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{\cancel{f(a)}^{0} + f'(c)(x-a)}{\cancel{g(a)}_{0} + g'(d)(x-a)} = \lim_{x \to a} \frac{f'(c)\cancel{(x-a)}}{g'(d)\cancel{(x-a)}}$$

$c$ and $d$ are both in $(a,x)$ $\Rightarrow$ may therefore replace $c$ and $d$ with $x$ in the fraction

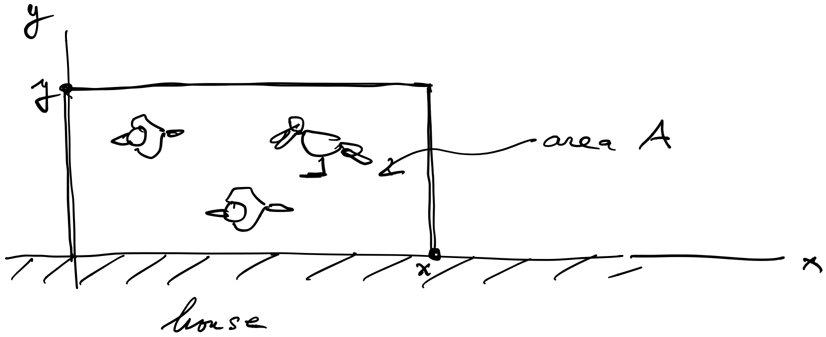
$$= \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

## Optimization problems.

Goal: Find best outcome, under the given constraints.

**Ex**: Build a fence along one wall of your house for your chicken.

Make area $A$ as large as possible, under the constraint that the length of the fence material is 100 m.




Optimize: $A = x \cdot y$

Constraint: $2y + x = 100$ m. length.

solve y for x:

$$2y = 100 - x \Rightarrow y = 50 - \frac{x}{2}.$$

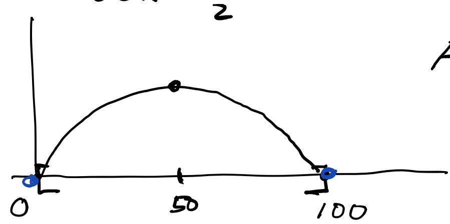
substitute

$$A = x \cdot \left(50 - \frac{x}{2}\right) = 50x - \frac{x^2}{2}$$

zeros: $x = 0$, $x = 100$

local, global max: $x = 50$.

local, global min: $x = 0, 100$




$A'(x) = 50 - x$

Critical values: $A'(x) = 50 - x = 0$

$\Rightarrow x = 50$.

Second derivative test: $A''(x) = -1$

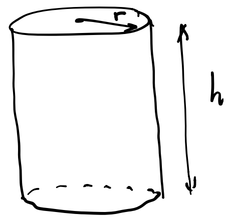
$\Rightarrow A''(50) = -1 < 0$

$\Rightarrow$ local max.

$\Rightarrow$ area is maximal when $x = 50$ m.

$$\Rightarrow A = 50 \cdot \left(\underbrace{50 - \frac{50}{2}}_{25}\right) = 1250 \text{ m}^2$$

**Ex** Build a can (cylindrical) holding $1\,\ell$ of beer, with the least amount of material.




**Volume**: $V = \pi r^2 h = 1\ell$, constraint

**Area**: $A = 2\cdot\pi r^2 + 2\pi r h$

11/6/2018

Optimization: Minimize the area, with volume constraint.

Solve for $h = \dfrac{1}{\pi r^2}$ using constraint equation.

Plug into $A(r) = 2\pi r^2 + \dfrac{2\pi r}{\pi r^2} = 2\pi r^2 + \dfrac{2}{r}$.

$$\Rightarrow A'(r) = 4\pi r - \frac{2}{r^2} = 0.$$

$$\Rightarrow 4\pi r = \frac{2}{r^2} \Rightarrow 4\pi r^3 = 2 \Rightarrow r^3 = \frac{2}{4\pi} = \frac{1}{2\pi}$$

$$\Rightarrow r = \frac{1}{\sqrt[3]{2\pi}} \quad \text{critical value}$$

Is it a local min?

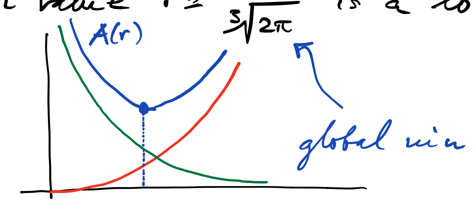
$$A''(r) = 4\pi + \frac{4}{r^3}$$

Plug in the critical value:

$$A''\left(\frac{1}{\sqrt[3]{2\pi}}\right) = 4\pi + \frac{4}{\left(\frac{1}{\sqrt[3]{2\pi}}\right)^3} = 4\pi + 4\cdot 2\pi = 4\pi + 8\pi = 12\pi > 0$$

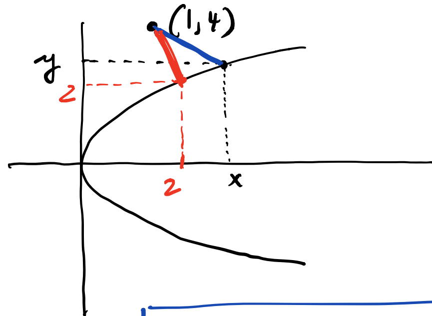
concave up.

$\Rightarrow$ crit value $r = \dfrac{1}{\sqrt[3]{2\pi}}$ is a local min.




$A(r) = 2\pi r^2 + \dfrac{2}{r}$

<u>Ex</u> Consider the parabola $y^2 = 2x$. Find the closest point on it to the point $(1,4)$




distance $d = \sqrt{(x-1)^2 + (y-4)^2}$ optimize!

Constraint: $y^2 = 2x \Rightarrow x = \frac{1}{2}y^2$

Plug into $d = \sqrt{\left(\frac{1}{2}y^2 - 1\right)^2 + (y-4)^2}$

$$S = d^2 = \left(\frac{1}{2}y^2 - 1\right)^2 + (y-4)^2$$

square of distance is minimal when $d$ is minimal.

$$S'(y) = 2\left(\frac{1}{2}y^2 - 1\right)\cdot y + 2(y-4) = 0.$$
$$= y^3 - \cancel{2y} + \cancel{2y} - 8$$
$$= y^3 - 8 = 0.$$

$$\Rightarrow y^3 = 8 \Rightarrow \underline{\underline{y = 2}}$$

$$S''(y) = 3y^2 \Rightarrow S''(2) = 3\cdot 2^2 = 12 > 0$$

Concave up.

$\Rightarrow$ local min.
global

# Antiderivatives.

$$F' = f$$ f is the derivative of F

F is an antiderivative of f.

**Ex** $x^9$ is an antiderivative of $9x^8$

$x^9+10$ — " — $9x^8$

**Thm** If $F(x)$ is an antiderivative of $f(x)$ on $(a,b)$, then any other antiderivative of $f$ on $(a,b)$ has the form

$$F(x) + C$$ arbitrary constant.

Therefore, the most general antiderivative of $f$ is

$$F(x) + C$$ arbitrary constant.

an arbitrary choice of an antiderivative

**Ex** Find general antiderivative of $f(x)$:

$f(x) = e^x \Rightarrow F(x) = e^x + C$

$f(x) = \frac{1}{x^2} \Rightarrow F(x) = -\frac{1}{x} + C.$

$f(x) = a^x \Rightarrow F(x) = \frac{1}{\ln a} a^x + C, \quad a > 0$

$f(x) = \frac{1}{x} \Rightarrow F(x) = \ln x + C, \quad x > 0$

$f(x) = x^r \Rightarrow F(x) = \frac{1}{r+1} x^{r+1} + C$
$r > 0$

$f(x) = \cos x \Rightarrow F(x) = \sin x + C$

$f(x) = \sin x \Rightarrow F(x) = -\cos x + C$

**Ex** $f(x) = \ln x$ , $x > 0$ $\Rightarrow$ find general antiderivative.

Look at $(x \cdot \ln x)' = x \cdot \frac{1}{x} + 1 \cdot \ln x = 1 + \ln x$.

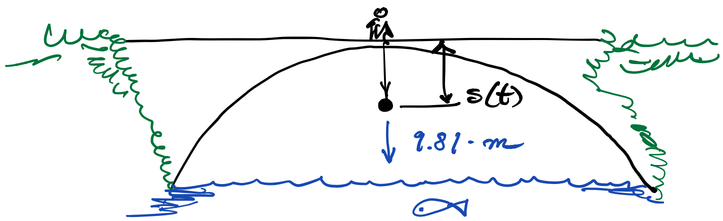
$$(x \cdot \ln x)' - 1 = \ln x.$$

$$= (x \cdot \ln x)' - (x)'$$

$$= (x \cdot \ln x - x)'$$

$$\Rightarrow \quad F(x) = x \cdot \ln x - x + C$$

**Ex**




Throw a rock at time $t=0$ from $s(0)=0$, with velocity $s'(0) = 1$ m/sec

Newton's gravitational constant $9.81$ m/sec$^2$

Newton's law: $\underbrace{m \cdot s''(t)}_{\text{mass} \,\cdot\, \text{acceleration}} = \underbrace{m \cdot g}_{\text{force}}$.

position at time $t$ : $s(t)$

velocity —"— : $s'(t)$

acceleration —"— : $s''(t)$

$$\cancel{m} \, s''(t) = \cancel{m} \cdot 9.81$$

antiderivative

$$\Rightarrow \quad s'(t) = 9.81 \cdot t + C$$

antiderivative

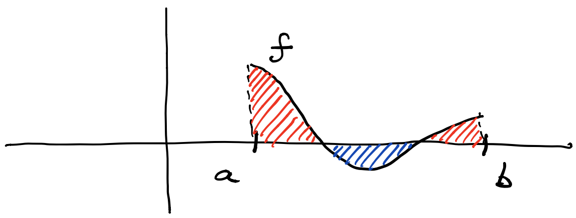
$$\Rightarrow \quad s(t) = 9.81 \cdot \frac{t^2}{2} + Ct + D$$

$$\left.\begin{array}{l} s'(0) = 1 = 9.81 \cdot 0 + C = C \Rightarrow C = 1 \\ s(0) = 0 = 9.81 \cdot \frac{0^2}{2} + 1 \cdot 0 + D = D \Rightarrow D = 0 \end{array}\right\} s(t) = 9.81 \frac{t^2}{2} + t$$

11/8/2018 .

# Integrals.

**Def**: The definite integral of a continuous function $f$ on $[a,b]$ is the <u>area</u> enclosed between the graph of $f$ and the x-axis.

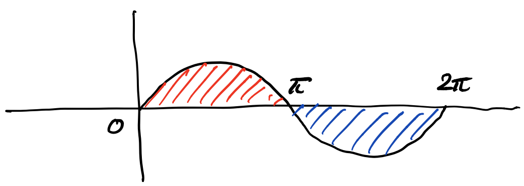



//// positive area.

//// negative area

If an area lies below x-axis, it is counted with a negative sign.

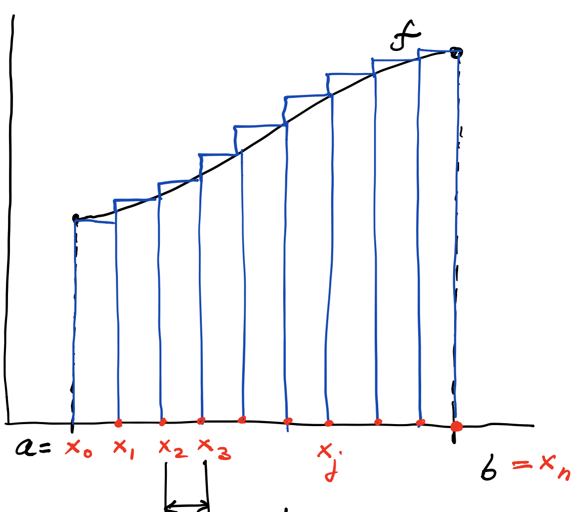
**Ex** $f(x) = \sin x$, $x$ in $[0, 2\pi]$




total area $= 0$.

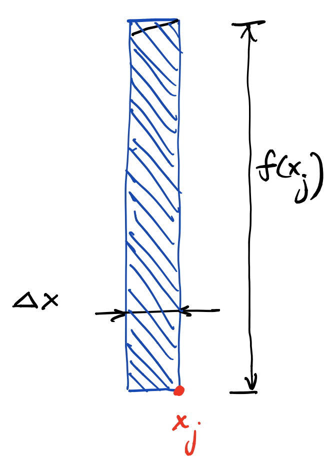
How to calculate this area?

**Approach # 1**




$\Delta x = \frac{b-a}{n}$

j-th rectangle




area $= f(x_j) \cdot \Delta x$

Sum of areas of all $n$ rectangles:

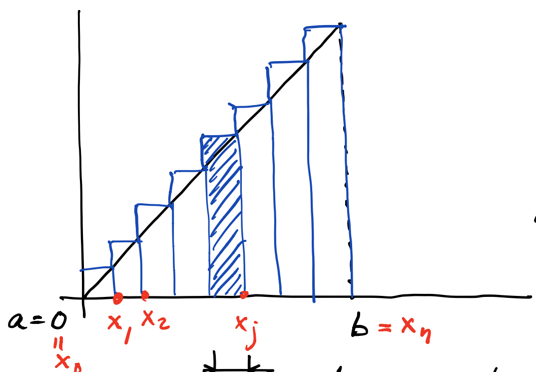
$$\underbrace{f(x_1)\cdot\Delta x + f(x_2)\cdot\Delta x + \dots\dots + f(x_n)\cdot\Delta x}_{\text{approximation of area}} \xrightarrow[n\to\infty]{\text{limit}} \text{area under } f(x)$$

Riemann sum.

$\shortparallel$

definite integral of $f$ over $[a,b]$

**Ex**




$$\Delta x = \frac{b-a}{n} = \frac{b}{n}$$

$f(x) = x$

area of $j$-th rectangle:

$$f(x_j)\cdot\Delta x = x_j \cdot \Delta x$$

we have $x_j = j\cdot\Delta x = j\cdot\frac{b}{n}$.

$\Rightarrow$ area of $j$-th rectangle:

$$x_j\cdot\Delta x = \underbrace{j\cdot\frac{b}{n}}_{x_j}\cdot\underbrace{\frac{b}{n}}_{\Delta x}$$

$$= j\cdot\frac{b^2}{n^2}$$

Sum of all $n$ rectangles:

$$f(x_1)\cdot\Delta x + f(x_2)\cdot\Delta x + \dots + f(x_n)\cdot\Delta x$$

$$= 1\cdot\frac{b^2}{n^2} + 2\cdot\frac{b^2}{n^2} + \dots + n\cdot\frac{b^2}{n^2}$$

$$= (1+2+3+\dots\quad+n)\,\frac{b^2}{n^2}$$

$$= (1+n)\cdot\frac{n}{2}\cdot\frac{b^2}{n^2}$$

number of pairs:

$$1+2+3+4+5+6+7+8+9+10+11+12+13 = 6\tfrac{1}{2}\cdot 14$$

14

14

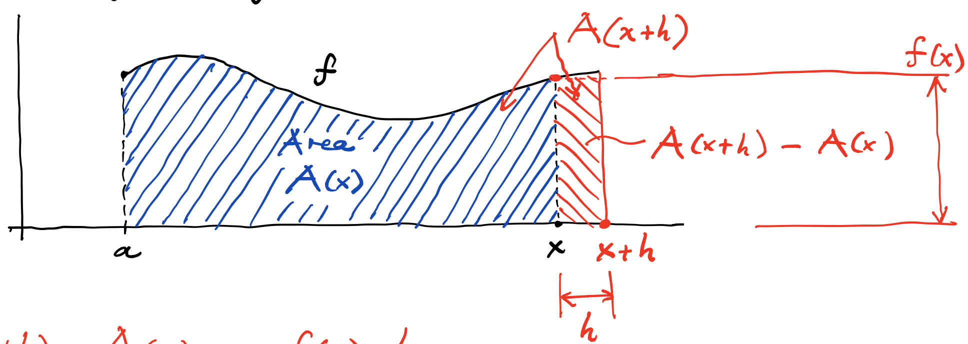
14

$$= \left(\frac{n}{2}+\frac{n^2}{2}\right)\frac{b^2}{n^2} = \underbrace{b^2\left(\frac{1}{2n}+\frac{1}{2}\right)}_{\text{area of all rectangles}} \xrightarrow[n\to\infty]{\text{limit}} \frac{b^2}{2}$$

integral of $f(x)=x$ from $a=0$ to $b$, using a Riemann sum.

## Approach #2

A more elegant way to calculate a definite integral.




$$/// = A(x+h) - A(x) = f(x) \cdot h$$

$$\frac{A(x+h)-A(x)}{h} = f(x)$$

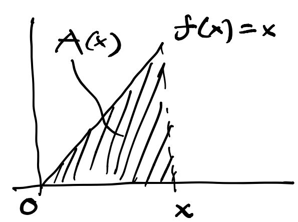
$$\Rightarrow A'(x) = \lim_{h \to 0} \frac{A(x+h)-A(x)}{h} = f(x)$$

$\Rightarrow$ $A(x)$ is an antiderivative of $f$ !!

**Ex** $f(x) = x$. Let $A(x)$ be the area under $f(x)$ over $[0, b]$

$A(x)$ is an antiderivative of $f(x) = x$.

$$\Rightarrow A(x) = \frac{1}{2} x^2 + C$$




$$A(0) = 0 = \frac{1}{2} 0^2 + C$$

$$\Rightarrow C = 0.$$

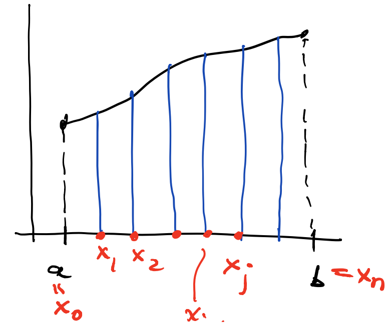
$$\Rightarrow A(x) = \frac{1}{2} x^2 \quad \Rightarrow \quad A(b) = \frac{b^2}{2}$$

**Def** (Definite integrals).

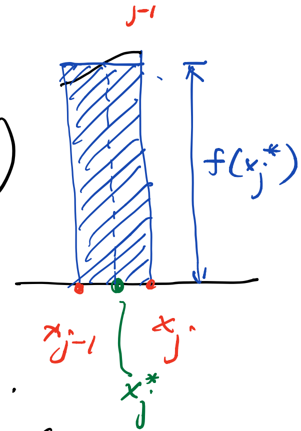
Assume $f$ continuous fct on $[a,b]$.

Let $\Delta x = \frac{b-a}{n}$ and $x_j = a + j \cdot \Delta x$.

Pick a sample point $x_j^*$ in $[x_{j-1}, x_j]$



$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\Big(f(x_1^*)\Delta x + f(x_2^*)\cdot\Delta x + \dots$$
$$\dots + f(x_n^*)\Delta x\Big)$$




is the <u>definite</u> <u>integral</u> of $f$ from $a$ to $b$, if this limit exists.

If it exists, we call $f$ <u>integrable</u> on $[a,b]$.

"Definite": boundary points $a, b$ are specified.

## <u>The fundamental theorem of calculus.</u>

Bridge between differentiation and integration.

Discovered by Barrow in 17th century (advisor of I. Newton)

<u>Thm</u> (FToC, Part I)

Assume $f$ continuous on $[a,b]$. Then,

$$A(x) = \int_a^x f(t)\,dt$$

t: integration variable, dummy variable

$\int_a^x f(\alpha)\,d\alpha$

is continuous for $x$ in $[a,b]$, and differentiable in $(a,b)$.

$$A'(x) = f(x)$$

(differentiation undoes what integration does).

<u>Ex</u>
$$A(x) = \int_1^x e^{t^2}\,dt$$
$$A'(x) = e^{x^2}$$

<u>Ex</u>
$$g(x) = \int_1^{x^2} e^{t^2}\,dt = A(x^2)$$
$$g'(x) = \left(A(x^2)\right)' \underset{\text{chain rule}}{=} \left(A'(x^2)\right)\cdot 2x = e^{x^4}\cdot 2x$$

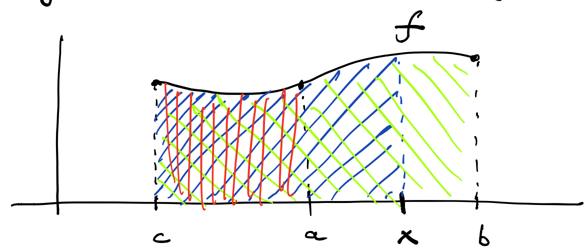
Thm (FToC, Part II).

If $f$ is continuous on $[a,b]$. Then,

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b$$

where $F$ is an arbitrary antiderivative of $f$ $(F' = f)$

WHY? Check:




$A(x) = \int_c^x f(t)\,dt$ is an antiderivative of $f$.

$= A(b) = \int_c^b f(t)\,dt$

$= A(a) = \int_c^a f(t)\,dt$.

$\int_a^b f(t)\,dt = A(b) - A(a)$

The most general antiderivative of $f$ has the form

$$F(x) = \underbrace{A(x)}_{\text{an antiderivative}} + \underbrace{C}_{\text{a constant.}}$$

$$\Rightarrow \int_a^b f(t)\,dt = A(b) - A(a) = (A(b) + \cancel{C}) - (A(a) + \cancel{C})$$
$$= F(b) - F(a)$$

Ex $\int_1^{10} \frac{1}{x}\,dx = \ln x \Big|_1^{10} = \ln 10 - \underbrace{\ln 1}_{=0}$.

Ex $\int_1^5 x^2\,dx = \frac{x^3}{3}\Big|_1^5 = \frac{5^3}{3} - \frac{1^3}{3} = \frac{125-1}{3} = \frac{124}{3}$

Integration rules: Next time.

## Review (Midterm II).

**Yes** :
- Pencil # 2, HB.
- Scratch paper
- Water, tissues
- 2 pages of handwritten, US letter size notes.

**No** :
- Calculator, computer, smart phones, etc
- No impersonators.

**Ex** : $\lim_{x\to\infty} (e^x + x)^{\frac{1}{x}}$

$$= \lim_{x\to\infty} e^{\frac{1}{x}\ln(e^x+x)} = e^{\lim_{x\to\infty} \frac{1}{x}\ln(e^x+x)} = e^1 = e$$

$$\lim_{x\to\infty} \frac{\ln(e^x+x)}{x} \underset{\text{de l'H}}{=} \lim_{x\to\infty} \frac{\frac{1}{e^x+x}\cdot(e^x+1)}{1}$$

$$= \lim_{x\to\infty} \frac{e^x+1}{e^x+x} \underset{\text{de l'H}}{=} \lim_{x\to\infty} \frac{e^x}{e^x+1}$$

$$= \lim_{x\to\infty} \frac{e^x}{e^x} = 1$$

**Ex** $\lim_{t\to\infty} t\ln\left(1+\frac{3}{t}\right) = \lim_{t\to\infty} \frac{\ln\left(1+\frac{3}{t}\right)}{\frac{1}{t}}$

$$\underset{\text{de l'H}}{=} \lim_{t\to\infty} \frac{\frac{1}{1+\frac{3}{t}}\left(-\cancel{\frac{3}{t^2}}\right)}{-\cancel{\frac{1}{t^2}}} = \lim_{t\to\infty} \frac{1}{1+\frac{3}{t}}\cdot 3 = 3$$

**Ex** Find $f(5.01)$ when $f(x) = 3x e^{2x-10}$

Linearization around $a = 5$.

$$f(a) = f(5) = 3\cdot 5\cdot e^{2\cdot 5-10} = 15$$

$$f'(x) = 3x\, e^{2x-10} \cdot 2 + 3e^{2x-10}$$

$$f'(5) = 3 \cdot 5 \cdot e^{10-10} \cdot 2 + 3e^{10-10}$$

$$= 30 + 3 = 33$$

Linearization: $y = f(a) + f'(a)(x-a)$. $\qquad x = 5.01$.

$$= 15 + \underbrace{33 \cdot 0.01}_{0.33} = \underline{\underline{15.33}}$$

Ex Assume $x, y$ satisfy $6y^2 + x^2 = 2 - x^3 \cdot e^{4-4y}$

Determine $y'$ when $x = -2$, $y = 1$

Check: $6 \cdot 1^2 + (-2)^2 = 6 + 4 = 10$

$$2 - (-2)^3 \underbrace{e^{4-4\cdot 1}}_{1} = 2 - (-8) = 10 \qquad \checkmark$$

Implicit differentiation:

$$12y \cdot y' + 2x = -3x^2 e^{4-4y} - (-4y')\, x^3 \cdot e^{4-4y}$$

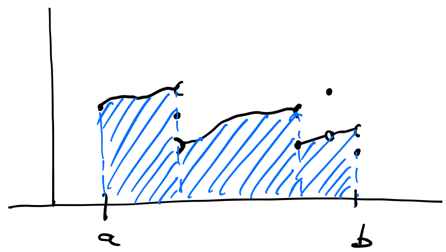
$$y' \cdot \left(12y - 4x^3 \cdot e^{4-4y}\right) = -3x^2 e^{4-4y} - 2x$$

$$y' = \frac{-3x^2 e^{4-4y} - 2x}{12y - 4x^3 \cdot e^{4-4y}} \qquad \begin{array}{l} x = -2 \\ y = 1 \end{array}$$

$$\Rightarrow y' = \frac{-3(-2)^2 e^0 - 2(-2)}{12 \cdot 1 - 4(-2)^3 \cdot e^0} = \frac{-12 + 4}{12 + 32} = \frac{-8}{44} = \underline{\underline{\frac{-2}{11}}}$$

11/27/2018

**Thm** If $f$ is continuous on $[a,b]$, and if it has only finitely many jump discontinuities (left- and right limits both exist), then $f$ is integrable.

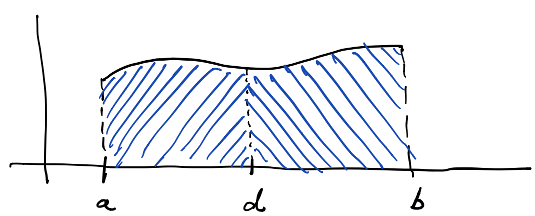



## Properties of definite integrals

**Thm**
$$\int_a^b c\, f(x)\,dx = c\int_a^b f(x)\,dx$$

$$\int_a^b \big(f(x) \pm g(x)\big)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

$$\int_a^b f(x)\,dx = \int_a^d f(x)\,dx + \int_d^b f(x)\,dx$$




$a<b$
$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

## Indefinite integrals.

Convenient notation for the most general antiderivative.

$$\int f(x)\,dx = F(x) + C$$

**Note:** A definite integral $\int_a^b f(x)\,dx$ is a number.

An indefinite integral $\int f(x)\,dx$ is a function.

(general antiderivative).

**Ex** $\int (x^2+x)\,dx = \frac{x^3}{3} + \frac{x^2}{2} + C$

**Interlude:** Derivatives of inverse trigonometric functions.

$\arcsin x = \sin^{-1} x$ $\quad (\sin(\arcsin x) = x, \ \arcsin(\sin x) = x)$.

Find $(\arcsin x)' = ?$

$x = \sin(\arcsin x)$.

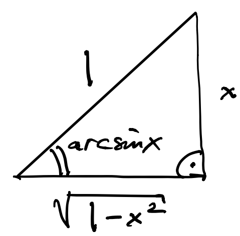
differentiate $\Downarrow$

$1 = \cos(\arcsin x) \cdot (\arcsin x)'$

$\Downarrow$

$(\arcsin x)' = \frac{1}{\cos(\arcsin x)}$

$= \frac{1}{\sqrt{1-x^2}}$

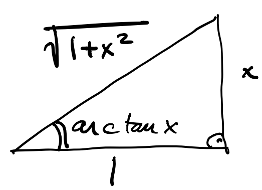



$\cos(\arcsin x) = \frac{\sqrt{1-x^2}}{1}$

Similarly,

$(\arccos x)' = \frac{-1}{\sqrt{1-x^2}}$

$(\arctan x)' = \frac{1}{1+x^2}$




because: $x = \tan(\arctan x)$

differentiate $\Downarrow$

$\cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}$

$1 = \frac{1}{\cos^2(\arctan x)} \cdot (\arctan x)'$

$\Downarrow$

$(\arctan x)' = \cos^2(\arctan x) = \frac{1}{1+x^2}$

**Ex**

$$\int a^x\,dx = \frac{1}{\ln a}\,a^x + C$$

$$\int \frac{1}{x}\,dx = \ln x + C, \qquad x>0$$

$$\int x^r\,dx = \frac{x^{r+1}}{r+1} + C, \qquad r \neq -1$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

$$\int \tan x\,dx = -\ln(\cos x) + C, \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

$$\int \ln x\,dx = x\ln x - x + C$$

$$\int \frac{-1}{\sqrt{1-x^2}}\,dx = \arccos x + C$$

Interpretation of FToC, part II

**Thm** (Net change theorem).

$$\underbrace{\int_a^b f'(x)\,dx}_{\substack{\text{rate of change,}\\ \text{integrated from}\\ a \text{ to } b}} = \underbrace{f(b)-f(a)}_{\substack{\text{total change of } f\\ \text{going from } a \text{ to } b.}}$$

## The substitution rule.

Helps to solve an integral where the function was obtained by differentiation using chain rule.

$$\int g'(f(x))\cdot f'(x)\,dx = g(f(x)) + C$$

because by chain rule, $\big(g(f(x))+C\big)' = g'(f(x))\cdot f'(x)$.

$$\Rightarrow \int_a^b g'(f(x))\, f'(x)\, dx = g(f(b)) - g(f(a)).$$

**Ex** $\displaystyle\int \frac{\cos x}{\sin^2 x}\, dx = \int g'(f(x))\, f'(x)\, dx = \int \frac{1}{\sin^2 x} \cos x\, dx$

$f(x) = \sin x \Rightarrow f'(x) = \cos x.$

$g'(f(x)) = \dfrac{1}{\sin^2 x} \Rightarrow g'(x) = \dfrac{1}{x^2} \Rightarrow g(x) = -\dfrac{1}{x} + C$

$$\Rightarrow \int \frac{\cos x}{\sin^2 x}\, dx = g(f(x)) + C = -\frac{1}{\sin x} + C$$

Substitution rule :

$$\int \underbrace{g'(f(x))}_{} \cdot \underbrace{f'(x)\, dx}_{}$$

$$\int g'(u) \quad du = g(u) + C$$

$$= g(f(x)) + C$$

$u = f(x)$

$du = f'(x)\, dx$ (differential).

11/29/2018

**Ex** $\displaystyle\int \frac{1}{x} \frac{1}{(\ln x)^3}\, dx = \int \frac{1}{u^3}\, du$

$$= -\frac{1}{2u^2} + C$$

$$= -\frac{1}{2(\ln x)^2} + C$$

$u = \ln x$

$du = \frac{1}{x}\, dx$

**Ex** $\displaystyle\int \sin^7 x \cos x\, dx = \int u^7\, du$

$$= \frac{u^8}{8} + C$$

$$= \frac{\sin^8 x}{8} + C$$

$u = \sin x$

$du = \cos x\, dx$

**Ex** $\int x^5 \cos(x^6+1)\,dx$

$$= \int \cos u \,\tfrac{1}{6}\,du$$

$$= \tfrac{1}{6}\sin u + C$$

$$= \tfrac{1}{6}\sin(x^6+1) + C$$

$u = x^6+1$

$du = 6x^5\,dx$

$\downarrow$

$\tfrac{1}{6}du = x^5 dx$

**Ex** $\int \sqrt{1+x^2}\; x^5\,dx$

$u = 1+x^2$

$du = 2x\,dx$

$$= \tfrac{1}{2}\int \sqrt{\underbrace{1+x^2}_{u}}\; x^4\; \underbrace{2x\,dx}_{du}$$

$x^2 = u-1$

$x^4 = (u-1)^2$

$= u^2-2u+1$

$$= \tfrac{1}{2}\int \underbrace{\sqrt{u}}_{\overset{\shortparallel}{u^{1/2}}}\,(u^2-2u+1)\,du$$

$$= \tfrac{1}{2}\int \left(u^{5/2} - 2u^{3/2} + u^{1/2}\right) du$$

$$= \tfrac{1}{2}\left(\tfrac{2}{7}u^{7/2} - 2\,\tfrac{2}{5}u^{5/2} + \tfrac{2}{3}u^{3/2}\right) + C$$

$$= \tfrac{1}{7}u^{7/2} - \tfrac{2}{5}u^{5/2} + \tfrac{1}{3}u^{3/2} + C$$

$$= \tfrac{1}{7}(1+x^2)^{7/2} - \tfrac{2}{5}(1+x^2)^{5/2} + \tfrac{1}{3}(1+x^2)^{3/2} + C$$

## Substitution for definite integrals.

**Thm** $\int_a^b g'(\underbrace{f(x)}_{u})\,\underbrace{f'(x)\,dx}_{du} = g(f(b)) - g(f(a))$

$u = f(x)$

$du = f'(x)\,dx$

$x=a \Rightarrow u = f(a)$

$x=b \Rightarrow u = f(b)$

$$\int_{f(a)}^{f(b)} g'(u)\,du = g\Big|_{f(a)}^{f(b)} \;\; \shortparallel \;\; g(f(b)) - g(f(a))$$

**Ex** $\displaystyle\int_0^4 \sqrt{2x+1}\,dx = \frac{1}{2}\int_1^9 \sqrt{u}\,du$

$u = 2x+1$
$du = 2\,dx$

$\displaystyle = \frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_1^9$

$x=0 \Rightarrow u=1$
$x=4 \Rightarrow u=9$

$\displaystyle = \frac{1}{3}\left(9^{3/2} - 1^{3/2}\right)$

$\displaystyle = \frac{1}{3}(27-1) = \underline{\frac{26}{3}}$ .

**Ex** $\displaystyle\int_1^2 \frac{1}{(3-5x)^2}\,dx = \frac{-1}{5}\int_{-2}^{-7}\frac{1}{u^2}\,du$

$u = 3-5x$
$du = -5\,dx$

$\displaystyle = \frac{1}{5}\int_{-7}^{-2}\frac{1}{u^2}\,du$

$x=1 \Rightarrow u=-2$
$x=2 \Rightarrow u=-7$

$\displaystyle = \frac{1}{5}\,\frac{-1}{u}\Big|_{-7}^{-2} = \frac{1}{5}\left(\frac{-1}{-2} - \frac{-1}{-7}\right)$

$\displaystyle = \frac{1}{5}\left(\frac{1}{2}-\frac{1}{7}\right) = \frac{1}{5}\,\frac{7-2}{14} = \underline{\frac{1}{14}}$ .

**Ex** $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/2}\frac{1}{\sqrt{1-\sin^2 u}}\cos u\,du$

$x = \sin u$
$(\Leftrightarrow u = \arcsin x)$

$dx = \cos u\,du$

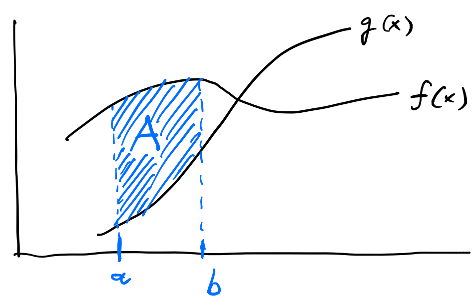
$\displaystyle = \int_0^{\pi/2}\frac{1}{\sqrt{\cos^2 u}}\cos u\,du$

$x=0 \Rightarrow \sin u = 0 \Rightarrow u=0$
$x=1 \Rightarrow \sin u = 1 \Rightarrow u = \frac{\pi}{2}$

$\displaystyle = \int_0^{\pi/2}\frac{1}{\cancel{\cos u}}\cancel{\cos u}\,du$

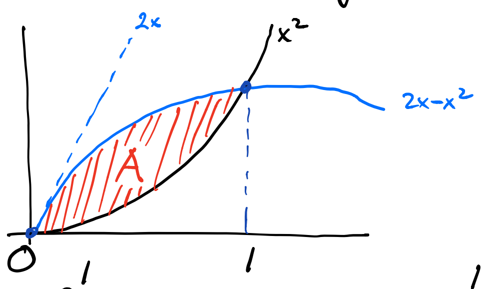
$\displaystyle = \int_0^{\pi/2} du = u\Big|_0^{\pi/2} = \frac{\pi}{2} - 0 = \underline{\frac{\pi}{2}}$

## Areas between curves




$$A = \int_a^b \left(f(x) - g(x)\right)dx$$

**Ex** Find the area enclosed by $y = x^2$ and $y = 2x - x^2$, $x \geqslant 0$




Intersection points:

$$x^2 = 2x - x^2$$

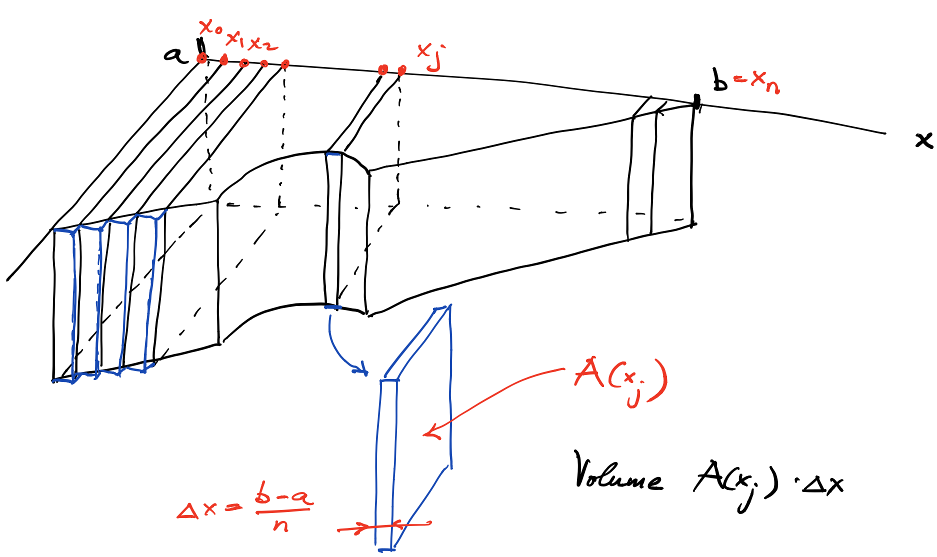
$$2x^2 = 2x \qquad x = 0, 1$$

$$A = \int_0^1 (2x - x^2 - x^2)\,dx = \int_0^1 (2x - 2x^2)\,dx$$

$$= \left(\cancel{2}\frac{x^2}{\cancel{2}} - 2\frac{x^3}{3}\right)\Big|_0^1$$

$$= 1 - \frac{2}{3} - 0 = \underline{\underline{\frac{1}{3}}}$$

## Volumes



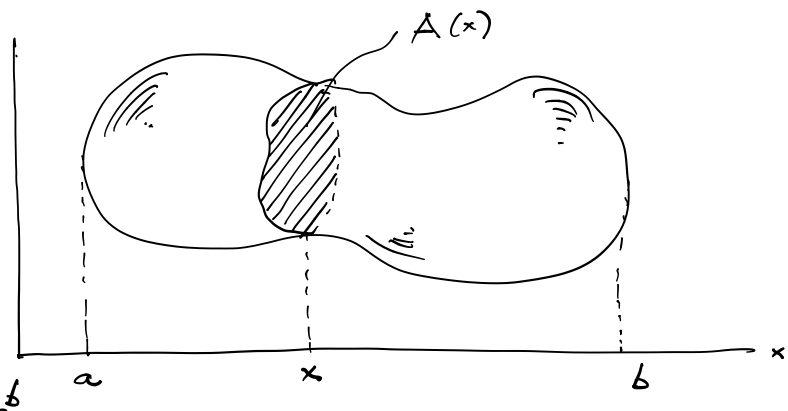

Volume $A(x_j) \cdot \Delta x$

$$\text{Volume} = \lim_{n\to\infty}\underbrace{\left(A(x_1)\Delta x + A(x_2)\Delta x + \cdots + A(x_n)\Delta x\right)}_{\text{Riemann sum.}}$$

$$= \int_a^b A(x)\,dx$$

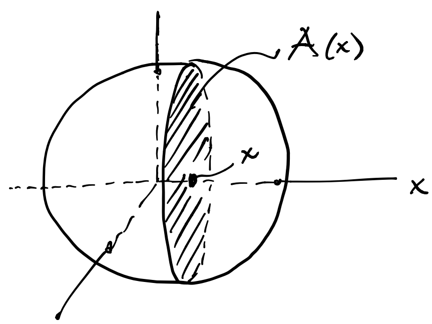
12/4/2018

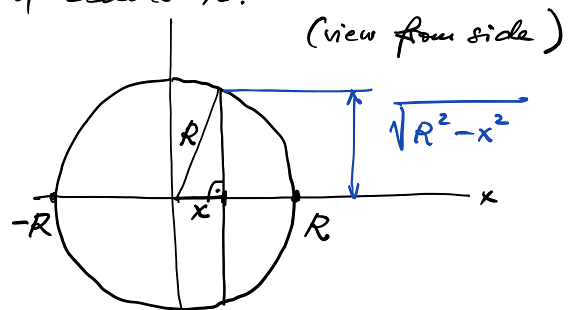
In general




$$\text{Volume} = \int_a^b A(x)\,dx$$

Ex Find the volume of a ball of radius $R$.



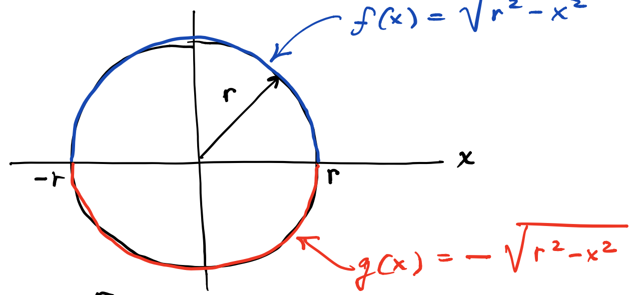

$$A(x) = \pi\left(\sqrt{R^2-x^2}\right)^2 = \pi\left(R^2-x^2\right)$$

$$\text{Volume} = \int_{-R}^{R} \pi\left(R^2-x^2\right)dx$$

$$= \pi\left(R^2 x - \frac{x^3}{3}\right)\Bigg|_{-R}^{R} = \pi\left(\underbrace{\left(R^2\cdot R - \frac{R^3}{3}\right)}_{\frac{2}{3}R^3} - \underbrace{\left(R^2\cdot(-R) - \frac{(-R)^3}{3}\right)}_{-\frac{2}{3}R^3}\right)$$

$$= \pi\cdot\frac{4}{3}R^3 = \frac{4\pi}{3}R^3$$

__Ex__ Find area of




$$\text{Area} = \int_{-r}^{r} \big(f(x) - g(x)\big)\,dx = \int_{-r}^{r}\Big(\sqrt{r^2-x^2} - \big(-\sqrt{r^2-x^2}\big)\Big)\,dx$$

$$= 2\int_{-r}^{r} \sqrt{r^2-x^2}\,dx.$$

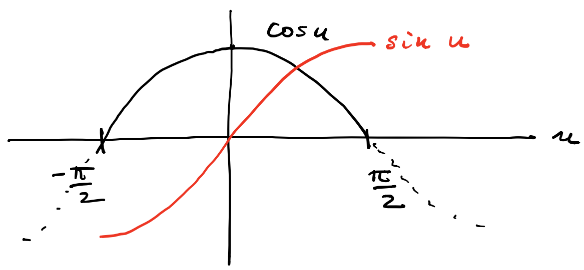
$x = r \sin u$

$dx = r \cos u \, du$

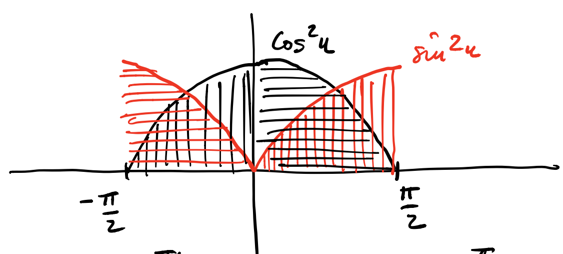
$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{\sqrt{r^2 - r^2\sin^2 u}}_{\sqrt{r^2(1-\sin^2 u)} = \sqrt{r^2\cos^2 u} = r\cos u} \; r\cdot\cos u\,du$$

$$= 2r^2 \int_{-\frac{\pi}{2}}^{\pi/2} \cos^2 u\,du.$$

$x = -r \Rightarrow \sin u = -1$
$\Rightarrow u = -\frac{\pi}{2}$
$x = r \Rightarrow \sin u = 1$
$\Rightarrow u = \frac{\pi}{2}$




$\Downarrow$ square




area equal to

$$\int_{-\pi/2}^{\pi/2} \cos^2 u\,du = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 u\,du$$

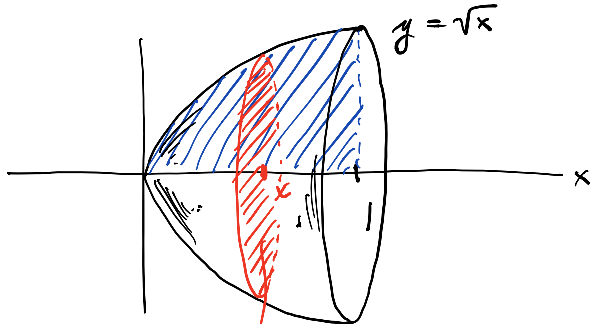
$$\Rightarrow \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 u\,du = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 u\,du + \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 u\,du$$

$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\pi/2} \underbrace{(\cos^2 u + \sin^2 u)}_{1}\, du$$

$$= \frac{1}{2}\, u \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{2}\Big(\underbrace{\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)}_{\pi}\Big) = \frac{\pi}{2}$$

$$\Rightarrow \text{Area} = 2r^2 \cdot \int_{-\frac{\pi}{2}}^{\pi/2} \cos^2 u\, du = 2r^2 \cdot \frac{\pi}{2} = \pi r^2$$

**Ex**



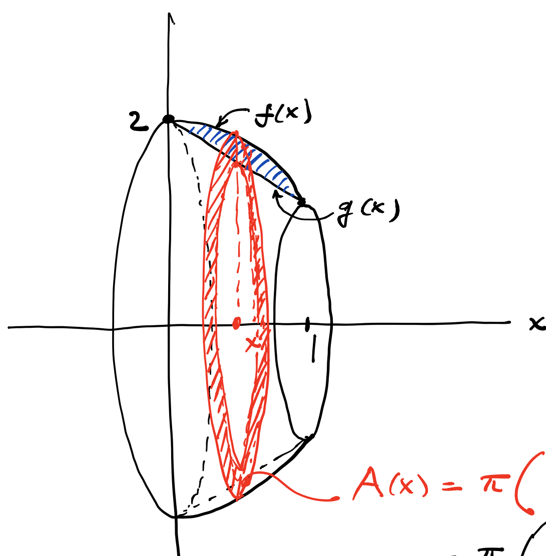

Rotate area [blue hatching] underneath $y = \sqrt{x}$ around x-axis.

$A(x) = \pi(\sqrt{x})^2 = \pi x$.

$$\text{Volume} = \int_0^1 \pi x\, dx$$

$$= \pi \frac{x^2}{2}\Big|_0^1 = \pi\left(\frac{1}{2} - \frac{0}{2}\right) = \underline{\underline{\frac{\pi}{2}}}.$$

**Ex**




$f(x) = 2 - x^2$
$g(x) = 2 - x$

rotate area [blue hatching] around x-axis.

$$A(x) = \pi(2-x^2)^2 - \pi(2-x)^2$$

$$= \pi\left(4 - 4x^2 + x^4 - (4 - 4x + x^2)\right)$$

$$= \pi\left(\cancel{4} - 4x^2 + x^4 - \cancel{4} + 4x - x^2\right)$$

$$= \pi\left(4x - 5x^2 + x^4\right)$$

$$\text{Volume} = \int_0^1 \pi\left(4x - 5x^2 + x^4\right)dx$$

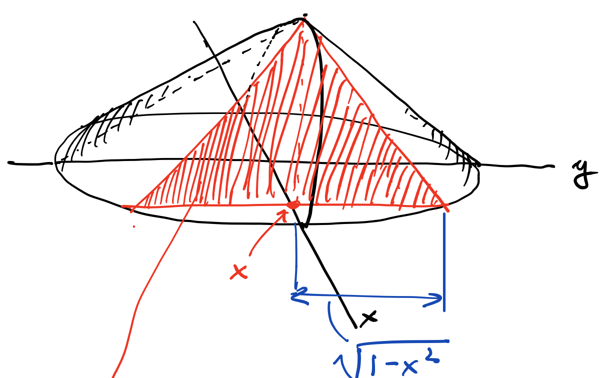
$$= \pi\left(4\frac{x^2}{2} - 5\frac{x^3}{3} + \frac{x^5}{5}\right)\Big|_0^1$$

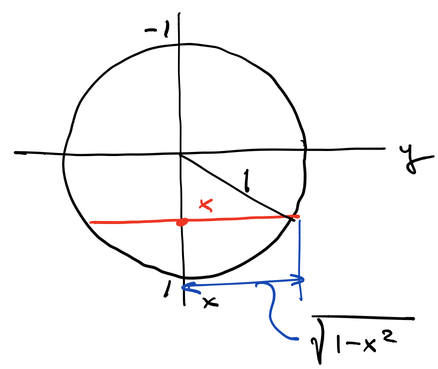
$$= \pi\left(2 - \frac{5}{3} + \frac{1}{5} - 0\right) = \pi\,\frac{30-25+3}{15} = \underline{\underline{\frac{8\pi}{15}}}$$

<u>Ex</u> Solid with a circular base of radius 1

Cross-sections perpendicular to x-axis are equilateral triangles.

From top:




$$A(x) = \underset{\substack{\uparrow\\ \text{triangle}}}{\frac{1}{2}}\,\underbrace{2\sqrt{1-x^2}}_{\text{base}} \cdot \underbrace{\frac{\sqrt{3}}{2} \cdot 2\sqrt{1-x^2}}_{\text{height of equilateral triangle.}}$$

$$= \sqrt{3}\cdot\left(1-x^2\right)$$

$$\text{Volume} = \int_{-1}^{1}\sqrt{3}\left(1-x^2\right)dx = \sqrt{3}\left(x - \frac{x^3}{3}\right)\Big|_{-1}^{1}$$

$$= \sqrt{3}\Big(\underbrace{\left(1 - \frac{1}{3}\right)}_{\frac{2}{3}} - \underbrace{\left(-1 - \frac{(-1)^3}{3}\right)}_{-\frac{2}{3}}\Big) = \underline{\underline{\sqrt{3}\cdot\frac{4}{3}}}$$

Final exam, Dec 15, 7-10 PM (Saturday).

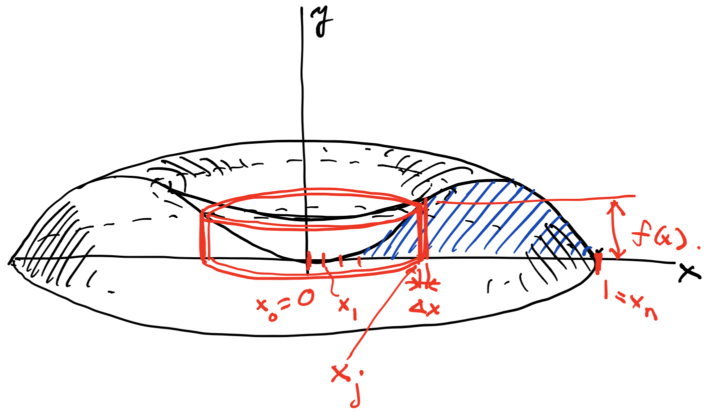
<u>YES</u>: Pencil, Eraser, Scratch paper, Tissues, Water, 3 pages of notes

<u>NO</u>: Calculator, Computer, etc

# Volumes using cylindrical shells.

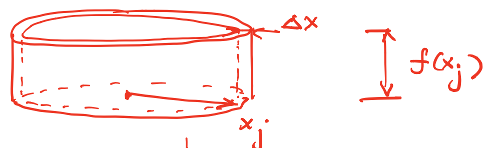
12/6/2018

**Ex**



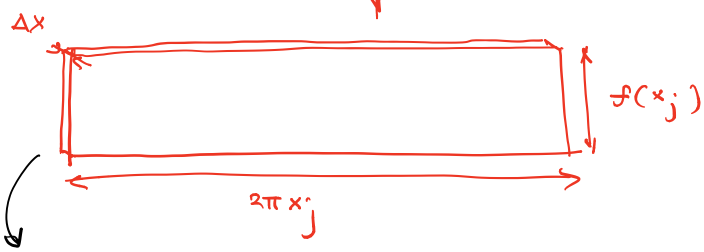

$f(x) = 3x^2 - 3x^3$
$0 \leq x \leq 1.$

rotate ////// around y-axis.




circumference $2\pi x_j$

cut and unfold




Volume $2\pi x_j \cdot f(x_j) \Delta x$

Total volume $= 2\pi \left( x_1 f(x_1) \Delta x + x_2 f(x_2) \Delta x + \dots + x_n f(x_n) \Delta x \right)$ ← Riemann Sum.

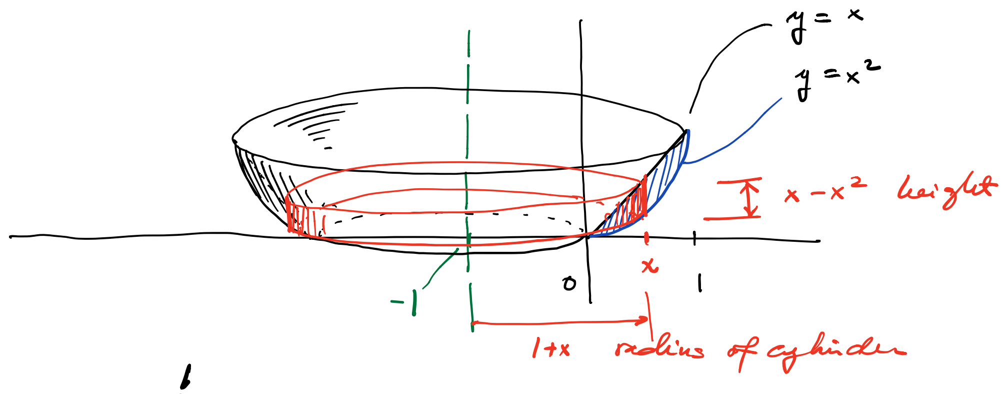
$$\xrightarrow{n \to \infty} 2\pi \int_0^1 x f(x)\, dx$$

$$\text{Volume} = 2\pi \int_0^1 (3x^2 - 3x^3)\, x \, dx$$

$$= 2\pi \int_0^1 (3x^3 - 3x^4)\, dx = 2\pi \left( 3\frac{x^4}{4} - 3\frac{x^5}{5} \right) \Big|_0^1$$

$$= 6\pi \left( \underbrace{\frac{1}{4} - \frac{1}{5}}_{\frac{5-4}{20}} - 0 \right) = \underline{\underline{\frac{3\pi}{10}}}$$

$\underline{\underline{Ex}}$

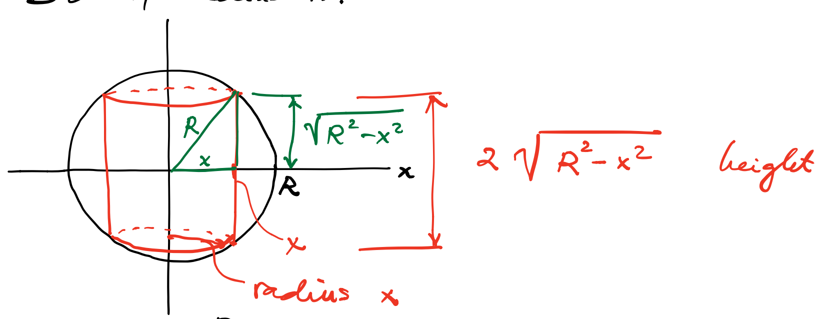



$$\text{Volume} = 2\pi \int_0^1 \underbrace{(1+x)}_{\text{radius}} \underbrace{(x - x^2)}_{\text{height}} \, dx$$

$$= 2\pi \int_0^1 (x - \cancel{x^2} + \cancel{x^2} - x^3) \, dx$$

$$= 2\pi \left( \frac{x^2}{2} - \frac{x^4}{4} \right) \Big|_0^1 = 2\pi \Big( \underbrace{\frac{1}{2} - \frac{1}{4}}_{\frac{1}{4}} - 0 \Big) = \underline{\underline{\frac{\pi}{2}}}$$

$\underline{\underline{Ex}}$ Ball of radius $R$.




$$\text{Volume} = 2\pi \int_0^R \underset{\substack{\uparrow \\ \text{radius}}}{x} \, \underbrace{2\sqrt{R^2 - x^2}}_{\text{height.}} \, dx$$

$$= 2\pi \int_0^R 2x \sqrt{R^2 - x^2} \, dx$$

$$= -2\pi \int_{R^2}^0 u^{1/2} \, du$$

$$= 2\pi \int_0^{R^2} u^{1/2} \, du$$

$u = R^2 - x^2$

$du = -2x\,dx$

$2x\,dx = -du$

$x = 0 \Rightarrow u = R^2$

$x = R \Rightarrow u = 0$

$$= 2\pi \frac{2}{3} u^{3/2} \Big|_0^{R^2} = \frac{4\pi}{3}\Big(\underbrace{(R^2)^{3/2}}_{R^3} - 0\Big) = \frac{4\pi}{3} R^3$$

---

**Ex** FT. C.

$$K(x) = \int_{g(x)}^{h(x)} f(t)\,dt = F(h(x)) - F(g(x))$$

where $F$ is an antiderivative of $f$ : $F' = f$.

$K'(x) = ?$

$$\Rightarrow \boxed{K'(x)} = F'(h(x)) \cdot h'(x) - F'(g(x)) \cdot g'(x).$$

$$= \boxed{f(h(x)) \cdot h'(x) - f(g(x)) \cdot g'(x).}$$

$\Rightarrow$ to obtain $K'(x)$, we can directly use $f$. No need to first find $F$, and then differentiate.

**Ex**
$$K(x) = \int_{x^2}^{x^4} \ln(\cos t)\,dt$$

$$K'(x) = \big(\ln(\cos(x^4))\big) \cdot 4x^3 - \big(\ln(\cos(x^2))\big) \cdot 2x$$

**Ex**
$$\lim_{x\to\infty} \left(\sqrt{x^2+1} - \sqrt{x^2+2}\right)$$

$$= \lim_{x\to\infty} \left(\sqrt{x^2+1} - \sqrt{x^2+2}\right) \frac{\sqrt{x^2+1} + \sqrt{x^2+2}}{\sqrt{x^2+1} + \sqrt{x^2+2}}$$

$$= \lim_{x\to\infty} \frac{(\sqrt{x^2+1})^2 - (\sqrt{x^2+2})^2}{\sqrt{x^2+1} + \sqrt{x^2+2}} = \lim_{x\to\infty} \frac{\cancel{x^2}+1-(\cancel{x^2}+2)}{\sqrt{x^2+1} + \sqrt{x^2+2}}$$

$$= \lim_{x\to\infty} \frac{-1}{\sqrt{x^2+1} + \sqrt{x^2+2}} = 0$$

**Ex** $f(x) = x^{x^2}$ $\Rightarrow$ find $f'(x) = ?$

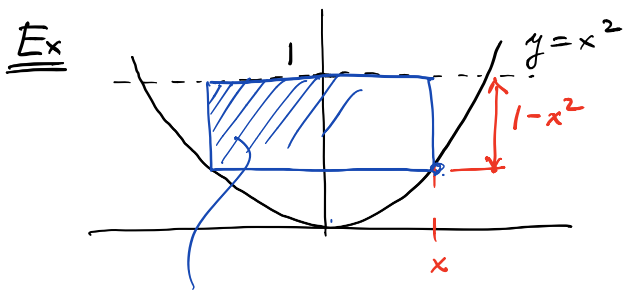
$$f(x) = \Big(\underbrace{e^{\ln x}}_{x}\Big)^{x^2} = e^{x^2 \ln x}.$$

$$f'(x) = e^{x^2 \ln x} \cdot \Big(2x \ln x + \underbrace{x^2 \cdot \frac{1}{x}}_{x}\Big).$$

$$= x^{x^2} \cdot x\,(2\ln x + 1).$$

$$= x^{x^2+1}\,(2\ln x + 1)$$

**Ex**




Find rectangle with largest area.

$$A(x) = 2x(1-x^2) = 2x - 2x^3$$

$$A'(x) = 2 - 6x^2 = 0 \Rightarrow x^2 = \frac{1}{3} \Rightarrow x = \pm\frac{1}{\sqrt{3}}$$

$$A''(x) = -12x$$

$$\Rightarrow A''\Big(\frac{1}{\sqrt{3}}\Big) = -\frac{12}{\sqrt{3}} < 0 \Rightarrow \text{local max}.$$