Integrable discretization and self-adaptive moving mesh method for a class of nonlinear wave equations

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Presentation at
Texas Analysis Mathematical Physics Symposium
Rice University, Houston, TX, USA

October 26, 2013

Outline

- A class of soliton equations with hodograph (reciprocal) transformation and motivation of our research
- Integrable semi-discrete analogues of the short pulse and coupled short pulse equations and its their self-adaptive moving mesh method
- Self-adaptive moving mesh method for the generalized Sine-Gordon equation
- Summary and further topics

- Existence of Lax pair (Lax integrability)

Integrability of nonlinear wave equations

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Integrability of nonlinear wave equations

- Existence of Lax pair (Lax integrability)
- Existence of infinity numbers of symmetries (conservation laws)
- Existence of N-soliton solution
- Pass the Painlevé Test (Painlevé integrability)
- Ask Hirota-sensei

Why integrable discretization?

- Nijhoff: The study of integrability of discrete systems forms at the present time the most promising route towards a general theory of difference equations and discrete systems.
- Hietarinta: Continuum integrability is well established and all easy things have already been done; discrete integrability, on the other hand, is relatively new and in that domain there are still new things to be discovered.

A class of integrable soliton equations share the following common features

- They are related to some well-known integrable systems through hodograph (reciprocal) transformation
- They admit bizarre solutions such as peakon, cuspon, loop or breather solutions.

Motivation

A class of integrable soliton equations share the following common features

- They are related to some well-known integrable systems through hodograph (reciprocal) transformation
- They admit bizarre solutions such as peakon, cuspon, loop or breather solutions.

Motivation of our research project

- Obtain integrable discrete analogues for this class of soliton equations
- Novel integrable numerical schemes for these soliton equations

The Camassa-Holm equation

$$u_t + 2\kappa^2 u_x - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}$$

R. Camassa, D.D. Holm, Phys. Rev. Lett. 71 (1993) 1661

Inverse scattering transform, A. Constantin, (2001)

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Inverse scattering transform, A. Constantin, (2001)

Short wave limit: $t \to \epsilon t, x \to x/\epsilon, u \to \epsilon^2 u$

The Hunter-Saxton equation

$$u_{txx} - 2\kappa^2 u_x + 2u_x u_{xx} + uu_{xxx} = 0$$

Hunter, & Saxton (1991): Nonlinear orientation waves in liquid crystals

Hunter & Zheng (1994): Lax pair, bi-Hamiltonian structure

FMO (2010): Integrable semi- and fully discretizations

The Degasperis-Procesi equation

$$u_t + 3\kappa^3 u_x - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx}\,,$$

A. Degasperis, M. Procesi, (1999)

Degasperis, Holm, Hone (2002)

N-soliton solution, Matsuno (2005)

The Degasperis-Procesi equation

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N-soliton solution, Matsuno (2005)

Short wave limit:

$$u_{txx} - 3\kappa^3 u_x + 3u_x u_{xx} + uu_{xxx} = 0$$

$$\partial_x(\partial_t + u\partial_x)u = 3\kappa^3 u$$

- **Reduced Ostrovsky equation**, L.A. Ostrovsky, *Okeanologia* 18, 181 (1978).
- **Vakhnenko equation**, V. Vakhnenko, *JMP*, 40, 2011 (1999)

Short pulse equation

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx}$$

$$\partial_x \left(\partial_t - \frac{1}{2} u^2 \partial_x \right) u = u$$

- Schäfer & Wayne(2004): Derived from Maxwell equation on the setting of ultra-short optical pulse in silica optical fibers.
- Sakovich & Sakovich (2005): A Lax pair of WKI type, linked to sine-Gordon equation through hodograph transformation;
- Brunelli (2006) Bi-Hamiltonian structure, Phys. Lett. A 353, 475478
- Matsuno (2007): Multisoliton solutions through Hirota's bilinear method
- FMO (2010): Integrable semi- and fully discretizations.

Coupled short pulse equation I

The coupled short pulse equations

$$\begin{cases} u_{xt} = u + \left(\frac{1}{2}uvu_x\right)_x \\ v_{xt} = v + \left(\frac{1}{2}uvv_x\right)_x \end{cases}$$

- Dimakis and Müller-Hoissen (2010), Derived from a bidifferential approach to the AKNS hierarchies.
- Matsuno (2011): Re-derivation, as well as its multi-soliton solution through Hirota's bilinear approach.
- Brunelli and Sakovich (2012) Bi-Hamiltonian structure

Coupled short pulse equation II

$$\begin{cases} u_{xt} = u + u u_x^2 + \frac{1}{2}(u^2 + v^2) u_{xx} \\ v_{xt} = v + v v_x^2 + \frac{1}{2}(u^2 + v^2) v_{xx} \end{cases}$$

$$\begin{cases} \partial_x \left(\partial_t - \frac{1}{2} \left(u^2 + v^2 \right) \partial_x \right) u = u - u_x v v_x \\ \partial_x \left(\partial_t - \frac{1}{2} \left(u^2 + v^2 \right) \partial_x \right) v = v - v_x u u_x \end{cases}$$

- B.F: J. Phys. A 45, 085202 (2012).
- Brunelli & Sakovich: Hamiltonian Integrability, arXiv:1210.5265, (2012).

The generalized sine-Gordon equation

The generalized sine-Gordon equation

$$u_{xt} = (1 + \nu \partial_x^2) \sin u$$

$$\partial_x (\partial_t - \nu \cos u \partial_x) u = \sin u \, .$$

- Proposed by A. Fokas through a bi-Hamiltonian method (1995)
- Matsuno gave a variety of soliton solutions such as kink, loop and breather solutions (2011)
- Under the short wave limit $\bar{u} = u/\epsilon, \bar{x} = (x-t)/\epsilon$, $\bar{t} = \epsilon t$, it converges to the short pulse equation.
- Under the long wave limit $\bar{u} = u, \bar{x} = \epsilon x$, $\bar{t} = t/\epsilon$, it converges to the sine-Gordon equation.

Integrable discretization and integrable numerical scheme

Equation	Integrable discretization	Self-adaptive moving mesh method
CH eq.	Yes	Yes
HS eq.	Yes	Numerical difficulty?
DP eq.	Yes	Under Construction
VE eq.	Yes	Yes
SP eq.	Yes	Yes
CSPI eq.	Yes	Yes
CSPII eq.	Yes	Yes
GsG eq.	Yes	Yes

Bilinear equations of the short pulse equation

Theorem (Matsuno 2007)

The short pulse equation

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx}$$

can be derived from bilinear equations

$$\begin{cases} \left(\frac{1}{2}D_sD_y - 1\right) f \cdot f = -\bar{f}^2\,, \\ \left(\frac{1}{2}D_sD_y - 1\right) \bar{f} \cdot \bar{f} = -f^2\,, \end{cases}$$

through the hodograph transformation

$$x(y,s) = y - 2\left(\ln \bar{f} f\right)_s\,, \quad t(y,s) = s$$

and the dependent variable transformation

$$u(y,s) = 2\mathrm{i}\left(\ln \frac{\bar{f}(y,s)}{f(y,s)}\right)_s\,.$$

Integrable semi-discrete short pulse equation

Theorem (FMO 2010, FIKMO2011)

The semi-discrete short pulse equation

$$\begin{cases} \frac{d}{ds}(u_{k+1}-u_k) = \frac{1}{2}(x_{k+1}-x_k)(u_{k+1}+u_k)\,, \\ \frac{d}{ds}(x_{k+1}-x_k) = -\frac{1}{2}(u_{k+1}^2-u_k^2)\,, \end{cases}$$

is derived from bilinear equations:

$$\begin{cases} \left(\frac{1}{a}D_s-1\right) f_{k+1}\cdot f_k = -\bar{f}_{k+1}\bar{f}_k, \\ \left(\frac{1}{a}D_s-1\right) \bar{f}_{k+1}\cdot \bar{f}_k = -f_{k+1}\bar{f}_k. \end{cases}$$

through discrete hodograph transformation and dependent variable transformation

$$u_k = 2\mathrm{i}\left(\ln\frac{\bar{f}_k}{f_k}\right)_s, \quad x_k = 2ka - 2(\log f_k g_k)_s, \quad \delta_k = x_{k+1}-x_k\,.$$

Bilinear equations of the coupled short pulse equation

Theorem (Matsuno 2011)

The coupled short pulse equation

$$\begin{cases} u_{xt} = u + \frac{1}{2}\left(uvu_x\right)_x \,, \\ v_{xt} = v + \frac{1}{2}\left(uvv_x\right)_x \,. \end{cases}$$

can be derived from bilinear equations

$$\begin{cases} D_s D_y f \cdot g_i = f g_i \,, \quad i = 1, 2 \\ D_s^2 f \cdot f = \frac{1}{2} g_1 g_2 \,, \end{cases}$$

through the hodograph and dependent variable transformations

$x(y,s) = y - 2\left(\ln f\right)_s$, $t(y,s) = s$, $u(y,s) = \frac{g_1(y,s)}{f(y,s)}$, $v(y,s) = \frac{g_2(y,s)}{f(y,s)}$

Theorem (FMO2013)

The semi-discrete coupled short pulse equation

$$\left\{\begin{array}{l} \frac{d}{ds}(u_{k+1}-u_k) = \frac{1}{2}(x_{k+1}-x_k)(u_{k+1}+u_k)\,, \\ \frac{d}{ds}(v_{k+1}-v_k) = \frac{1}{2}(x_{k+1}-x_k)(v_{k+1}+v_k)\,, \\ \frac{d}{ds}(x_{k+1}-x_k) = -\frac{1}{2}(u_{k+1}v_{k+1}-u_kv_k)\,, \end{array}\right.$$

is derived from bilinear equations:

$$\left\{\begin{array}{l} \frac{1}{a}D_s(g^{(i)}_{k+1}\cdot f_k - g^{(i)}_k\cdot f_{k+1}) = g^{(i)}_{k+1}f_k + g^{(i)}_k f_{k+1}\,, \quad i=1,2 \\ D_s^2 f_k\cdot f_k = \frac{1}{2}g^{(1)}_k g^{(2)}_k\,, \end{array}\right.$$

through discrete hodograph transformation and dependent variable transformations $x_k = 2ka - 2(\ln f_k)_s$, $u_k = \frac{g^{(1)}_k}{f_k}$, $v_k = \frac{g^{(2)}_k}{f_k}$.

Pfaffian solution to semi-discrete coupled short pulse equation

Theorem

The semi-discrete coupled short pulse equation has the following pfaffian solution

$$\begin{aligned} f_k &= \mathrm{Pf}(a_1, \cdots, a_{2N}, b_1, \cdots, b_N, c_1, \cdots, c_N)_k \,, \\ g_k^{(i)} &= \mathrm{Pf}(d_0, \beta_i, a_1, \cdots, a_{2N}, b_1, \cdots, b_N, c_1, \cdots, c_N)_k \,, \end{aligned}$$

where

$$(a_i, a_j)_k = \frac{p_i - p_j}{p_i + p_j} \varphi_i^{(0)}(k) \varphi_j^{(0)}(k) \,, (a_i, b_j)_k = \delta_{i,j}, (a_i, c_j)_k = \delta_{i,j+N} \,,$$

$$(d_n, a_i)_k = \varphi_i^{(n)}(k) \,, (a_i, d^k)_k = \varphi_i^{(n)}(k+1) \,, (b_i, c_j) = -\frac{1}{4} \frac{(p_i p_{N+j})^2}{p_i^2 - p_{N+j}^2} \,.$$

$$(b_i, \beta_1) = (c_i, \beta_2) = 1 \,, \quad (d_0, d^k) = 1, \quad (d_{-1}, d^k) = -a \,.$$

$$\varphi_i^{(n)}(k) = p_i^n \left(\frac{1 + a p_i}{1 - a p_i} \right)^k e^{\xi_i}, \quad \xi_i = \frac{1}{p_i} s + \xi_{i0} \,.$$

We apply the semi-implicit Euler scheme to the semi-discrete short pulse equation

$$\begin{cases} \frac{d}{ds}(u_{k+1}-u_k) = \frac{1}{2}\delta_k(u_{k+1}+u_k)\,, \\ \frac{d}{ds}(x_{k+1}-x_k) = -\frac{1}{2}(u_{k+1}^2-u_k^2)\,, \end{cases}$$

as follows

$$\begin{cases} p_k^{n+1} = p_k^n + \frac{1}{2}\delta_k^n(u_{k+1}^n+u_k^n)\Delta t\,, \\ \delta_k^{n+1} = \delta_k^n - \frac{1}{2}\left((u_{k+1}^{n+1})^2-(u_k^{n+1})^2\right)\Delta t\,, \end{cases}$$

where $p_k^n = u_{k+1}^n - u_k^n$, $\delta_k^n = x_{k+1}^n - x_k^n$.

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where $p_k^n = u_{k+1}^n - u_k^n$, $\delta_k^n = x_{k+1}^n - x_k^n$.

- The quantity $\sum \delta_k$, which corresponds to one of the conserved quantities in the short pulse equaion is conserved.
- Although the semi-implicit Euler is a first-order integrator, it is symplectic. In other words, this scheme is symplectic for another quantity, the Hamiltonian, of the short pulse equation.
- The mesh is evolutive and self-adaptive, so we name it **self-adaptive**

Coupled short pulse equation

$$\begin{cases} u_{xt} = u + \frac{1}{2}\left(uvu_x\right)_x, \\ v_{xt} = v + \frac{1}{2}\left(uvv_x\right)_x. \end{cases}$$

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Integrable semi-discrete analogue

$$\begin{cases} \frac{d}{ds}(u_{k+1} - u_k) = \frac{1}{2}\delta_k(u_{k+1} + u_k)\,, \\ \frac{d}{ds}(v_{k+1} - v_k) = \frac{1}{2}\delta_k(v_{k+1} + v_k)\,, \\ \frac{d}{ds}(x_{k+1} - x_k) = -\frac{1}{2}(u_{k+1}v_{k+1} - u_k v_k)\,, \end{cases}$$

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Self-adaptive moving mesh scheme

$$\begin{cases} p_k^{n+1} = p_k^n + \frac{1}{2}\delta_k^n(u_{k+1}^n + u_k^n)\Delta t\,, \\ q_k^{n+1} = q_k^n + \frac{1}{2}\delta_k^n(v_{k+1}^n + v_k^n)\Delta t\,, \\ \delta_k^{n+1} = \delta_k^n - \frac{1}{2}\left(u_{k+1}^{n+1}v_{k+1}^{n+1} - u_k^{n+1}v_k^{n+1}\right)\Delta t\,. \end{cases}$$

where $p_k = u_{k+1} - u_k$, $q_k = v_{k+1} - v_k$.

Numerical solution to one-loop solution

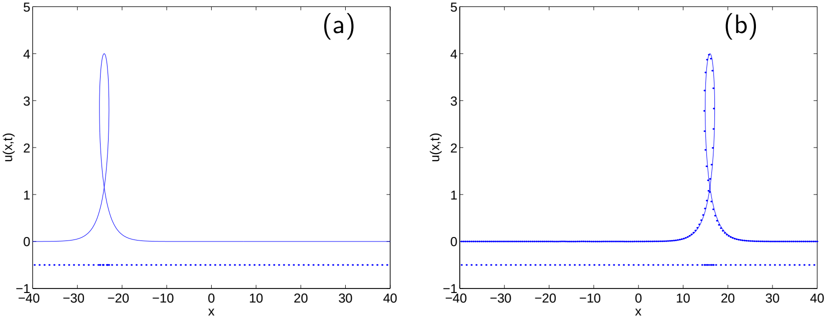

Figure : One-loop solution to the SP equation for $\boldsymbol{p_1 = 1.0}$; (a) t=0; (b) t=10.0.

Two-loop interaction

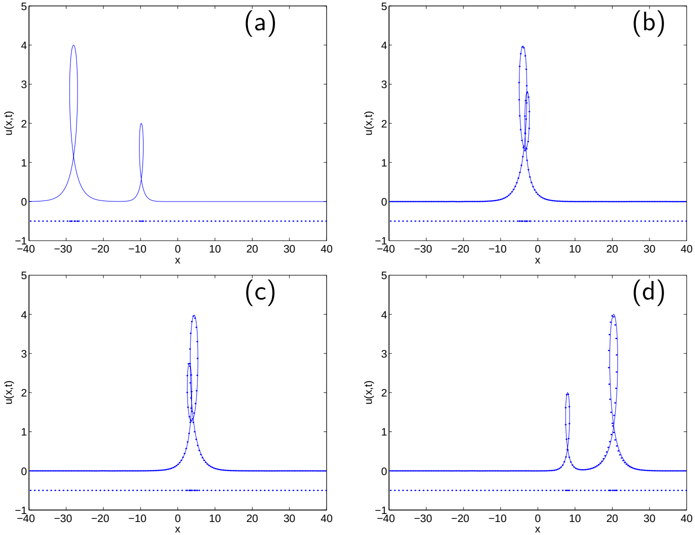

Figure : Two loop interaction;(a) t=0; (b) t=6.0; (c) t=8; (d) t=12.

One breather solution

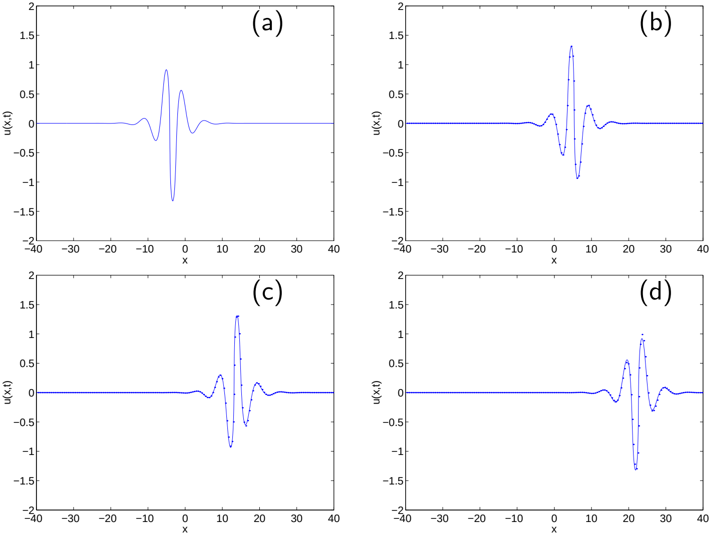

Figure : One breather solution; (a) t=0; (b) t=10.0; (c) t=20; (d) t=30.

Loop-breather Interaction

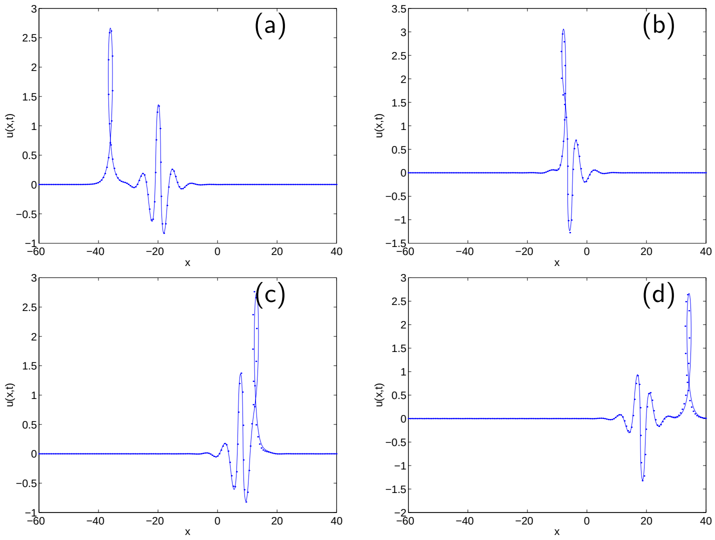

Figure : Loop-breather interaction; (a) t=0; (b) t=16; (c) t=28; (d) t=40.

Numerical solution for one-loop solution of coupled short pulse equation

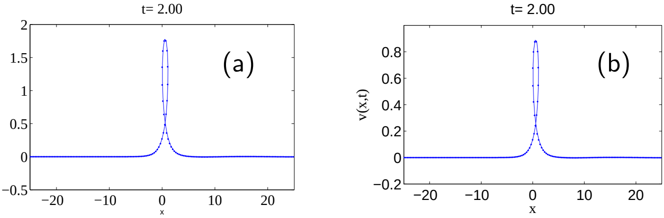

Figure : One-loop solution to the CSP equation (a) $x - u$ at $t = 2$; (b) $x - v$ at $t = 2.0$.

Numerical solution for two-loop solution of coupled short pulse equation

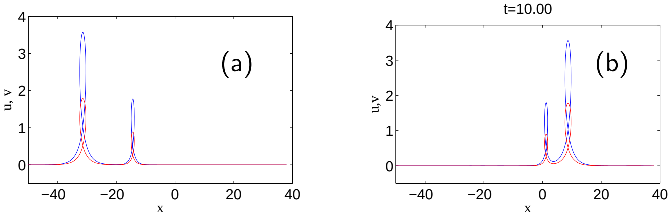

Figure : Two-loop solution to the CSP equation; (a) t=0; (b) t=10.0.

A semi-discrete system obtained from generalized sine-Gordon equation

The generalized sine-Gordon equation

$$u_{tx} = (1 + \nu \partial_x^2) \sin u$$

$$\partial_x (\partial_t - \nu \cos u \partial_x) u = \sin u \, .$$

A semi-discrete system

$$\begin{cases} \frac{d}{ds}(u_{k+1} - u_k) = \frac{1}{2}\delta_k (\sin u_{k+1} + \sin u_k) \, , \\ \frac{d}{ds}(x_{k+1} - x_k) = -\nu (\cos u_{k+1} - \cos u_k) \, , \end{cases}$$

Numerical solution for generalized sine-Gordon equation

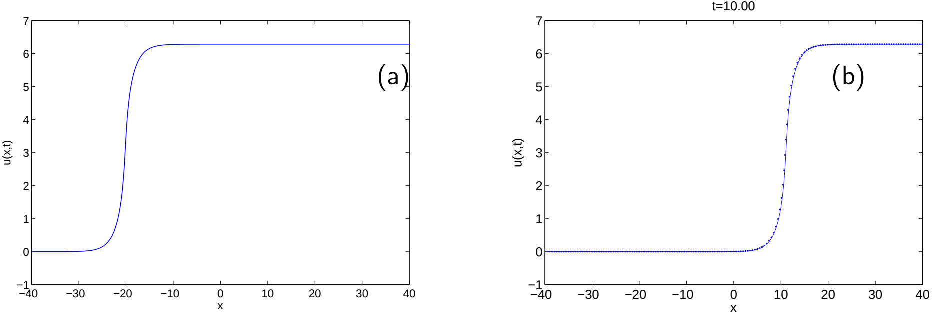

Figure : Regular kink solution to the generalized sine-Gordon equation(a) $t = 0$; (b) $t = 10.0$.

Numerical solution for generalized sine-Gordon equation

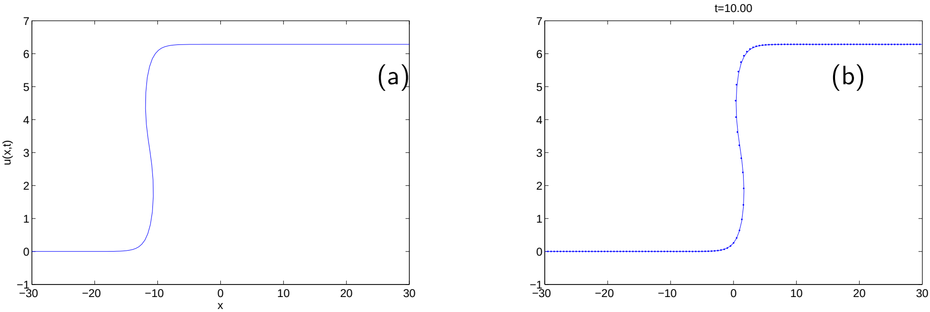

Figure : Irregular kink solution to the generalized sine-Gordon equation(a) $t = 0$; (b) $t = 10.0$.

Summary and further topics

- A novel numerical method: integrable self-adaptive moving mesh method, is born from integrable discretizations of a class of soliton equations with hodograph transformation

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- A novel numerical method: integrable self-adaptive moving mesh method, is born from integrable discretizations of a class of soliton equations with hodograph transformation
- A self-adaptive moving mesh method is not necessarily to be integrable
- Further topic 1: High order symplectic numerical method for the implementation of the self-adaptive moving mesh method
- Further topic 2: self-adaptive moving mesh method for soliton equations without hodograph transformation and non-integrable wave equations