Linear stability and instability of self-interacting spinor field

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October, 2013

Einstein: $E^2 = p^2 + m^2$, Schrödinger: $(i\partial_t)^2\psi = (-i\nabla)^2\psi + m^2\psi$

$E = \sqrt{m^2 + p^2} \approx m + \frac{p^2}{2m}$, [*Schrödinger*[26]]: $i\partial_t\psi = \frac{1}{2m}(-i\nabla)^2\psi$

[*Dirac*[28]]: $E = \sqrt{p^2 + m^2} = \alpha \cdot p + \beta m$,

$$i\partial_t\psi = \underbrace{(-i\alpha \cdot \nabla + \beta m)}_{D_m}\psi, \qquad \psi(x,t) \in \mathbb{C}^4, \quad x \in \mathbb{R}^3$$

α_j $(1 \le j \le 3)$ and β are self-adjoint and such that $D_m^2 = -\Delta + m^2$

Standard choice: $\alpha_j = \begin{bmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{bmatrix}$ (Pauli matrices), $\beta = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix}$

Self-interacting spinors

Models of self-interacting spinor field:
[*Ivanenko*[38], *Finkelstein et al.*[51], *Finkelstein et al.*[56], *Heisenberg*[57]] [...]

Massive Thirring model [*Thirring*[58]] in (n+1)D:

$$\mathscr{L}_{\mathrm{MTM}} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + \left(\bar{\psi}\gamma_\mu\psi \ \bar{\psi}\gamma^\mu\psi\right)^{\frac{k+1}{2}} \qquad k > 0 \tag{V-V}$$

Soler model [*Soler*[70]] in (n+1)D:

$$\mathscr{L}_{\mathrm{Soler}} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + (\bar{\psi}\psi)^{k+1} \tag{S-S}$$

In (1+1)D: massive Gross-Neveu model [*Gross & Neveu*[74], *Lee & Gavrielides*[75]]

Soler model: NLD with scalar-scalar self-interaction

$$i\partial_t\psi = \underbrace{(-i\alpha\cdot\nabla + m\beta)}_{D_m}\psi - (\bar\psi\psi)^k\beta\psi, \qquad \psi\in\mathbb{C}^N, \quad x\in\mathbb{R}^n$$

- [*Soler*[70], *Cazenave & Vázquez*[86]]: existence of solitary waves in $\mathbb{R}^3$,

$$\psi(x,t) = \phi_\omega(x)e^{-i\omega t}, \qquad \omega\in(0,m)$$

- Attempts at stability: [*Bogolubsky*[79], *Alvarez & Soler*[86], *Strauss & Vázquez*[86]] ...

- Numerics [*Alvarez & Carreras*[81], *Alvarez & Soler*[83], *Berkolaiko & Comech*[12]] suggest that (all?) solitary waves in 1D cubic Soler model are stable

- Assuming *linear stability*, one tries to prove asymptotic stability [*Pelinovsky & Stefanov*[12], *Boussaid & Cuccagna*[12]]

Nonrelativistic limit of NLD: $\omega \to m$

Solitary wave: $\psi(x,t) = \begin{bmatrix} \phi(x) \\ \rho(x) \end{bmatrix} e^{-i\omega t}$; $\qquad \phi,\ \rho \in \mathbb{C}^2$

$$i\dot{\psi} = \Big(-i \begin{bmatrix} 0 & \sigma \cdot \nabla \\ \sigma \cdot \nabla & 0 \end{bmatrix} + m\beta \Big)\psi - (\bar{\psi}\psi)^k \beta\psi,$$

$$\omega \begin{bmatrix} \phi \\ \rho \end{bmatrix} \approx -i\sigma \cdot \nabla \begin{bmatrix} \rho \\ \phi \end{bmatrix} + m \begin{bmatrix} \phi \\ -\rho \end{bmatrix} - |\phi|^{2k} \begin{bmatrix} \phi \\ -\rho \end{bmatrix}$$

If $\omega \lesssim m$: $\quad 2m\rho \approx -i\sigma \cdot \nabla\phi$, $\quad \phi$ satisfies NLS:

$$-(m-\omega)\phi = -\frac{1}{2m}\Delta\phi - |\phi|^{2k}\phi.$$

Scaling: $\phi(x) = \epsilon^{1/k}\Phi(\epsilon x)$, $\qquad \epsilon = \sqrt{m-\omega}$

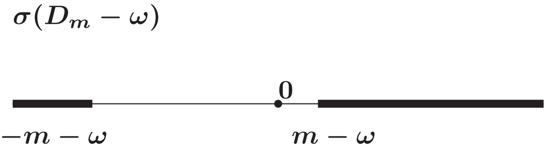
$$-\Phi = -\frac{1}{2m}\Delta\Phi - |\Phi|^{2k}\Phi$$

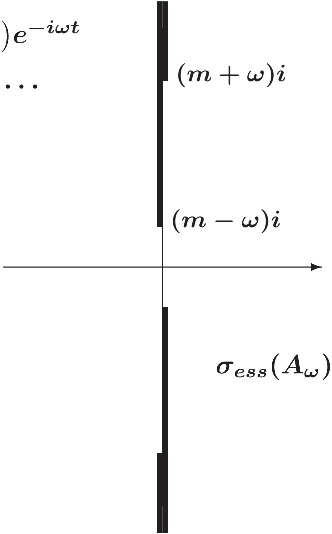
NLD: linearization at a solitary wave

Given $\phi_\omega(x)e^{-i\omega t}$, consider $\psi(x,t) = \big(\phi_\omega(x) + r(x,t)\big)e^{-i\omega t}$

Linearized eqn on $r(x,t) \in \mathbb{C}^N$, $\quad i\partial_t r = D_m r - \omega r + \dots$

$$\partial_t \begin{bmatrix} \mathrm{Re}\, r \\ \mathrm{Im}\, r \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & D_m - \omega + \dots \\ -D_m + \omega + \dots & 0 \end{bmatrix}}_{A_\omega} \begin{bmatrix} \mathrm{Re}\, r \\ \mathrm{Im}\, r \end{bmatrix}$$

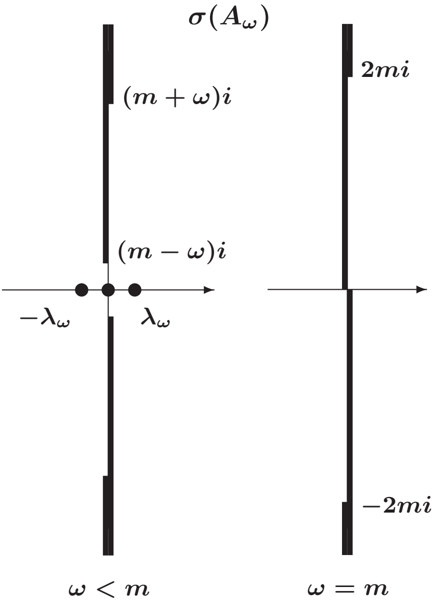




Linear instability of NLD

Theorem 1 ([*Comech & Guan & Gustafson*[12]]).

> If NLS$_k$ is linearly unstable, then
> for $\omega \lesssim m$, $\exists\ \ \pm\lambda_\omega \in \sigma_d(A_\omega)$,
> $$\mathrm{Re}\,\lambda_\omega > 0, \qquad \lambda_\omega \underset{\omega\to m}{\longrightarrow} 0$$



1D, above quintic

2D, above cubic

3D cubic and above

Proof: Rescale; use Rayleigh-Schrödinger perturbation theory. □

Linear stability of NLD

Theorem 2 ([*Boussaid & Comech*[12]]).

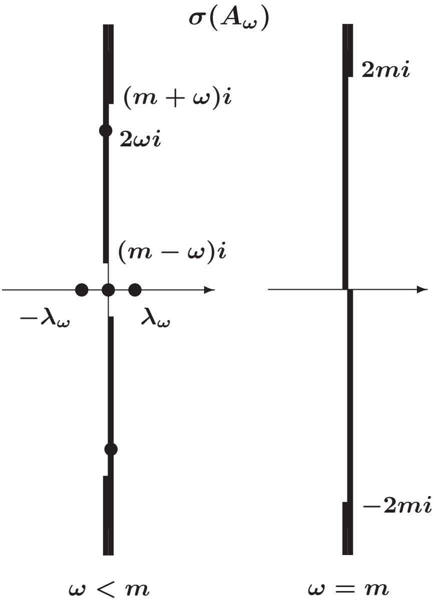
Assume $\lambda_\omega \in \sigma_p(A_\omega)$, $\omega \lesssim m$

1. $\lambda_\omega \underset{\omega\to m}{\longrightarrow} \{0\,;\ \pm 2mi\}$.

2. If $\mathrm{Re}\,\lambda_\omega \ne 0$, then $\lambda_\omega \underset{\omega\to m}{\longrightarrow} 0$,

$$\Lambda := \lim_{\omega\to m} \frac{\lambda_\omega}{(m-\omega)} \in \sigma_p(\mathrm{NLS}_k)$$

3. $\Lambda \ne 0$ unless critical case:
   2D quintic; 3D cubic



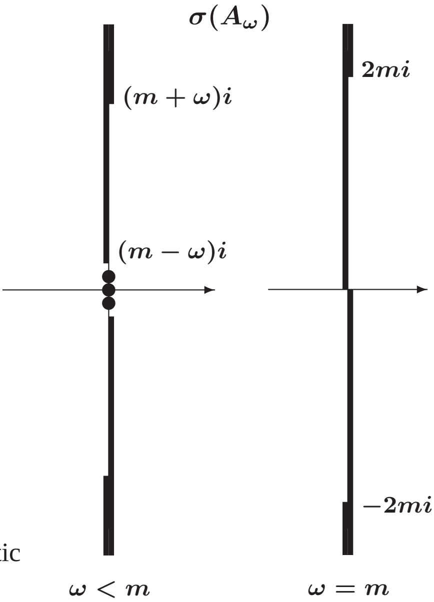
Proof: Limiting absorption principle [*Agmon*[75], *Berthier & Georgescu*[87]] □

Linear stability of NLD

Corollary 3 ([*Boussaid & Comech*[12]]).

> 1D cubic:
>
> $\phi_\omega e^{-i\omega t}$ are linearly stable for $\omega \lesssim m$

Remark 4. Also true for 1D cubic and 2D quintic ("charge-critical NLS")



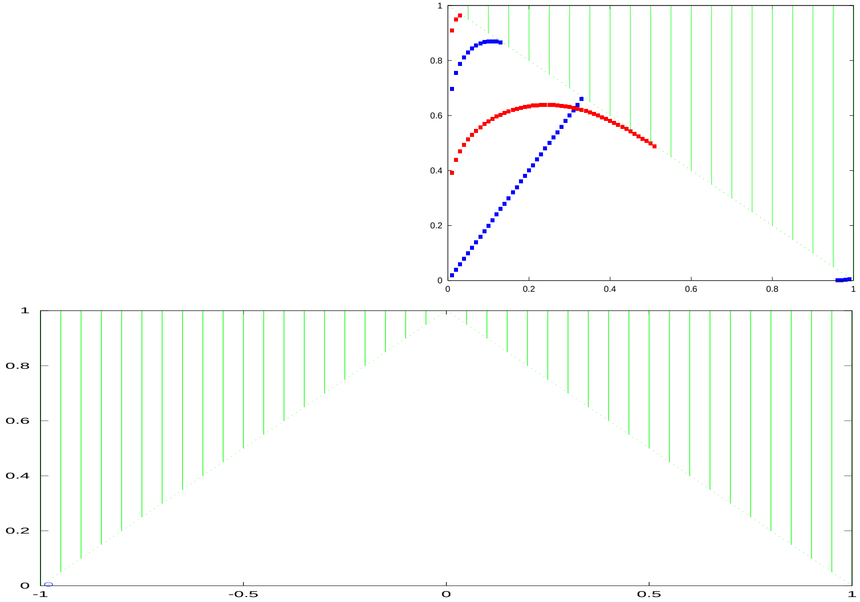


Figure 1: Upper half of the spectral gap. TOP: 1D cubic Soler
BOTTOM: 1D cubic massive Thirring

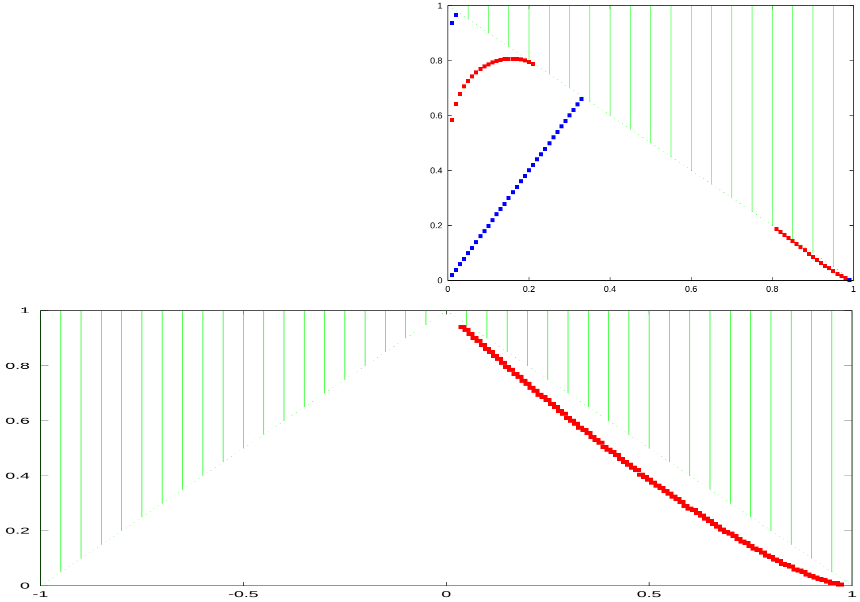


Figure 2: 1D quintic ("charge critical"). TOP: Soler; BOTTOM: massive Thirring

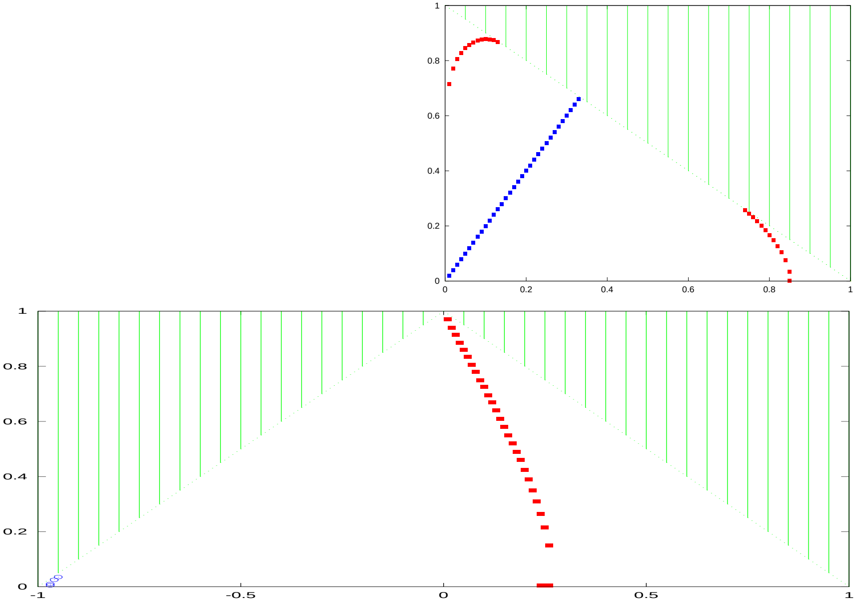


Figure 3: 1D, seventh order. Soler and MTM

Bifurcations from σ_{ess}

Let $\Omega \in (0, m)$

Theorem 5 ([*Boussaid & Comech*[12]]).

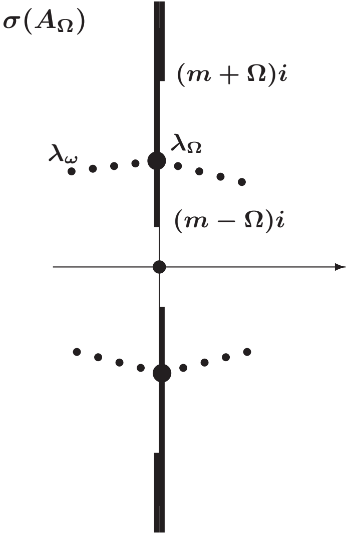
If $\lambda_\omega \in \sigma_p(A_\omega)$, $\operatorname{Re}\lambda_\omega \neq 0$,

$$\lambda_\omega \underset{\omega\to\Omega}{\longrightarrow} \lambda_\Omega \in i\mathbb{R}$$

then $\lambda_\Omega \in \sigma_p(A_\Omega)$, $|\lambda_\Omega| \le m + \Omega$

Moreover,

$$\lambda \in \sigma_p \cap i\mathbb{R} \quad \Rightarrow \quad |\lambda| \le m + |\Omega|$$



Theorem 6 ([[*Berkolaiko & Comech & Sukhtyaev*[13]]).

‖ $Q'(\omega) = 0$ and $E(\omega) = 0$ correspond to the boundary of the linear instability region

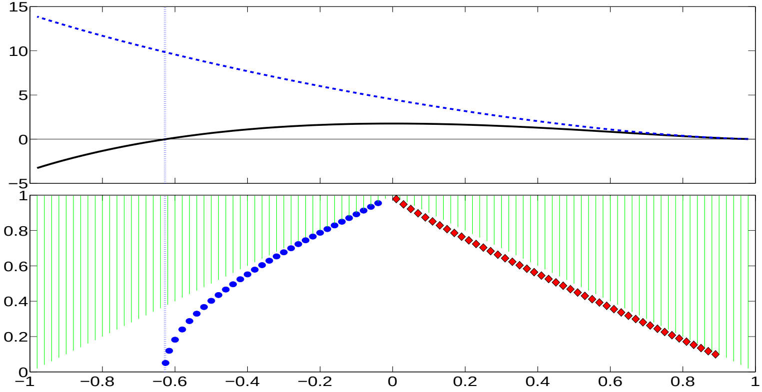


Figure 4: quadratic MTM. TOP: Charge (......) and energy (—) as functions of ω. BOTTOM: Purely imaginary eigenvalues (•, ♦) of the linearized equation in the spectral gap.

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