Cyclic vectors in Dirichlet-type spaces

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Broader Impacts of the problem of cyclicity

- Invariant subspace problem and cyclic vectors:
  Does every bounded operator T on a Hilbert space $\mathcal{H}$ have a non-trivial closed *invariant* subspace (i.e. $T(W) \subset W$)?
  NO, IF one can find an operator T such that every $\mathbf{0} \neq \varphi \in \mathcal{H}$ is *cyclic* (i.e. $\mathcal{H} = \operatorname{clos}\operatorname{span}\{T^n \varphi : n \in \mathbb{N}\}$).
- Structure (basic building blocks) of a function space determined by its cyclic vectors
- Brown–Shields conjecture
- For physicists, the cyclicity of an operator means that the spectrum has multiplicity one

One complex variable

Dirichlet-type spaces and cyclic vectors

- Consider the *Dirichlet-type spaces* $\mathcal{D}_\alpha$, i.e. bounded analytic functions on the unit disk $\mathbb{D} \subset \mathbb{C}$ with norm $\|f\|^2_{\mathcal{D}_\alpha} = \sum_{k=0}^\infty (k+1)^\alpha |a_k|^2 < \infty$, where $f(z) = \sum_{k=0}^\infty a_k z^k$
- Bergman $A^2 = \mathcal{D}_{-1}$; Hardy $H^2 = \mathcal{D}_0$; and Dirichlet $\mathcal{D} = \mathcal{D}_1$
- A vector f is *cyclic* (under the forward shift) for $\mathcal{D}_\alpha$ if

$$\mathcal{D}_\alpha = \overline{\mathrm{span}\{z^k f(z) : k \in \mathbb{N} \cup \{0\}\}}$$

- The constant function $\mathbf{1}$ is cyclic for $\mathcal{D}_\alpha$
- $f \in \mathcal{D}_\alpha$ cyclic, implies $f(z) \neq 0$ for $z \in \mathbb{D}$

"The fewer zeros the easier is cyclicity."

Optimality

- Note f is cyclic in $\mathcal{D}_\alpha$ iff

$$N_n(f,\alpha) := \inf_{p_n} \|p_n f - 1\|^2_{\mathcal{D}_\alpha} \to 0 \quad \text{as } n \to \infty$$

- If $f(z) = 1 - z$, then $p_n =$ (order n Taylor poly. of $1/f$) yields

$$\|p_n f - 1\|^2_{\mathcal{D}_\alpha} = n + 2$$

Two types of results:

- Optimal sequence of polynomials p_n
- The optimal rate of decay of these norms $N_n(f,\alpha)$ as $n \to \infty$

Example of explicit optimal approximants

For $f(z) = 1 - z$, optimal for

$$H^2: \qquad C_n(z) = \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) z^k,$$

$$\mathcal{D}: \qquad R_n(z) = \sum_{k=0}^{n} \left(1 - \frac{H_{k+1}}{H_{n+2}}\right) z^k, \qquad H_n = \sum_{k=2}^{n} \frac{1}{k}\,,$$

$$A^2: \qquad S_n(z) = \sum_{k=0}^{n} \left(1 - \frac{k(k+3)}{(n+1)(n+4)}\right) z^k.$$

Rate of decay

Let $H_n = \sum_{k=2}^{n} \frac{1}{k}$ and note that $H_n \approx \log n$ for large n.

Definition

For $\alpha < 1$, we set $\varphi_\alpha(n) = n^{\alpha-1}$, $n \in \mathbb{N}$.
For $\alpha = 1$, we use $\varphi_1(n) = 1/H_n$, $n \in \mathbb{N}$.

Theorem (Bénéteau–Condori–L.–Seco–Sola, J. d'A. accepted)

Suppose $f \in \mathcal{D}_\alpha$, $\alpha \leq 1$, can be extended analytically to some strictly bigger disk. Suppose also that f does not vanish in $\mathbb{D}$. Then there exists a constant C_0 so that the optimal norm satisfies

$$N_n(f, \alpha) \leq C_0 \varphi_\alpha(n+1).$$

Moreover, for polynomial f with zero on $\mathbb{T}$, and $\alpha = 1, 0, -1$, there is a constant C_1 so that

$$C_1 \varphi_\alpha(n+1) \leq N_n(f, \alpha).$$

Polynomials that have no zeros in $\mathbb{D}$ are cyclic in $\mathcal{D}_\alpha$ for $\alpha \leq 1$.

Partial result on the Brown–Shields conjecture

Outer

- Vectors in H^2 are cyclic iff they are outer
- For $\alpha \geq 0$: If f cyclic in $\mathcal{D}_\alpha$, then f outer

Logarithmic capacity

- Non-tangentially $f^*(\zeta) = \lim_{z \to \zeta \in \mathbb{T}} f(z)$
- For $f \in \mathcal{D}$, f^* exists outside a set of logarithmic capacity zero
- Zero set $Z(f) = \{\zeta \in \mathbb{T} : f^*(\zeta) = 0\}$
- Brown–Shields: If $f \in \mathcal{D}$ is cyclic, then $Z(f)$ has capacity zero

Brown–Shields Conjecture (1984)

A vector $f \in \mathcal{D}$ is cyclic iff it is outer and has $Z(f)$ capacity zero.

Brown–Cohn: For any closed set of logarithmic capacity zero $E \subset \mathbb{T}$, there exists a cyclic function f in $\mathcal{D}$ with $Z(f) = E$.

Two weak versions of the Brown–Shields conjecture:

Theorem (Hedenmalm–Shields 1990, Richter–Sundberg 1994)

A vector $f \in \mathcal{D}$ is cyclic, if it is outer and $Z(f)$ is countable.

Theorem (El-Fallah–Kellay–Ransford 2006)

The condition 'countable' can be replaced by one which is closer to 'capacity zero', but VERY complicated.

Theorem (Bénéteau–Condori–L.–Seco–Sola, J. d'A. accepted)

Suppose $f \in \mathcal{D}$ *and* $\log f \in \mathcal{D}$. *Then* f *is cyclic in* $\mathcal{D}$.

Theorem (Bénéteau–Condori–L.–Seco–Sola, J. d'A. accepted)

Let $f \in H^\infty$ *and* $q = \log f \in \mathcal{D}_\alpha$, $\alpha \le 1$. *Suppose there exist polynomials* q_n *of degree* $\le n$ *that approach* q *in* $\mathcal{D}_\alpha$ *norm with*

$$\sup_{z \in \mathbb{D}} \mathrm{Re}(q(z) - q_n(z)) + \log \|q_n - q\| \le C$$

for some constant $C > 0$. *Then* f *is cyclic in* $\mathcal{D}_\alpha$.

Brown–Cohn's examples satisfy above assumptions.

Two complex variables

Dirichlet-type space on the bidisk

- Bidisk $\mathbb{D}^2 = \{(z_1, z_2) \in \mathbb{C}^2 \colon |z_1| < 1, |z_2| < 1\}$
- Holomorphic $f \colon \mathbb{D}^2 \to \mathbb{C}$ belongs to the *Dirichlet-type space* $\mathfrak{D}_\alpha$ if its power series $f(z_1, z_2) = \sum_{k=0}^\infty \sum_{l=0}^\infty a_{k,l} z_1^k z_2^l$ satisfies

$$\|f\|_\alpha^2 = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} (k+1)^\alpha (l+1)^\alpha |a_{k,l}|^2 < \infty$$

- Function $f \in \mathfrak{D}_\alpha$ is *cyclic*, if

$$\mathfrak{D}_\alpha := \overline{\mathrm{span}\{z_1^k z_2^l f \colon k = 0, 1, \ldots; l = 0, 1, \ldots\}}$$

- Let $\mathfrak{P}_n$, $n \in \mathbb{N}$, be the polynomials of the form

$$p_n = \sum_{k=0}^{n} \sum_{l=0}^{n} c_{k,l} z_1^k z_2^l$$

- f is cyclic iff $\mathfrak{N}_n(f, \alpha) := \inf_{p_n \in \mathfrak{P}_n} \|p_n f - 1\|_{\mathfrak{D}_\alpha}^2 \overset{n\to\infty}{\to} 0$

Reductions to functions of one variable

Reduction to functions of one variable

- Consider

$$\mathcal{J}_{\alpha,M,N} := \left\{ f \in \mathfrak{D}_\alpha \colon f = \sum_{k=0}^{\infty} a_k z_1^{Mk} z_2^{Nk} \right\},$$

e.g. $f(z_1, z_2) = 1 - z_1 z_2 \in \mathcal{J}_{\alpha,1,1}$

- Consider the mappings

$$\begin{aligned} L_{M,N}\colon \quad \mathcal{D}_{2\alpha} \to \mathfrak{D}_\alpha \quad &\text{via} \quad L_{M,N}(F)(z_1, z_2) = F(z_1^M \cdot z_2^N), \\ R_{M,N}\colon \mathcal{J}_{\alpha,M,N} \to \mathcal{D}_{2\alpha} \quad &\text{via} \quad R_{M,N}(f)(z) = f(z^{1/M}, 1) \end{aligned}$$

- If $f \in \mathcal{J}_{\alpha,M,N}$, there exist constants such that

$$c_2 \|R(f)\|_{\mathcal{D}_{2\alpha}} \leq \|f\|_\alpha \leq c_1 \|R(f)\|_{\mathcal{D}_{2\alpha}}$$

Note the change from $\mathfrak{D}_\alpha$ for bidisk to $\mathcal{D}_{2\alpha}$ for disk!

Theorem (Bénéteau–Condori–L.–Seco–Sola, submitted 2013)

Let $f \in \mathcal{J}_{\alpha,M,N}$ *have the property that* $R(f) = f(z^{1/M}, 1)$ *is a function that admits an analytic continuation to the closed unit disk, whose zeros lie in* $\mathbb{C} \setminus \mathbb{D}$.
Then f *is cyclic in* $\mathfrak{D}_\alpha$, *and there exists a constant* $C = C(\alpha, f, M, N)$ *such that*

$$\mathfrak{N}_n(f, \alpha) \leq C \varphi_{2\alpha}(n+1).$$

This result is sharp in the sense that, if $R(f)$ *has at least one zero on* $\mathbb{T}$, *then there exists* $c = c(\alpha, f, M, N)$ *such that for large* n:

$$c\varphi_{2\alpha}(n+1) \leq \mathfrak{N}_n(f, \alpha).$$

Here $\varphi_{2\alpha}(n) = \left\{ \begin{array}{ll} n^{2\alpha-1} & \text{for } 2\alpha < 1 \\ 1/\sum_{k=2}^{n} \frac{1}{k} & \text{for } 2\alpha = 1 \end{array} \right\}$ increases if $\alpha > 1/2$.

Examples

- Functions like $f(z_1, z_2) = 1 - z_1$, $f(z_1, z_2) = (1 - z_1 z_2)^N$, $N \in \mathbb{N}$, and $f(z_1, z_2) = z_1^2 z_2^2 - 2(\cos\theta) z_1 z_2 + 1$, $\theta \in \mathbb{R}$, satisfy the assumptions of the theorem
- Polynomial $g(z_1, z_2) = 1 - z_1 z_2$ is not cyclic in $\mathfrak{D}_\alpha$ for $\alpha > 1/2$, although it is only zero for $z_1 = z_2 = 1$
- Notice that g is outer, but its zero set $\{z_1 = z_2 = 1\}$ has non-zero logarithmic capacity

Open problems

- The *Brown-Shields conjecture* for functions on the bidisk: Is the condition that $f \in \mathfrak{D}$ is outer and the zero set of f (on the boundary) has logarithmic capacity 0 sufficient for f to be cyclic?

- Sub-problem: Characterize the *cyclic polynomials* $f \in \mathfrak{D}_\alpha$ for each $\alpha \leq 1$.