Character-automorphic Hardy classes in Widom domains and a solution of Kotani–Last's problem

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October, 2013

1. Domains without Cauchy formula for Smirnov class functions

Let $\Omega = \mathbb{C} \setminus E, E \subset \mathbb{R}$, be a multiply (infinitely) connected domain. E is a closed set of positive length. We deal with multiple-valued holomorphic (meromorphic) functions f in Ω such that $|f|$ is single-valued. Then of course for $\omega(\gamma) \in \mathbb{R}$, γ is a closed loop in Ω:

$$f \circ \gamma(z) = e^{2\pi i\omega(\gamma)} f(z),\ \ \gamma \in \Gamma = \text{fundamental group of } \ \Omega.$$

Then

$$\alpha(\gamma) := e^{2\pi i\omega(\gamma)} : \Gamma \to \mathbb{T}$$

is a character of fundamental group Γ. The group of characters will be called Γ^*. So our main object will be holomorphic (meromorphic) functions which are character-automorphic:

$$f \circ \gamma = \alpha(\gamma) f.$$

2. Smirnov class

Holomorphic character-automorphic function f in Ω is called of Smirnov class if it is of bounded characteristic, namely, $f = h_1/h_2$, h_i are bounded character-automorphic holomorphic functions in Ω, and on the top of that h_2 does not have inner part in its inner-outer factorization. If by $\mathfrak{z} : \mathbb{D} \to \Omega$ we denote the universal covering map, $\mathfrak{z}(0) = \infty, \mathfrak{z}'(0) > 0$. One may understand the inner-outer factorization in terms of inner-outer factorization (due to A. Beurling) of the analytic function $h = g \circ \mathfrak{z}$ in the disc $\mathbb{D}$. Hardy classes H^p of holomorphic functions with $|h|^p$ having finite harmonic majorant, acquire extra feature:
$h \circ \gamma(z) = \alpha(\gamma)h(z), z \in \mathbb{D}$, γ's are elements of Fuchsian group of Möbius maps of $\mathbb{D}$ to itself, we call this Fuchsian group Γ, it is isomorphic to fundamental group of Ω, and $\Omega = \mathbb{D}/\Gamma$. As before $\alpha \in \Gamma^*$, the group of characters. So $g \to h = g \circ \mathfrak{z}$ makes a single valued function h from multiple valued g, but h has some "periodicity" property in the disc. We of course call such functions character-automorphic (w.r.to Fuchsian Γ) in $\mathbb{D}$.

3. Cauchy formula

$$f(\zeta) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(\eta)}{\eta - \zeta}\, d\eta$$

is valid for many holomorphic functions in the disc, but not for all. Obviously we need $f(\eta) \in L^1(\mathbb{T}, m)$ (m is Lebesgue measure on $\mathbb{T}$). But this is not enough, ($M_s(\mathbb{T})$= singular measures on $\mathbb{T}$.)

$$h(z) = e^{\frac{1+z}{1-z}}, \text{ or } h(z) = e^{\int_{\mathbb{T}} \frac{1+ze^{i\theta}}{1-ze^{i\theta}}\, d\mu(\theta)}, \ \mu \in M_s(\mathbb{T})$$

are all in $L^\infty(\mathbb{T})$, moreover $|h(e^{i\phi}| = 1$ for m-a.e. $e^{i\phi} \in \mathbb{T}$ but the Cauchy formula is false for them. V.I. Smirnov found a simple necessary and sufficient condition for having Cauchy formula over the boundary: 1) $h \in L^1$(on the boundary), 2) $h \in$ Smirnov class in the domain. He did this for simply connected domains with finite length boundary. Finitely connected domains are ok too. Jumping ahead: some very good infinitely connected domains fail to have this property. These will be our main culprits.

4. Domains without Cauchy formula. No DCT domains

We saw that $h \in$ Smirnov class in Ω is crucial even for the simplest $\Omega = \mathbb{D}$. But there are simple and very good in all other respects domains $\Omega = \mathbb{C} \setminus E$, where the Cauchy formula fails for very good (Smirnov class and $L^1(\partial\Omega)$) functions. Here E will be a sequence of segments on $\mathbb{R}$ converging to 0, and also $[0,1] \subset E$. Then $|E| < \infty$, and we build the example Benedicks' theorem, 1980:

Theorem

Let $O = (\mathbb{C} \setminus [0,\infty)) \setminus \cup_{m=1}^{\infty}[-m-d_m, -m+d_m]$*, such that* $d_m \le 1/4, d_k \asymp d_m,\ k \asymp m$*. Consider Martin function*

$$M(z) := \lim_{y\to\infty} \frac{G(z,iy)+G(z,-iy)}{2G(0,iy)}$$

Then $M(z) \approx |y|$ *iff* $\sum_{m=1}^{\infty} \frac{\log 1/d_m}{m^2} < \infty$.

Growth $|y|$ is maximal possible for $\Omega \supset \mathbb{C}_+$. Martin functions are extremal points of the cone of positive harmonic functions in Ω.

5. This is counterintuitive.

The maximal growth should be reserved for "thick" boundaries at ∞, e. g. like $O = \mathbb{C} \setminus ([0,\infty) \cup (-\infty, -1])$. In Benedicks' theorem above, d_m can be chosen

$$d_m \approx e^{-\sqrt{m}}, \text{ or } d_m \approx e^{-m^{1-\delta}}$$

easily. This is the choice we will make. Domain with such d_m looks "almost" like $O = \mathbb{C} \setminus [0,\infty)$, for which Martin function has a much slower growth: $M(z) \approx \sqrt{|y|}$. Making the domain only slightly smaller with sub-exponentially small d_m as above "boosts" Martin function to $M(z) \approx |y|$. How to use this effect? Consider $c \in (-1 + d_1, 0)$ and map just constructed $O = \mathbb{C} \setminus E$ by $w = \frac{1}{z-c}$ onto $\Omega = \mathbb{C} \setminus \tilde{E}$, $\tilde{E} := w(E)$. It is a set formed by $[0, |c^{-1}|]$ and a sequence of sub-exp. small segments converging to 0, the length of the m-th segment is $\approx e^{\sqrt{m}}/m^2$.

6. Good function in Ω without Cauchy formula

Put $F(z) := \cos\sqrt{c} - \cos\sqrt{z}$, with an obvious choice of the branch of $\sqrt{z}$ it is analytic function in O. And as $F(x+i0) = F(x-i0), x \in E$ (we use that it is a cos!) and we use that $\approx e^{\sqrt{m}}/m^2$–smallness kills growth of cosh:

$$1)\ \int_E \frac{|F(x)|}{(x-c)^2}dx < \infty\,, 2)\ \oint \frac{F(x)}{(x-c)^2}dx = 0,\ 2)\ F(c) = 0, F'(c) \neq 0.$$

Changing variable we gat $\Phi(w) = F(\frac{1}{w} + c)$ in Ω with the compact boundary $\tilde{E}$ such that

$$1)\int_{\tilde{E}} |\Phi(u)|\, du < \infty, 2)\oint_{\tilde{E}} \Phi(u)du = 0, 3)\Phi(w) \approx \frac{F'(c)}{w} + O(\frac{1}{w^2}), w \approx \infty.$$

Put $G(w) := (w - w_0)\Phi(w) - F'(c), w_0 \in \Omega$. Then $G(\infty) = 0$ and $\int_{\tilde{E}} |G| du < \infty$, but Cauchy formula does **not**, however, hold:

$$\oint_{\tilde{E}} \frac{G(w)dw}{w - w_0} = \oint_{\tilde{E}} \Phi(w)dw = 0 \neq -F'(c) = G(w_0).$$

7. Why Φ and G are in Smirnov class in $\Omega = \mathbb{C} \setminus \tilde{E}$?

To be in Smirnov class is a conformal invariant property. So it is enough to check that $F(z) = \cos\sqrt{c} - \cos\sqrt{z} \in Smirnov(O)$. But

$F = C - \cos\sqrt{z}$; $\cos\sqrt{z} = \dfrac{e^{2i\sqrt{z}} + 1}{e^{i\sqrt{z}}}$, ratio of two bounded functions in O.

Notice that $\log|\text{Denominator}| \approx \sqrt{|y|} << M(z) \approx |y|$ by Benedicks' theorem. But the inner part of the Denominator $e^{i\sqrt{z}}$ can hide only at infinity and can be only of the type $e^{-a(M(z)+i\tilde{M}(z))}$, where $\tilde{M}$ is the harmonic conjugate to M and $a > 0$. If so, then it must be that $\log|\text{Denominator}| \approx a|y|$, $a > 0$. Contradiction.

8. Our Ω is a very good domain. It is a Widom domain.

Question: In which domains any character $\alpha \in \Gamma^*$ arises as a character of nice character automorphic function? Widom answered in Ann. of Math. 1971:

$$\forall \alpha \in \Gamma^* \exists h \in H^\infty(\alpha), h \neq 0 \text{ iff } \sum_{\nabla G(c)=0} G(c) < \infty.$$

Here $G(z) = G(z, a)$, we let $a = \infty \in \Omega$. Let $\{c_i\}$ be critical points of $G(z)$. So Ω is Widom iff the character-automorphic Blaschke product

$$\Delta_\Omega := e^{-\sum_{i=1}^\infty G(z,c_i) + i\widetilde{G}(z,c_i)} \text{ converges } z \in \Omega$$

One of the main player will be

$$\Delta := \Delta_\Omega \circ \mathfrak{z} : \mathbb{D} \to \mathbb{D}$$

the character-automorphic Blaschke product in $\mathbb{D}$. Its character will be denoted by letter ν, $\nu \in \Gamma^*$.

9. Widomness and finite entropy

Theorem

For a plain domain such that $E := \partial\Omega \subset \mathbb{R}$ TFAE: 1) Ω is a Widom domain, 2) there is a conformal map of $\mathbb{C}_+$ onto a comb domain such that E goes to its base, gaps go to "teeth", and the comb has locally rectifiable boundary, 3) the entropy of harmonic measure is finite: $\int_{\partial\Omega} \log \omega(x)\,\omega(x)dx < \infty$, ω being the density of $d\omega(x,\infty)$ with respect to the length dx, 4) $\int_0 Betti(G(z) > t)dt < \infty$.

Sketch of proof $1) \Rightarrow 3)$. Put $\omega(x) := \frac{\partial G}{\partial n}(x)$, $\Omega' = \Omega \setminus D(0,R)$. Then

$$\int_E \omega_0(x) \log \omega_0(x) dx = \int_E \frac{\partial G}{\partial n} \log \frac{\partial G}{\partial n} dx = Const + \int_E G \frac{\partial}{\partial n} \log \frac{\partial G}{\partial n} +$$

$$\int_{\Omega'} \Delta G \log |\nabla G(z)| - \int_{\Omega'} G \Delta \log |\nabla G(z)| = Const + \sum_{i=1}^{\infty} G(c_i, 0).$$

Theorem

1) $\inf_{\alpha \in \Gamma^*} \sup_{f \in H^\infty(\alpha), \|f\|_\infty \le 1} |f(0)| = |\Delta(0)| > 0$ *iff* Ω *is Widom.*
2) Let $\Omega = \mathbb{C} \setminus E, E \subset \mathbb{R}, E := [b_0, a_0] \setminus \cup_{j=1}^\infty (a_j, b_j), a_0 = 1.$
Then $\theta := -\tilde{G}(z) + iG(z)$ *is the conformal map of* $\mathbb{C}_+$ *onto comb* $(-\pi, 0, \infty)$ *with teeth of the height* $G(c_i)$*. It maps gaps* (a_j, b_j) *into* j*-th tooth of the comb.*

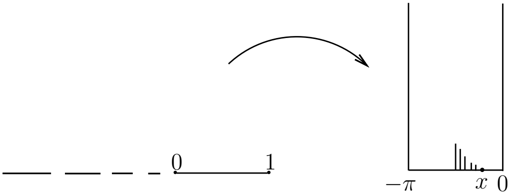
Notations. $b_\Omega := e^{i\theta(z)}$, $b = b_\Omega \circ \mathfrak{z}$. It is a ch.-autom. Blaschke product in $\mathbb{D}$ w.r. to Fuchsian group Γ:
$b \circ \gamma(\zeta) = \mu(\gamma) b(\zeta), \zeta \in \mathbb{D}, \gamma \in \Gamma$. Letter μ denotes the character of b. So

$$\Delta \in H^\infty(\nu),\ b \in H^\infty(\mu)$$

$$\mu(\gamma_j) =: e^{-2\pi i \omega_j},\ \omega_j = \omega_\Omega([b_j, a_0], \infty),$$

where γ_j corresponds to a loop going through the gap (a_j, b_j) and, say, point 2014.

10a. Picture of the conformal map $\theta : \mathbb{C}_+ \to$ Hedgehog



11. Group orbits

Fix $z_0 \in \Omega$. Put $orb(\zeta_0) = \mathfrak{z}^{-1}(z_0) = \{\gamma(\zeta_0)\}_{\gamma\in\Gamma}$ be the orbit of a point in $\mathbb{D}$ under the Fuchsian group. One can define the Blaschke product with zeros on this orbit: $\log|b_{z_0}(\zeta)|^{-1} = G(\mathfrak{z}(\zeta), z_0)$. If, as always, c_j are critical points of $G(z) = G(z, \infty)$, then

$$\Delta(\zeta) = \Pi_{j=1}^{\infty} b_{c_j}(\zeta),$$

$$b(\zeta) = b_\infty(\zeta).$$

Simple facts. $b_\Omega = e^{-G - i\tilde{G}}$, $b'_\Omega = |\nabla G|$ on E. Pommerenke: b' is Smirnov for Widom domains, the inner part of b'_Ω is Δ_Ω, the inner part of b' is Δ. Let ϕ be analytic ch.-automorphic in Ω. Then (denoting $D(\Omega) := Smirnov(\Omega)$)

$\phi \in D(\Omega) \Leftrightarrow \phi(b'_\Omega)_{out} \in D(\Omega) \Leftrightarrow \frac{\phi b'_\Omega}{\Delta_\Omega} \in D(\Omega) \Leftrightarrow \frac{\phi b'_\Omega}{\Delta_\Omega b_\Omega} \in D(\Omega) \Leftrightarrow$

In fact, b_Ω in denominator can only introduce Blaschke zeros, no singular inner parts. But the only zero got cancelled out:

$b'_\Omega(z) \approx \frac{1}{z^2}$, $b_\Omega(z) \approx \frac{1}{z}$ at ∞.

The display line can be rewritten in terms of $\mathbb{D}$ as in the following slide:

12. Change of variable

Let $f = \phi \circ \mathfrak{z}$, then $f \in H^1(\nu) \Leftrightarrow \int_{\mathbb{T}} |f| dm < \infty, f \in D(\mathbb{D})$. This is the same that

$$\phi \in D(\Omega), \text{ and } \int_E |\phi| d\omega(x) < \infty.$$

Therefore, this is the same that (recall that $|b'_\Omega| = |\nabla G|$ on E)

$$F := \frac{\phi b'_\Omega}{\Delta_\Omega b_\Omega} \in D(\Omega), \text{ and } \int_E |F| dx < \infty$$

Also change of variable in integral without absolute values gives $|b'_\Omega(x)| dx = d\omega(x) = dm(\theta)$ if $\mathfrak{z}(e^{i\theta}) = x$, and $b'_\Omega(x)/b_\Omega(x) = i|b'_\Omega(x)|$, thus we have

$$\frac{1}{2\pi i} \oint F(x) dx = \frac{1}{2\pi} \int_0^{2\pi} \frac{f}{\Delta} d\theta, \; F(z) \approx \frac{f(0)/\Delta(0)}{z} + O(\frac{1}{z^2}).$$

Definition. We say that domain $\Omega = \mathbb{C} \setminus E, \infty \in \Omega$, has no DCT if there exist F analytic in Ω, $F \in D(\Omega)$, $\int_E |F(x)|\, dx < \infty$, $F(z) \approx \frac{A}{z} + O(\frac{1}{z^2})$, but

$$\oint_E F(x)dx \neq A.$$

By considering $G(z) := (z - z_0)F(z) - A$ we see that no DCT is exactly the same that the Cauchy formula does not hold for some Smirnov class functions summable on the boundary. By the slides 11, 12 we proved the following theorem:

14. No DCT domains

Theorem

TFAE: 1) Domain Ω has no DCT, 2) the Cauchy formula fails for some functions from Smirnov class integrable on the boundary, 2) for some functions $f \in H^1(\nu)$ in $\mathbb{D}$, where ν is the character of Δ constructed on slide 8 the following formula must fail:

$$\int_{\mathbb{T}} \frac{f(\zeta)}{\Delta(\zeta)} dm(\zeta) = \frac{f(0)}{\Delta(0)},$$

4) for some functions $f \in H^1_0(\nu)$ (meaning $f(0) = 0$) in $\mathbb{D}$, where ν is the character of Δ constructed on slide 8 the following formula must fail:

$$\int_{\mathbb{T}} \frac{f(\zeta)}{\Delta(\zeta)} dm(\zeta) = 0.$$

Proof was done. Notice 3) $\Leftrightarrow$ 4) by $f_1 := f - f(0)\Delta \in H^1_0(\nu)$ iff $f \in H^1(\nu)$.

15. Our $\Omega = w(O)$, $w = 1/(z - c)$ is a Widom domain

Fix a character $\alpha \in \Gamma^*$. Notice that Benedicks construction allows to have Ω Widom. In fact Widomness is conformal invariant, so $O = (\mathbb{C} \setminus [0, \infty)) \setminus \cup_{m=1}\infty[-m - d_m, m + d_m]$, small d_m, has to be Widom. This is so by the theorem of Koosis.

Theorem

Domain $\Omega = \mathbb{C} \setminus E$, $E \subset \mathbb{R}$, $a \in \Omega$ has Martin function $M(z)$ such that $M(z) \approx |y|$ iff

$$\int_{\mathbb{R}} G(x, a) dx < \infty.$$

As our choice of d_m ensures the this growth of Martin function, slide 5, we conclude that $\int_{\mathbb{R}} G(x, c) dx < infty$. Critical points e_i of $G(z, c)$ lie by one in each E-complementary interval (gap) $L_i := (A_i, B_i)$, G is concave on L_i, $G(e_i, c) = \max_{x \in L_i} G(x, c)$. Therefore, $\sum_i G(e_i, c) L_i \leq \int_{\mathbb{R}} G(x, c) dx < infty$. But obviously $|L_i| \geq 1/2$ as all $d_m \leq 1/4$. So $O \in$ Widom, so is $\Omega = w(O)$.

16. Natural Hardy spaces: $\hat{H}^2(\alpha)$

Functions from the usual Hardy space H^2 in the disc, which have character-automorphic property:

$$h \circ \gamma(\zeta) = \alpha(\gamma) h(\zeta), \zeta \in \mathbb{D}, \gamma \in \text{Fuchsian}\Gamma, \alpha \in \Gamma^*$$

form a closed subspace of the usual H^2. We call it hat-space. It the largest natural space of ch.-automorphic functions in H^2 with character automorphism α.
Recall that with the usual duality annihilator of H^2 is $\bar{H}_0^2$. Can it be that annihilator of $\hat{H}^2(\alpha)$ is something like $\hat{H}_0^2(\alpha^{-1})$? Not at all. First of all our ubiquitous Widom function Δ intervenes. Slide 8.

17. Check spaces $\check{H}^2(\alpha)$

Theorem

The annihilator to $\hat{H}^2_0(\alpha)$ consists of functions $\Delta\bar{\phi}_1$ such that $\phi_1 \in \hat{H}^2(\alpha^{-1}\nu)$ and such that

$$\int_{\mathbb{T}} \frac{\phi_1\phi_0}{\Delta} dm = 0, \ \forall \phi_0 \in \hat{H}^2_0(\alpha).$$

The fact that $\forall \alpha \in \Gamma^*$ this annihilator is equal to the whole $\Delta\overline{\hat{H}^2(\alpha^{-1}\nu)}$ is equivalent to

$$\int_{\mathbb{T}} \frac{f}{\Delta} dm = 0, \ \forall f \in \hat{H}^1_0(\nu).$$

Corollary

If Ω is a Widom domain with no DCT, then annihilator of $\hat{H}^2_0(\nu\alpha^{-1})$ for a certain $\alpha \in \Gamma^*$ is a proper closed subspace of $\Delta\overline{\hat{H}^2(\alpha)}$. **Call it** $\Delta\overline{\check{H}^2(\alpha)}$.

In fact, we saw (slide 14) that no DCT means the existence of $f \in \hat{H}^1_0(\nu)$ such that $\int_{\mathbb{T}} \frac{f}{\Delta} dm \neq 0$. Let us factorize this $f = h_0 h_1$, $h_1 := (f)^{1/2}_{out}$, then automatically h_1 is modulo automorphic: $|h_1 \circ \gamma| = |h_1|$, then so is h_0, then they are both character-automorphic.

Let the character of h_1 be α, then the character of h_0 has to be $\alpha^{-1}\nu$. Then $h_0 \in \hat{H}^2_0(\alpha^{-1}\nu)$, $h_1 \in \hat{H}^2(\alpha)$, but $\Delta\bar{h_1}$ is not in annihilator of $\underline{\hat{H}^2_0(\alpha^{-1}\nu)}$. So the annihilator of $\hat{H}^2_0(\alpha^{-1}\nu)$ is strictly smaller than $\Delta \hat{\bar{H}}^2(\alpha)$. This is why it deserves a new name: and the space $\check{H}^2(\alpha)$ appears.

The space $\check{H}^2(\alpha)$ is the smallest natural closed subspace of H^2 having α-automorphic property. Domain is no DCT iff

$$\exists \alpha \in \Gamma^* : \check{H}^2(\alpha) \subsetneq \hat{H}^2(\alpha).$$

19. Check spaces $\check{H}^2(\alpha)$

We just repeat what has been already said: $\check{H}^2(\alpha)$ is the collection of character automorphic functions f from H^2 with character α, such that

$$\int_{\mathbb{T}} \frac{fg_0}{\Delta} dm = 0, \ \forall g_0 \in \hat{H}^2_0(\alpha^{-1}\nu).$$

Symmetrically, we will see that $\hat{H}^2(\alpha)$ is the collection of character automorphic functions f from H^2 with character α, such that

$$\int_{\mathbb{T}} \frac{fg_0}{\Delta} dm = 0, \ \forall g_0 \in \check{H}^2_0(\alpha^{-1}\nu).$$

Theorem

1) $\Delta\overline{\check{H}^2(\alpha)}$ *is the annihilator of* $\hat{H}^2_0(\alpha^{-1}\nu)$.
2) $\Delta\overline{\check{H}^2_0(\alpha)}$ *is the annihilator of* $\hat{H}^2(\alpha^{-1}\nu)$.
3) $\Delta\overline{\hat{H}^2(\alpha)}$ *is the annihilator of* $\check{H}^2_0(\alpha^{-1}\nu)$.
4) $\Delta\overline{\hat{H}^2(\alpha)}$ *is the annihilator of* $\check{H}^2_0(\alpha^{-1}\nu)$.
5) $\check{H}^2(\alpha)$ *is the closure of* $P^\alpha(\Delta H^\infty)$, *where* P^α *projection* $L^1(\mathbb{T})$ *onto* $L^1(\alpha)$ *is given by*

$$P^\alpha(f) := \sum_{\gamma\in\Gamma} \frac{\alpha^{-1}(\gamma)|\gamma'| f\circ\gamma}{\sum_{\gamma\in\Gamma}|\gamma'|}, \zeta \in \mathbb{T}.$$

6) $\mathfrak{z}\hat{H}^2_0(\alpha) \subset \hat{H}^2(\alpha)$.
7) $\mathfrak{z}\check{H}^2_0(\alpha) \subset \check{H}^2(\alpha)$.

21. Divisibility property I.

1) $\hat{H}^2_0(\alpha) = b\hat{H}^2(\alpha\mu^{-1})$.
2) $\check{H}^2_0(\alpha) = b\check{H}^2(\alpha\mu^{-1})$. Only 2) should be proved. Let us prove that if $\phi = b\tilde{\phi}$ and ϕ is in check space, then $\tilde{\phi}$ is also in check space. First prove

$$\int \frac{\phi\Phi}{\Delta} = 0,\ \forall \Phi \in \hat{H}^2(\alpha^{-1}\nu). \qquad (1)$$

Write $\Phi = \Phi - \frac{\Phi(0)\hat{k}^{\alpha^{-1}}\Delta}{\Delta(0)\hat{k}^{\alpha^{-1}}(0)} + \frac{\Phi(0)\hat{k}^{\alpha^{-1}}\Delta}{\Delta(0)\hat{k}^{\alpha^{-1}}(0)} =: \Phi_1 + \Phi_2$. Then $\Phi_1 \in \hat{H}^2_0(\alpha^{-1}\nu)$, so $\int \frac{\phi\Phi_1}{\Delta} = 0$ by the definition of check space. And

$$\int_{\mathbb{T}} \frac{\phi\Phi_2}{\Delta} = c\int_{\mathbb{T}} \phi\hat{k}^{\alpha^{-1}} = c\int_{\mathbb{T}} b\tilde{\phi}\hat{k}^{\alpha^{-1}} = b\tilde{\phi}\hat{k}^{\alpha^{-1}}(0) = 0\,.$$

So (1) is proved. Rewrite it as $\int_{\mathbb{T}} \frac{\tilde{\phi}b\Phi}{\Delta} = 0$, but $b\Phi$ runs over all $\hat{H}^2_0(\alpha^{-1}\nu\mu)$ as division is possible in hat spaces. So $\tilde{\phi}$ belongs to $\check{H}^2(\alpha\mu^{-1})$ by the definition of what is check.

22. Divisibility property II.

Theorem

1) $\mathfrak{z}\hat{H}^2_0(\alpha) \subset \hat{H}^2(\alpha)$.
2) $\mathfrak{z}\check{H}^2_0(\alpha) \subset \check{H}^2(\alpha)$.

Again only 2) should be proved. We know that $\check{H}^2_0(\alpha) = b\check{H}^2(\alpha\mu^{-1})$. Also it is clear that $\mathfrak{z}b = (zb_\Omega) \circ \mathfrak{z}$, so $\mathfrak{z}b \in H^\infty(\mu)$. We now see that $\mathfrak{z}\check{H}^2_0(\alpha) = \mathfrak{z}b\check{H}^2(\alpha\mu^{-1}) \subset H^\infty(\mu)\check{H}^2(\alpha\mu^{-1})$. The space $H^\infty(\mu)$ multiplies hat spaces obviously. So by the description of the annihilators $\int_{\mathbb{T}} \phi_1\phi_0\Delta = 0$ on slide 17, it also multiplies check spaces.
We are done with 2).

We call a closed subspace $H^2(\alpha)$, **our** Hardy space with character α if

$$1)\ \check{H}^2(\alpha) \subset H^2(\alpha) \subset \hat{H}^2(\alpha),$$

$$2)\ \mathfrak{z} H^2_0(\alpha) \subset H^2(\alpha).$$

Check spaces and hat spaces are **our** Hardy spaces.

Theorem

Any **our** *Hardy space defines a reflectionless Jacobi matrix $J(H^2(\alpha))$ with spectrum E. If E is weakly homogeneous in the sense of Poltoratski–Remling, then $J(H^2(\alpha))$ is purely absolutely continuous.*

Several slides contain the sketch of the proof.

24. Duality formulae

Let $\hat{e}^\alpha$ be a normalized in L^2 reproducing kernel of $\hat{H}^2(\alpha)$, $\hat{e}^\alpha = \hat{k}^\alpha/\sqrt{\hat{k}^\alpha(0)}$; Let $\check{e}^\alpha$ be a normalized in L^2 reproducing kernel of $\check{H}^2(\alpha)$, $\check{e}^\alpha = \check{k}^\alpha/\sqrt{\check{k}^\alpha(0)}$.

Theorem

1) $\Delta\overline{\check{e}^{\alpha^{-1}\nu}} = \hat{e}^\alpha$ *on* $\mathbb{T}$*;*

2) $\sqrt{\hat{k}^\alpha(0)\cdot\check{k}^{\alpha^{-1}\nu}(0)} = \Delta(0)$*;*

3) $\hat{e}^\alpha(0)\cdot\check{e}^{\alpha^{-1}\nu}(0) = \Delta(0)$*.*

Proof: using slide 20 we can write
$L^2(\alpha) = \hat{H}^2_0(\alpha)\oplus\Delta\overline{\check{H}^2(\alpha^{-1}\nu)} = \hat{H}^2_0(\alpha)\oplus\{\Delta\overline{\check{e}^{\alpha^{-1}\nu}}\}\oplus\Delta\overline{\check{H}^2_0(\alpha^{-1}\nu)}$
and $L^2(\alpha) = \hat{H}^2(\alpha)\oplus\Delta\overline{\check{H}^2_0(\alpha^{-1}\nu)} = \hat{H}^2_0(\alpha)\oplus\{\hat{e}^\alpha\}\oplus\Delta\overline{\check{H}^2_0(\alpha^{-1}\nu)}$.
Comparison gives 1): $\bar{\Delta}\check{e}^{\alpha^{-1}\nu} = \overline{\hat{e}^\alpha}$ on $\mathbb{T}$. Multiply on $\hat{e}^\alpha$ and integrate on $\mathbb{T}$: $1 = \int|\hat{e}^\alpha|^2dm = \int\frac{\check{e}^{\alpha^{-1}\nu}\cdot\hat{e}^\alpha}{\Delta}dm$

Repeating: $1 = \int |\hat{e}^\alpha|^2 dm = \int \frac{\check{e}^{\alpha^{-1}\nu} \cdot \hat{e}^\alpha}{\Delta} dm$. The RHS can be written as

$$1 = \int_{\mathbb{T}} \left(\hat{e}^\alpha - \frac{\hat{e}^{\alpha^{-1}\nu} \Delta}{\hat{e}^{\alpha^{-1}\nu}(0)\Delta(0)} \hat{e}^\alpha(0) \right) \cdot \frac{\check{e}^{\alpha^{-1}\nu}}{\Delta} dm + C \int_{\mathbb{T}} \hat{e}^{\alpha\nu^{-1}} \check{e}^{\alpha^{-1}\nu} dm =$$

$$0 + \frac{\hat{e}^\alpha(0)}{\hat{e}^{\alpha^{-1}\nu}(0)\Delta(0)} \hat{e}^{\alpha\nu^{-1}}(0) \check{e}^{\alpha^{-1}\nu}(0) = \hat{e}^\alpha(0) \cdot \check{e}^{\alpha^{-1}\nu}(0).$$

We got 0 in the first term because the big bracket expression is $\in \hat{H}^2_0(\alpha)$ and $\check{e}^{\alpha^{-1}\nu} \in \check{H}^2(\alpha^{-1}\nu)$, see slides 19, 20. Hence we proved 3) of the previous theorem. But 2) is the same as 3). Theorem is proved.

26. Construction of $J(H^2(\alpha))$

Given **our** Hardy space $H^2(\alpha)$ define $H^2(\alpha\mu^{-1})$ as

$$bH^2(\alpha\mu^{-1}) := H^2_0(\alpha),$$

that is: by division. It is a well defined closed subspace of $\hat{H}^2(\alpha\mu^{-1})$, and superspace of $\check{H}^2(\alpha\mu^{-1})$ because division by b preserves check and hat.

One need to check that thus defined $H^2(\alpha\mu^{-1})$ is also **our** Hardy space.

One need to check that $f_0 \in H^2_0(\alpha\mu^{-1})$ implies $\mathfrak{z} f_0 \in H^2(\alpha\mu^{-1})$. By definition the latter means exactly $\mathfrak{z} b f_0 \in H^2_0(\alpha)$. For that it is enough to check that $\mathfrak{z} b f_0 \in H^2(\alpha)$ (notice double zero of bf_0 at 0). But $H^2(\alpha)$ was assumed to be **our** space. Therefore, of course $\mathfrak{z} b f_0 \in H^2(\alpha)$ if $bf_0 \in H^2_0(\alpha)$. But $f_0 \in H^2_0(\alpha\mu^{-1}) \subset H^2(\alpha\mu^{-1})$, so $bf_0 \in H^2_0(\alpha)$ by the definition of $H^2(\alpha\mu^{-1})$. We are done.

27. Construction of $J(H^2(\alpha))$

We just proved

Theorem

If $H^2(\alpha)$ is **our** *Hardy space, then all Hardy spaces in the next chain of equalities are also* **our** *Hardy spaces: 1)* $H^2(\alpha) = \{e^\alpha\} \oplus bH^2(\alpha\mu^{-1}) = \{e^\alpha\} \oplus \{be^{\alpha\mu^{-1}}\} \oplus b^2 H^2(\alpha\mu^{-2}) = \{e^\alpha\} \oplus \{be^{\alpha\mu^{-1}}\} \oplus \{b^2 e^{\alpha\mu^{-2}}\} \oplus b^3 H^2(\alpha\mu^{-3}) = \dots$, *where $e^{\alpha\mu^{-n}}$ is a normalized reproducing kernel of* **our** *Hardy space $H^2(\alpha\mu^{-n})$. 2) These vectors form the basis in $H^2(\alpha)$. 3) e^α is is orthogonal to $\mathfrak{z} b^k e^{\alpha\mu^{-k}}$ for all $k \geq 2$.*

Now negative direction: call $e_k := b^k e^{\alpha\mu^{-k}}, k \geq 0$.
$e^\alpha(0) \geq \check{e}^\alpha(0) \geq \Delta(0) > 0$. Hence $\mathfrak{z} e_0$ has a simple pole at 0. By 3) of the Theorem above $\mathfrak{z} e_0$ is orthogonal to $b^2 H^2(\alpha\mu^{-2})$. Hence

$$\mathfrak{z} e_0 = p_0 e_{-1} + q_0 e_0 + p_1 e_1,$$

where e_{-1} is orthogonal to e_0, e_1, and thus to all $e_k, k \geq 0$, e_{-1} has a simple pole at zero.

28. Construction of $J(H^2(\alpha))$

By definition

$$H^2(\alpha\mu) := bH^2(\alpha) \oplus \{be_{-1}\}.$$

Again one can prove that this is **our** Hardy space.

Theorem

$e_{-1} = b^{-1}e^{\alpha\mu}$, where $e^{\alpha\mu}$ is the normalized reproducing kernel of $H^2(\alpha\mu)$ at 0.

Proof: it is enough to check that be_{-1} is proportional to $k^{\alpha\mu}$. But if $f = c_0 be_{-1} + c_1 be_0 + \dots$ then $f(0) = c_0(be_{-1})(0)$. But by orthogonality $\langle f, be_{-1}\rangle = c_0$. Therefore, $k^{\alpha\mu} = (be_{-1}) \cdot (be_{-1})(0)$.

29. Construction of $J(H^2(\alpha))$

Starting now with **our** $H^2(\alpha\mu)$ we build **our** $H^2(\alpha\mu^2)$ and $e_{-2} = b^{-2} e^{\alpha\mu^2}$, etc. Finally we get

Theorem

Starting with **our** $H^2(\alpha)$ *one builds the chain of* **our** $H^2(\alpha\mu^k)$, $k \in \mathbb{Z}$, *such that their normalized reproducing kernels* $e^{\alpha\mu^k}$ *give us the orthonormal basis* $e_k^\alpha := b^{-k} e^{\alpha\mu^k}$, $k \in \mathbb{Z}$, *and the operator of multiplication on* $\mathfrak{z}$ *(real function on* $\mathbb{T}$*) in the space* $L^{(}\alpha)$ *has a three-diagonal Jacobi form in the basis* $\{e_k\}_{k\in\mathbb{Z}}$. *Moreover,* $\mathfrak{z} e_n^\alpha = p_n(\alpha) e_{n-1}^\alpha + q_n(\alpha) e_n^\alpha + p_{n+1}^\alpha e_{n+1}^\alpha$, *where* $p_n(\alpha) = \mathbb{P}(\alpha\mu^{-n})$, $q_n(\alpha) = \mathbb{Q}(\alpha\mu^{-n})$, *and* $\mathbb{P}(\alpha) = (\mathfrak{z} b)(0)\sqrt{\frac{k^\alpha(0)}{k^{\alpha\mu}(0)}}$, $\mathbb{Q}(\alpha) = \dots$. *This matrix is reflectionless.*

Proof: For the formula, take $n = 1$ and decompose $\mathfrak{z} e_n^\alpha = p_n(\alpha) e_{n-1}^\alpha + q_n(\alpha) e_n^\alpha + p_{n+1}^\alpha e_{n+1}^\alpha$ near $\zeta = 0$.

30. Reflectionlessness of $J(H^2(\alpha))$

Skip index α. It is known that

$$r_+(z) := \langle (J_+ - z)^{-1} e_0, e_0 \rangle = -\frac{1}{z - q_0 - \frac{p_1^2}{z - q_1 - \dots}}.$$

But from the previous slide

$$-\frac{e_0}{p_0 e_{-1}}(\zeta) = -\frac{1}{\zeta - q_0 - \frac{p_1^2}{\zeta - q_1 - \dots}}.$$

We get

$$r_+(\mathfrak{z}(\zeta)) = -\frac{e_0}{p_0 e_{-1}}(\zeta),\ \zeta \in \mathbb{D}\ (\text{and}\ \zeta \in \mathbb{T}\ a.e.) \tag{2}$$

Exactly as for hat we have orthogonality and check: exactly so for any **our** spaces $H_0^2(\alpha)$, $H^2(\alpha)$ we get that their annihilators are

$$\Delta \overline{\tilde{H}^2(\alpha^{-1}\nu)},\ \Delta \overline{\tilde{H}_0^2(\alpha^{-1}\nu)} = \Delta \overline{b\tilde{H}^2(\mu^{-1}\alpha^{-1}\nu)}.$$

And all space above are **our** Hardy spaces.

30a. Reflectionlessness of $J(H^2(\alpha))$

Then we have the dual basis of normalized reproducing kernels $\tilde{e}_n := b^n e^{\mu^{-n}\alpha^{-1}\nu}$. Exactly the same Duality formulae, slides 24, 25 will hold:

$$b\tilde{e}_n = \Delta\overline{e_{-n-1}} \text{ on } \mathbb{T}. \qquad (3)$$

We also get **the inversion** of matrix:

$$\tau J(H^2(\alpha)) = J(\tilde{H}^2(\mu^1\alpha^{-1}\nu)),\ \tau p_n = p_{-n}, \tau q_n = q_{-n-1}\,.$$

Therefore, denoting $r_-(z) = \langle (J_- - z)^{-1}e_{-1}, e_{-1}\rangle$ we get

$$r_-(\mathfrak{z}(\zeta)) = -\frac{\tilde{e}_0}{p_0\tilde{e}_{-1}}(\zeta),\ \zeta \in \mathbb{T}\ a.e. \qquad (4)$$

Hence, $\frac{1}{r_+(\mathfrak{z}(\zeta))} = -\frac{p_0 e_{-1}}{e_0}(\zeta) = -\frac{p_0\overline{\tilde{e}_0}}{\overline{\tilde{e}_{-1}}} = p_0^2\overline{r_-(\mathfrak{z}(\zeta))}$ a. e. $\zeta \in \mathbb{T}$. So $\frac{1}{r_+(x)} = p_0^2\overline{r_-(x)}$ a. e. $d\omega_\Omega(x)$, which is mutually absolutely continuous with Lebesgue measure $dx|E$ for Widom domains. Reflectionlessness is proved.

31. Poltoratski–Remling condition: $J(E) = \{J(H^2(\alpha))\}$. Bijection.

Our E contains $[0,1]$ and small intervals accumulating to 0. Automatically

$$\int_E \frac{dx}{|x|} = \infty.$$

Let $J(E)$ denote all reflectionless Jacobi matrices with spectrum E. Here is the corollary of Poltoratski–Remling weak homogeneity criterion.

Theorem

Let E be as above (countable sequence of intervals converging to 0 and integral above diverges). Then all $J(E)$ are purely absolutely continuous.

In particular this is our situation by a trivial reason that $[0,1] \subset E$.

32. $J(E) = \{J(H^2(\alpha))\}$. Bijection.

Theorem

Let $\Omega = \mathbb{C} \setminus E, E \subset \mathbb{R}$, be a Widom domain. And let J be a reflectionless matrix with the spectrum E. Then a) there exists a unique factorization

$$r_+ \circ \mathfrak{z} = -\frac{1}{p_0} \frac{e_0}{e_{-1}} \tag{5}$$

such that $p_0(e_{-1}(\zeta)\overline{e_0(\zeta)} - e_0(\zeta)\overline{e_{-1}(\zeta)}) = \sqrt{(\mathfrak{z} - a_0)(\mathfrak{z} - b_0)} \prod_{j\ge 1} \frac{\sqrt{(\mathfrak{z}-a_j)(\mathfrak{z}-b_j)}}{\mathfrak{z}-c_j}$, for $\zeta \in \mathbb{T}$, where e_0 and be_{-1} are of Smirnov class with mutually simple singular parts and $e_0(0) > 0$. b) $(e_0)_{inn}$ is Blaschke product, which is a divisor of Πb_{x_j}, where $x_j \in (a_j, b_j)$ are poles of $\frac{1}{R_{0,0}} = \frac{1}{r_+} - p_0^2 r_-$, $R_{0,0} := \langle (J - z)^{-1} e_0, e_0 \rangle$. c) $e_0 \in \hat{H}^2(\alpha)$ for some $\alpha \in \Gamma^*$. d) If in addition J has purely a. c. spectrum , then there exists **our** *Hardy space $H^2(\alpha)$ such that $J = J(H^2(\alpha))$.*

33. Inversion formula

Let e_{-1}, e_0 denote standard vectors in $\ell^2(\mathbb{Z})$. For an arbitrary two-sided Jacobi matrix J span$\{e_{-1}, e_0\}$ is a cyclic subspace. The spectral 2×2 matrix measure $d\sigma$ is defined by

$$R(z) = \mathcal{E}^*(J-z)^{-1}\mathcal{E} = \int \frac{d\sigma(x)}{x-z},$$

where $\mathcal{E} : \mathbb{C}^2 \to \ell^2$, by $\mathcal{E}((a,b)) = ae_{-1} + be_0$. And by general inversion formula

$$R(z) = \begin{bmatrix} R_{-1,-1} & R_{-1,0} \\ R_{0,-1} & R_{0,0} \end{bmatrix}(z) = \begin{bmatrix} r_-^{-1}(z) & p_0 \\ p_0 & r_+^{-1}(z) \end{bmatrix}^{-1}. \qquad (6)$$

In particular,

$$-\frac{1}{R_{0,0}(z)} = -\frac{1}{r_+(z)} + p_0^2 r_-(z). \qquad (7)$$

34. e_0

If r_+ is as in (5) and reflectionlessness holds, then

$$\Im \frac{1}{R_{0,0}} = 2\Im \frac{1}{r_+} = \frac{p_0(e_{-1}(\zeta)\overline{e_0(\zeta)} - e_0(\zeta)\overline{e_{-1}(\zeta)})}{i|e_0(\zeta)|^2}. \qquad (8)$$

But $R_{0,0}$ is purely imaginary on E a. e. by (7) and reflectionlessness. And it is real on $\mathbb{R}\setminus E$. Also $R_{0,0}$ is of positive imaginary part in $\mathbb{C}_+$. Such function can be restored by its purely imaginary values on E $R_{0,0}(z) = \frac{-1}{\sqrt{z-a_0)(z-b_0)}}\Pi_{j=1}^{\infty}\frac{z-x_j}{\sqrt{(z-a_j)(z-b_j)}} = \frac{-1}{\sqrt{z-a_0)(z-b_0)}}\Pi_{j=1}^{\infty}\frac{z-x_j}{z-c_j}\frac{z-c_j}{\sqrt{(z-a_j)(z-b_j)}}$ Put $W(z) = \Pi_{j=1}^{\infty}\frac{z-x_j}{z-c_j}$. Comparing two formulae above and Wronski formula on slide 32 we get that $|e_0(\zeta)|^2 = W\circ\mathfrak{z}$. This defines uniquely the outer part of e_0. Furthermore, r_+ is of positive imaginary part in $\mathbb{C}_+$, and all its poles are in gaps (a_j, b_j), not more than one in each. Therefore one can apply Sodin–Yuditskii theorem that says that such functions in Widom Ω satisfy that $r_+\circ\mathfrak{z}$ has its inner part only the ration of two Blaschke products. So $(e_0)_{inn}$ is a Blaschke product (over some poles of $1/R_{0,0}$

Automatically e_0 has the form

$$e_0(\zeta) = \Pi_{j\geq 1} b_{x_j}^{\frac{1+\varepsilon_j}{2}} \sqrt{\frac{W \circ \mathfrak{z}\Delta(\zeta)}{\Pi_{j\geq 1 b_{x_j}(\zeta)}}} = inner \cdot outer.$$

Conversely, define e_0 by this formula (x_j are zeros of r_+) and define $p_0 e_{-1}(\zeta)$ then by (5). Then Wronski formula of Theorem on slide 32 follows. In fact, it follows from (8) of the previous slide, i. e. from reflectionless, and from the fact that e_0 defined above automatically satisfies $|e_0(\zeta|^2 = W \circ \mathfrak{z}$. Of course we use the formula for $R_{0,0}$ from the previous slide again.

36. $J(E) \subset \{J(H^2(\alpha))\}$. Sketch.

Let e_{-1}, e_0 denote standard vectors in $\ell^2(\mathbb{Z})$. For an arbitrary two-sided Jacobi matrix J span$\{e_{-1}, e_0\}$ is a cyclic subspace. The spectral 2×2 matrix measure $d\sigma$ is defined by

$$R(z) = \mathcal{E}^*(J - z)^{-1}\mathcal{E} = \int \frac{d\sigma(x)}{x - z},$$

where $\mathcal{E} : \mathbb{C}^2 \to \ell^2$, by $\mathcal{E}((a, b)) = ae_{-1} + be_0$. Let us make the correspondence of standard vectors in ℓ^2 to elements of $L^2(d\sigma)$:

$$e_n \to \begin{bmatrix} -p_0 Q_n^+ \\ P_n^+ \end{bmatrix}, \quad e_{-n-1} \to \begin{bmatrix} P_n^- \\ -p_0 Q_n^- \end{bmatrix},$$

where $P_n^\pm$ and $Q_n^\pm$ are orthogonal polynomials of the first and second kind generated by $J_\pm$. The operator J becomes the operator multiplication by independent variable in $L^2(d\sigma)$.

37. $J(E) \subset \{J(H^2(\alpha))\}$. Sketch.

Theorem

Assume, in addition, that $J \in J(E)$ has absolutely continuous spectrum. Then the map

$$\begin{bmatrix} F(x) \\ G(x) \end{bmatrix} \to f(\zeta) = e_{-1}(\zeta) F \circ \mathfrak{z} + e_0(\zeta) G \circ \mathfrak{z}, \quad \begin{bmatrix} F(x) \\ G(x) \end{bmatrix} \in L^2(d\sigma), \tag{9}$$

acts unitary from $L^2_{d\sigma}$ to $L^2(\alpha)$, $\alpha = \pi(J)$. Moreover, the composition map

$$\mathcal{F} : \ell^2 \to L^2(d\sigma) \to L^2(\alpha)$$

is such that $H^2_J := \mathcal{F}(\ell^2_+)$ possesses the properties

$$\check{H}^2(\alpha) \subseteq H^2_J \subseteq \hat{H}^2(\alpha), \quad \mathfrak{z}(H^2_J)_0 \subset H^2_J.$$

In other words, this $J = J(H^2(\alpha))$ with **our** *Hardy space.*

38. $J(E) \subset \{J(H^2(\alpha))\}$. Sketch.

Sketch: Looking at slide 32 and defining the dual functions $\tilde{e}_0, \tilde{e}_{-1}$ we consider

$$\Psi = -p_0 \begin{bmatrix} \tilde{e}_0 & 0 \\ 0 & e_0 \end{bmatrix}, \ \Phi = \begin{bmatrix} \tilde{e}_{-1} & -e_0 \\ -\tilde{e}_0 & e_{-1} \end{bmatrix}. \tag{10}$$

And by general inversion formula

$$\begin{bmatrix} R_{-1,-1} & R_{-1,0} \\ R_{0,-1} & R_{0,0} \end{bmatrix}(z) = \begin{bmatrix} r_-^{-1}(z) & p_0 \\ p_0 & r_+^{-1}(z) \end{bmatrix}^{-1}. \tag{11}$$

In particular,

$$-\frac{1}{R_{0,0}(z)} = -\frac{1}{r_+(z)} + p_0^2 r_-(z). \tag{12}$$

we get

$$R \circ \mathfrak{z} = \Psi \Phi^{-1}.$$

39. $J(E) \subset \{J(H^2(\alpha))\}$. Sketch.

Now if $f(\zeta) = e_{-1}(\zeta) F \circ \mathfrak{z} + e_0(\zeta) G \circ \mathfrak{z}$, $\begin{bmatrix} F(x) \\ G(x) \end{bmatrix} \in L^2(d\sigma)$ then

$$\begin{bmatrix} f(\zeta) \\ \Delta(\zeta) f(\bar{\zeta})/b(\zeta) \end{bmatrix} = \begin{bmatrix} e_{-1} & e_0 \\ \tilde{e}_0 & \tilde{e}_{-1} \end{bmatrix} \begin{bmatrix} F \\ G \end{bmatrix} \circ \mathfrak{z} = \Phi^{-1} \begin{bmatrix} F \\ G \end{bmatrix} \circ \mathfrak{z} \cdot \det \Phi.$$

Therefore, we have

$$\frac{1}{2}\left(\int_{\mathbb{T}} |f(\zeta)|^2 dm(\zeta) + \int_{\mathbb{T}} |f(\bar{\zeta})|^2 dm(\zeta)\right) = \int_E \begin{bmatrix} F \\ G \end{bmatrix}^* (x) \sigma'_{a.c}(x) dx \begin{bmatrix} F \\ G \end{bmatrix} (x),$$

because $\sigma_{a.c} = (\Phi^{-1})^* \Phi^{-1} \det \Phi$ from $\sigma'_{a.c} = \frac{1}{2\pi i}(R - R^*)$ and the formula for R at the end of the previous slide. Then of course we get That this map is an isometry from $L^2(d\sigma)$ onto $L^2(\alpha)$ if *sigma* is absolutely continuous.

40. $J(E) \subset \{J(H^2(\alpha))\}$. Sketch.

On the previous slide we used that $det\Phi(x)\omega(x)dx = 1$, where harmonic measure density ω has the formula

$$\omega(x) = \frac{1}{\sqrt{(x-a_0)(b_0-x)}} \Pi_{j\geq 1} \frac{x-c_j}{\sqrt{(x-a_j)(x-b_j)}}$$

and $\det \Phi(x) = e_{-1}\tilde{e}_{-1} - e_0\tilde{e}_0 = e_{-1}\overline{e_0} - e_0\overline{e_{-1}} = \textit{reciprocal}$ see slide 32, Wronski relationship, and use the fact that dual $\tilde{e}$ can be defined by flipping the matrix and that they will satisfy duality relation for reflectionless J: $\frac{p_0 e_{-1}}{p_0 e_0} = \frac{p_0\overline{\tilde{e}_0}}{\overline{\tilde{e}_{-1}}}$.

41. $J(E) \subset \{J(H^2(\alpha))\}$. Sketch.

Let

$$e_n(\zeta) = \mathcal{F}(e_n) = -p_0 e_{-1}(\zeta) Q_n^+(\mathfrak{z}) + e_0(\zeta) P_n^+(\mathfrak{z}),\ \ n \geq 0.$$

Since $-Q_n^+/P_n^+$ is the Padé approximation for r_+, this function has zero of exact multiplicity n at the origin. Thus, it is of Smirnov class, and therefore belongs to $\hat{H}^2(\alpha)$. It proves $H_J^2 \subseteq H^2(\alpha)$ and $\mathfrak{z}(H_J^2)_0 \subset H_J^2$. The latter because of the definition of H_J^2 as the span of $\{e_n(\zeta)\}_{n\geq 0}$ as above and because with this definition of $e_n(\zeta)$ the Jacobi 3-terms relationship obviously holds (and with coefficients coming from the initial matrix J of course).
To show that $\check{H}^2(\alpha) \subset H_J^2$ we pass to the dual representation for the flipped matrices

$$\Delta(\zeta) e_{-n-1}(\bar{\zeta})/b(\zeta) = -p_0 \tilde{e}_{-1}(\zeta) Q_n^-(\mathfrak{z}) + \tilde{e}_0(\zeta) P_n^-(\mathfrak{z}),\ \ n \geq 0.$$

Theorem

Let Ω be a Widom domain. $\forall \beta \in \Gamma^* \ \exists w \in H^\infty(\beta)$ Blaschke product such that $\forall \alpha \in \Gamma^*$, $w\hat{H}^2(\alpha) \subset \check{H}^2(\alpha\beta)$.

Proof: Consider linear functional Λ on $\bar{\Delta}\hat{H}^1(\beta^{-1}\nu)$ given by

$$\Lambda(\bar{\Delta} f) = f(0)$$

and extend to $L^1(\mathbb{T})$. We get $w_0 \in L^\infty(\mathbb{T})$ such that

$$\int_{\mathbb{T}} \frac{w_0 \hat{H}^1(\beta^{-1}\nu)}{\Delta} dm = f(0).$$

So for $w_1 := \Delta \bar{w}_0$ we have

$$\langle \hat{H}^1(\beta^{-1}\nu), w_1 \rangle = f(0).$$

Put $w_2 := P^{\beta^{-1}\nu} w_1$, then

$$\langle \hat{H}^1(\beta^{-1}\nu), w_1 \rangle = f(0).$$

Hence, $\langle \Delta h_0, w_2 \rangle = \langle P^{\beta^{-1}\nu} \Delta h_0, w_2 \rangle = 0$ for all $h_0 \in H_0^\infty$ (as $(P^\alpha \Delta h)(0) = \Delta(0) h(0)$). Therefore $w_3 := \bar{w}_2 \Delta \in H^\infty(\beta)$. And

$$\int_{\mathbb{T}} \frac{w_3 f}{\Delta} = f(0), \, \forall f \in \hat{H}^1(\beta\nu).$$

Test on $f = \Delta f_1, f_1 \in H^\infty(\beta^{-1})$. Then $w_3(0) f_1(0) = \int_{\mathbb{T}} \frac{w_3 \Delta f_1}{\Delta} = f(0) = f_1(0)\Delta(0)$. So

$$w_3(0) = \Delta(0).$$

So

$$\int_{\mathbb{T}} \frac{w_3 f}{\Delta} = \frac{w_3(0)}{\Delta(0)} f(0), \, \forall f \in \hat{H}^1(\beta\nu).$$

Consider finally $w := w_3 / \|w_3\|_\infty$, then again

$$\int_{\mathbb{T}} \frac{wf}{\Delta} = \frac{w(0)}{\Delta(0)} f(0), \, \forall f \in \hat{H}^1(\beta\nu), \, w \in H^\infty(\beta), \, \|w\|_\infty = 1.$$

44. For Widom domain $\check{H}^2(\alpha) = \hat{H}^2(\alpha)$ for a. e. α

$\|w_3\|_\infty \le \|w_1\|_\infty = \|w_0\|_\infty - \|\Lambda\| \le 1$. Therefore, the last functional $\Lambda/\|w_3\|_\infty$ has norm ≥ 1. Hence there exists $f \in \hat{H}^1(\beta^{-1}\nu)$, $\|f\|_1 = 1$ such that

$$\frac{w(0)f(0)}{\Delta(0)} = \int_{\mathbb{T}} \frac{wf}{\Delta} dm \ge 1 \tag{13}$$

Factorize $f = h_1 H_2$, $h_1 \in \hat{H}^2(\alpha_0)$ for some $\alpha_0 \in \Gamma^*$, $h_2 \in \hat{H}^2(\alpha_0^{-1}\beta^{-1}\nu)$, $\|h_1\|_2 = \|h_2\|_2 = 1$. We can write these functions as follows

$$h_1 = h_1(0)\frac{\hat{k}^{\alpha_0}}{\sqrt{\hat{k}^{\alpha_0}(0)}} + H_1,\ H_1 \in \hat{H}_0^2(\alpha_0)$$

$$h_2 = h_1(0)\frac{\hat{k}^{\alpha_0^{-1}\beta^{-1}\nu}}{\sqrt{\hat{k}^{\alpha_0^{-1}\beta^{-1}\nu}(0)}} + H_2,\ H_2 \in \hat{H}_0^2(\alpha_0^{-1}\beta^{-1}\nu)$$

45. $w\hat{H}^2(\alpha) \subset \check{H}^2(\alpha\beta)$

On fact, it is very easy to see the following result:

Theorem

Function $w \in H^\infty(\beta)$ *satisfies*

$$\int_{\mathbb{T}} \frac{wf}{\Delta} = \frac{w(0)}{\Delta(0)} f(0),\ \forall f \in \hat{H}^1(\beta\nu) \tag{14}$$

iff $w\hat{H}^2(\alpha) \subset \check{H}^2(\alpha\beta)$.

In fact, given (14) we have

$$\int_{\mathbb{T}} \frac{wh_1 \cdot h_2}{\Delta}\, dm = 0$$

for all $h_1 \in \hat{H}^2(\alpha), h_2 \in \hat{H}^2_0(\alpha^{-1}\beta^{-1}\nu)$. By definition of check spaces on slide 19 this means that $w\hat{H}^2(\alpha) \subset \check{H}^2(\alpha\beta)$. Conversely, if $w\hat{H}^2(\alpha) \subset \check{H}^2(\alpha\beta)$, we just factorize $f \in \hat{H}^1_0(\beta\nu)$ to get (14) for $\forall f \in \hat{H}^1_0(\beta\nu)$. Then $f \in \hat{H}^1(\beta\nu)$ is done, see slide 14.

Therefore,

$$1 = \Big(\frac{|h_1(0)|^2}{\hat{k}^{\alpha_0}(0)} + \|H_1\|_2^2\Big)\Big(\frac{|h_2(0)|^2}{\hat{k}^{\alpha_0^{-1}\beta^{-1}\nu}(0)} + \|H_2\|_2^2\Big).$$

Taking into account (13) from slide 44, we get

$$\frac{|h_1(0)|}{\sqrt{\hat{k}^{\alpha_0}(0)}}\frac{|h_2(0)|}{\sqrt{\hat{k}^{\alpha_0^{-1}\beta^{-1}\nu}(0)}} \le 1 \le \frac{w(0)h_1(0)h_2(0)}{\Delta(0)}$$

And we obtained

$$w(0) \ge \frac{\Delta(0)}{\sqrt{\hat{k}^{\alpha_0}(0)\hat{k}^{\alpha_0^{-1}\beta^{-1}\nu}(0)}} = \frac{\check{e}^{\alpha_0\beta}(0)}{\hat{e}^{\alpha_0}(0)}. \qquad (15)$$

We used here duality formula fro slide 25.

Theorem

$w(0) = \inf_{\alpha\in\Gamma^*} \frac{\check{e}^{\alpha\beta}(0)}{\hat{e}^{\alpha}(0)}$.

47. For Widom domain $\check{H}^2(\alpha) = \hat{H}^2(\alpha)$ for a. e. α

Proof: we need now only

$$w(0) \le \frac{\check{e}^{\alpha\beta}(0)}{\hat{e}^{\alpha}(0)}. \qquad (16)$$

Proof: we come back to relationship (14) on slide 45:

$$\frac{w(0)}{\Delta(0)} f(0) = \int_{\mathbb{T}} \frac{wf}{\Delta}, \ \forall f \in \hat{H}^1(\beta\nu), \ w \in H^\infty(\beta), \ \|w\|_\infty = 1.$$

Take an arbitrary $\alpha \in \Gamma^*$ and test this relationship on

$$f = \hat{e}^{\alpha} \cdot \hat{e}^{\alpha^{-1}\beta^{-1}\nu}.$$

Then $\frac{w(0)\hat{e}^{\alpha}(0)\cdot\hat{e}^{\alpha^{-1}\beta^{-1}\nu}(0)}{\Delta(0)} \le \int |w|\,|f| dm \le 1$. This means that

$$w(0) \le \frac{\Delta(0)}{\hat{e}^{\alpha}(0)\cdot\hat{e}^{\alpha^{-1}\beta^{-1}\nu}(0)} = \frac{\check{e}^{\alpha\beta}(0)}{\hat{e}^{\alpha}(0)}.$$

In the last equality we again used duality formula from slide 24. So (16) is done.

Lemma

w as in (14) of slide 45 or (the same) as i Theorem 20 is a Blaschke product.

Proof: it will be easy to prove that w is an inner function. Lower semi-continuity of the RHS in Theorem 20, slide 46, mean that inf is min, and let α_0 be where it is attained.

$0 \le \|w\hat{e}^{\alpha_0} - \check{e}^{\alpha_0\beta}\|_2^2 + \|(1-|w|^2)^{1/2}\hat{e}^{\alpha_0}\|_2^2 = 2 - 2\langle w\hat{e}^{\alpha_0}, \check{e}^{\alpha_0\beta}\rangle = 2\left(1 - w(0)\frac{\hat{e}^{\alpha_0}(0)}{\check{e}^{\alpha_0\beta}(0)}\right) = 0$ the penultimate equality is because $\check{e}^{\alpha_0\beta} = \frac{\check{k}^{\alpha_0\beta}}{\check{e}^{\alpha_0\beta}(0)}$.

Therefore, $|w| = 1$ a. e. on $\mathbb{T}$, so w is inner.

To prove that it is a Blaschke product is more complicated.

Fortunately all is ready for that. We just saw

$$w = \frac{\check{e}^{\alpha_0\beta}}{\hat{e}^{\alpha_0}} = \frac{(\check{e}^{\alpha_0\beta})_{inn}}{(\hat{e}^{\alpha_0})_{inn}}. \qquad (17)$$

So w_{inn} divides $(\check{e}^{\alpha_0\beta})_{inn}$. Choose $J = J(\check{H}^2(\alpha_0\beta))$. We use r_+ for this operator. Function e_0 below exactly coincides with $\check{e}^{\alpha_0\beta}$. We combine a Theorem of Sodin–Yuditskii and formula (2) from slide 30: $r_+ \circ \mathfrak{z} = -\frac{e_0}{p_0 e_{-1}}$. We already mentioned and used the following Sodin–Yuditskii's theorem:

Theorem

Let Ω be a Widom domain. Let F be meromorphic in Ω, analytic and with positive imaginary part in $\mathbb{C}_+$ and let its poles satisfy the Blaschke condition in Ω. Then $F \circ \mathfrak{z}$ is of bounded characteristic, and F_{inn} is the ratio of two Blaschke products.

Function r_+ is exactly like this, all its poles are in gaps of $E = \partial\Omega$, at most one in each gap of $E = \partial\Omega$, so Blaschke condition on poles is obvious from the fact that our Ω is a Widom domain. Obviously we conclude that $(\check{e}^{\alpha_0\beta})_{inn} = (e_0)_{inn}$ divides the Blaschke product in the numerator of $(r_+)_{inn}$, so it is a Blaschke product itself.

50. For Widom domain $\check{H}^2(\alpha) = \hat{H}^2(\alpha)$ for a. e. α

Now we are ready to prove that Widomness of Ω implies

$$\hat{H}^2(\alpha) = \check{H}^2(\alpha), \ \textit{for } d\alpha a.e.\alpha.$$

Take $\beta = id \in \Gamma^*$ and choose function w as before: $w\hat{H}^2(\alpha) \subset \check{H}^2(\alpha)$. In Widom domain we proved it is necessarily a Blaschke product: $w = \Pi_{j\geq 1} b_{x_j}$. We denote by $\gamma_j^{-1}\Gamma^*$ the character of b_{y_j}. Then

$$\beta_n := \gamma_1 \ldots \gamma_n \to id \text{ in } \Gamma^*.$$

We know $b_{y_1} w_1 \hat{H}^2(\alpha) \subset \check{H}^2(\alpha)$, so $b_{y_1} w_1 \hat{H}^2(\alpha) \subset \check{H}^2_{y_1}(\alpha)$. Now use divisibility theorem (for y_1 not for 0) from slide 21: $w_1 \hat{H}^2(\alpha) \subset \check{b}_{y_1} H^2(\alpha\gamma_1)$. Hence,

$$w_1 \hat{H}^2(\alpha) \subset \check{H}^2(\alpha\gamma_1), \ldots, w_n \hat{H}^2(\alpha) \subset \check{H}^2(\alpha\gamma_1 \ldots \gamma_n) = \check{H}^2(\alpha\beta_n).$$

Theorems on slides 45, 46 imply then that $\forall\alpha\ w_n(0) \leq \frac{\check{e}^{\alpha\beta_n}(0)}{\hat{e}^{\alpha}(0)}$.

51. Finishing the proof that $\check{H}^2(\alpha) = \hat{H}^2(\alpha)$ for a. e. α for Widom domain

Again $\forall\alpha$ $w_n(0) \le \frac{\check{e}^{\alpha\beta_n}(0)}{\hat{e}^{\alpha}(0)}$. So

$$1 \ge \int_{\Gamma^*} \frac{\check{e}^{\alpha}(0)}{\hat{e}^{\alpha}(0)} d\alpha = \int_{\Gamma^*} \frac{\check{e}^{\alpha\beta_n}(0)}{\hat{e}^{\alpha\beta_n}(0)} d\alpha = \int_{\Gamma^*} \frac{\check{e}^{\alpha\beta_n}(0)}{\hat{e}^{\alpha}(0)} \frac{\hat{e}^{\alpha}(0)}{\hat{e}^{\alpha\beta_n}(0)} d\alpha \ge$$

$$\ge w_n(0) \int_{\Gamma^*} \frac{\hat{e}^{\alpha}(0)}{\hat{e}^{\alpha\beta_n}(0)} d\alpha$$

$$1 \ge \liminf_{n\to\infty} w_n(0) \int_{\Gamma^*} \frac{\hat{e}^{\alpha}(0)}{\hat{e}^{\alpha\beta_n}(0)} d\alpha = \lim_{n\to\infty} w_n(0) \liminf_{n\to\infty} \int_{\Gamma^*} \frac{\hat{e}^{\alpha}(0)}{\hat{e}^{\alpha\beta_n}(0)} d\alpha.$$

$\lim_{n\to\infty} w_n(0) = 1$ because w is a Blaschke product $w = b_{y_1} \dots b_{y_n} w_n$. By Fatou's lemma and upper-continuity of hats $\liminf_n \int_{\Gamma^*} \frac{\hat{e}^{\alpha}(0)}{\hat{e}^{\alpha\beta_n}(0)} d\alpha \ge \int_{\Gamma^*} \liminf_n \frac{\hat{e}^{\alpha}(0)}{\hat{e}^{\alpha\beta_n}(0)} d\alpha = \frac{\hat{e}^{\alpha}(0)}{\hat{e}^{\alpha}(0)} d\alpha = 1$.
Therefore,

$$\check{e}^{\alpha}(0) = \hat{e}^{\alpha}(0)\, a.e.$$

So $\check{H}^2(\alpha) = \hat{H}^2(\alpha)$ a. e.

52. There exists $J \in J(E)$, which is not almost periodic

Let $\Theta \subset \Gamma^*$ of irregular points α, that is α: $\check{H}^2(\alpha) \neq \hat{H}^2(\alpha)$.
We know that $\Theta \neq \emptyset$ and the set of regular points is not empty an set, $\mathcal{R} := \Gamma \setminus \Theta \neq \emptyset$ for Widom domains without DCT.
Fix $\alpha \in \Theta$. Denote $\check{J} := J(\check{H}^2(\alpha)), \hat{J} := J(\hat{H}^2(\alpha))$. Fix any $\beta \in \mathcal{R}$ and find subsequence $[|m_n\}$ such that

$$\alpha\mu^{-m_n} \to \beta \text{ in } \Gamma^*.$$

We have $\pi : J(E) \to \Gamma^*$ (Abel map) because every $J \in J(E)$ is $J(H^2(\alpha)$. On the other hand, it is a continuous map as in classical theories. Clearly

$$\pi(S^{m_n}\check{J}S^{-m_n}) = \pi(S^{m_n}\hat{J}S^{-m_n}) = \alpha\mu^{-m_n} \to \beta.$$

Passing to subsequence twice we WLOG think that these sequences weakly converge to some reflectionless J_1, J_2. We saw that $J_1 = J(H^2(\beta)), J_2(H^2(\beta))$. But there is only one **our** space $H^2(\beta)$ as $\beta \in \mathcal{R}$. So $J_1 = J_2 =: J_0$.

53. Almost all $J \in J(E)$ are not almost periodic

Now if both $\check{J}$ and $\hat{J}$ were almost periodic, then passing to subsequence of $\{m_n\}$ (but keeping the notation), we would get Then $\|S^{m_n}\check{J}S^{-m_n} - J_0\| \to 0, \|S^{m_n}\hat{J}S^{-m_n} - J_0\| \to 0$.

$$0 < \|\check{J} - \hat{J}\| = \|S^{m_n}\check{J}S^{-m_n} - \|S^{m_n}\hat{J}S^{-m_n}\| \to 0.$$

Contradiction. So we have a non-almost periodic element from $J(E)$. With purely abs. continuous spectrum (all of them are like that).

Consider any invariant ergodic probability measure σ on $J(E)$. Push it forward by π. Measure $\pi^*\sigma$ is then μ-invariant. But $\mu(\gamma_j) = e^{2\imath i\omega_j}$. In generic position of E these ω_j are rationally independent. So we have only unique μ-invariant ergodic measure. So

$$d\alpha = \text{Haar measure} = \pi^*\sigma.$$

So $\sigma(\pi^{-1}\Theta) = 0$.

54. Almost all $J \in J(E)$ are not almost periodic

Let J_0 be a non a. p. matrix. Take a sequence of open neighborhoods $\{V_n\}$ of J_0, $\cap_n V_n = \{J_0\}$.

Theorem

Let $\Omega - \mathbb{C} \setminus E, E \subset \mathbb{R}$, be a Widom domain such that all Jacobi matrices from $J(E)$ are purely absolutely continuous. Then for any open set V in $J(E)$ (open in the weak topology), one has $\sigma(V) > 0$.

Consider $TJ := SJS^{-1}$, and $\Sigma_n := \cup_m T^{-m} V_n$. By ergodicity of σ and by Theorem above, $\sigma(\Sigma_n) = 1 \, \forall n$. Put $\Sigma := \cap_n \Sigma_n$. Then $\sigma(\Sigma) = 1$. Also $\forall J \in \Sigma$ there is a subsequence $\{m_n\}$ such that $T^{m_n} J \to J_0$ weakly. If J were a. p. then a subsequence of m_n (keep the same notation) would give $\|T^{m_n} J - J_0\| \to 0$. But then for the norm-topology orbits we have

$$\overline{orb J_0}^{\|\cdot\|} \subset \overline{orb J}^{\|\cdot\|}$$

A. p. of J then would imply a. p. of J_0. But J_0 is not a. p. So Σ, $\sigma(\Sigma) = 1$, all consists of non a. p. matrices.

As we already proved (the end of slide 53) that

$$\sigma(\pi^{-1}\mathcal{R}) = 1$$

and that

$$\sigma(\text{non a. p.}) = 1,$$

we can find $\tilde{J}$ such that it is non a. p. and such that $\pi(\tilde{J}) = \tilde{\beta} \in \mathcal{R} = \Gamma^* \setminus \Theta$. Suppose that $J \in J(E)$ is a. p. Let $\pi(J) = \gamma$. Find subsequence $\{m_n\}$ such that

$$\gamma\mu^{-m_n} \to \tilde{\beta}.$$

Then take a weakly converging subsequence (keep the name) such that $T^{m_n}J$ weakly converges to some (of course reflectionless) matrix. As $\tilde{\beta}$ is regular, then there is only one **our** space $H^2(\tilde{\beta}) \Rightarrow T^{m_n} \to \tilde{J}$. Passing to subsequence once more and using that J is a. p. we get that $\|T^{m_n}J - \tilde{J}\| \to 0$. But then $\overline{orb\tilde{J}}^{\|\cdot\|} \subset \overline{orbJ}^{\|\cdot\|}$ Almost periodicity of J then would imply a. p. of $\tilde{J}$. This contradicts the choice of $\tilde{J}$. We are done.

56. Picture of the Abel map $\pi : J(E) rightarrow \Gamma^*$