Existence and stability of traveling pulse solutions for the FitzHugh-Nagumo equation

(**with G. Arioli**)

(1) **CAP Examples**

(2) **The FitzHugh-Nagumo model**

(3) **New results**

(4) **Existence of pulse solutions**

(5) **Stability**

(6) **Eigenvalues**

(7) **Some details**

(8) **More details**

Favored problems for computer-assisted proofs
are equations with no free parameters.

Some appear in renormalization, like

The hierarchical model

$$\boldsymbol{F^2} * \text{Gaussian} \equiv \boldsymbol{F} \quad \text{(mod scaling)}$$

The Feigenbaum-Cvitanović equation

$$\boldsymbol{F} \circ \boldsymbol{F} \equiv \boldsymbol{F} \quad \text{(mod scaling)}$$

MacKay's fixed point equation (commuting area-preserving maps)

$$\begin{bmatrix} \boldsymbol{G} \\ \boldsymbol{F} \circ \boldsymbol{G} \end{bmatrix} \equiv \begin{bmatrix} \boldsymbol{F} \\ \boldsymbol{G} \end{bmatrix} \quad \text{(mod scaling)}$$

A related equation for Hamiltonians on $\mathbb{T}^2 \times \mathbb{R}^2$

$$\boldsymbol{H} \circ \text{Nontrivial} \equiv \boldsymbol{H} \quad \text{(mod trivial)}$$

Current developments in CAPs
focus on problems that (cannot be solved by hand and) involve

- exploring new areas of application,
- developing new methods,
- testing the boundaries of what is feasible.

Low regularity problems and
boundary value problems on nontrivial domains, like

$$\Delta u = f(u) \quad \text{on } \Omega \,, \qquad f = g \quad \text{on } \partial\Omega \,.$$

Beginnings by [M.T. Nakao, N. Yamamoto, . . . 1995+].

Orbits in dissipative PDEs like Kuramoto-Sivashinsky,

$$\partial_t u + 4\partial_x^4 u + \alpha\partial_x^2 u + 2\alpha u \partial_x u = 0 \,.$$

Periodic: [P. Zgliczyński 2008-10; G. Arioli, H.K. 2010].
Chaotic: a long term goal.

Existence and stability of waves and patterns.
A good starting point is the Fitzhugh-Nagumo equation in 1 spatial dimension,

$$\begin{aligned} \partial_t w_1 &= \partial_x^2 w_1 + f(w_1) - w_2 \,, \\ \partial_t w_2 &= \epsilon(w_1 - \gamma w_2) \,. \end{aligned}$$

See page 5.

Plus all the exciting work by

M. Berz , R. Castelli , S. Day , R. de la Llave , D. Gaidashev , M. Gameiro , A. Haro , J.M. James, T. Johnson, S. Kimura, J.P. Lessard, K. Mischaikow, M. Mrozek, M.T. Nako, S. Oishi, M. Plum, S.M. Rump, W. Tucker, J.B. van den Berg, D. Wilczak, N. Yamamoto, P. Zgliczyński,

and many others.

Motivation for our current work:

[D. Ambrosi, G. Arioli, F. Nobile, A. Quarteroni 2011] proposed and studied numerically an improved version of the Fitzhugh-Nagumo equation:

$$\partial_t\big((1-\beta w_1)w_1\big) = \partial_x\big((1-\beta w_1)^{-1}\partial_x w_1\big) + (1-\beta w_1)f(w_1) - (1-\beta w_1)w_2\,,$$
$$\partial_t\big((1-\beta w_1)w_2\big) = \epsilon(1-\beta w_1)\big(w_1-\gamma w_2\big)\,.$$

Existence of a pulse solution: proved in [D. Ambrosi, G. Arioli, H.K. 2012].
Stability: ?

The FitzHugh-Nagumo equations in one spatial dimension are

$$\begin{aligned} \partial_t w_1 &= \partial_x^2 w_1 + f(w_1) - w_2\,, \\ \partial_t w_2 &= \epsilon(w_1 - \gamma w_2)\,, \end{aligned}$$

with $\epsilon, \gamma \geq 0$ and

$$f(r) = r(r-a)(1-r)\,, \qquad 0 < a < \tfrac{1}{2}\,.$$

They describe the propagation of electrical signals in biological tissues.
$w_1 = w_1(x,t)$ action potential (voltage difference across cell membrane).
$w_2 = w_2(x,t)$ gate variable (fraction of ion channels that are open, slow recovery).
ϵ^{-1} recovery time.

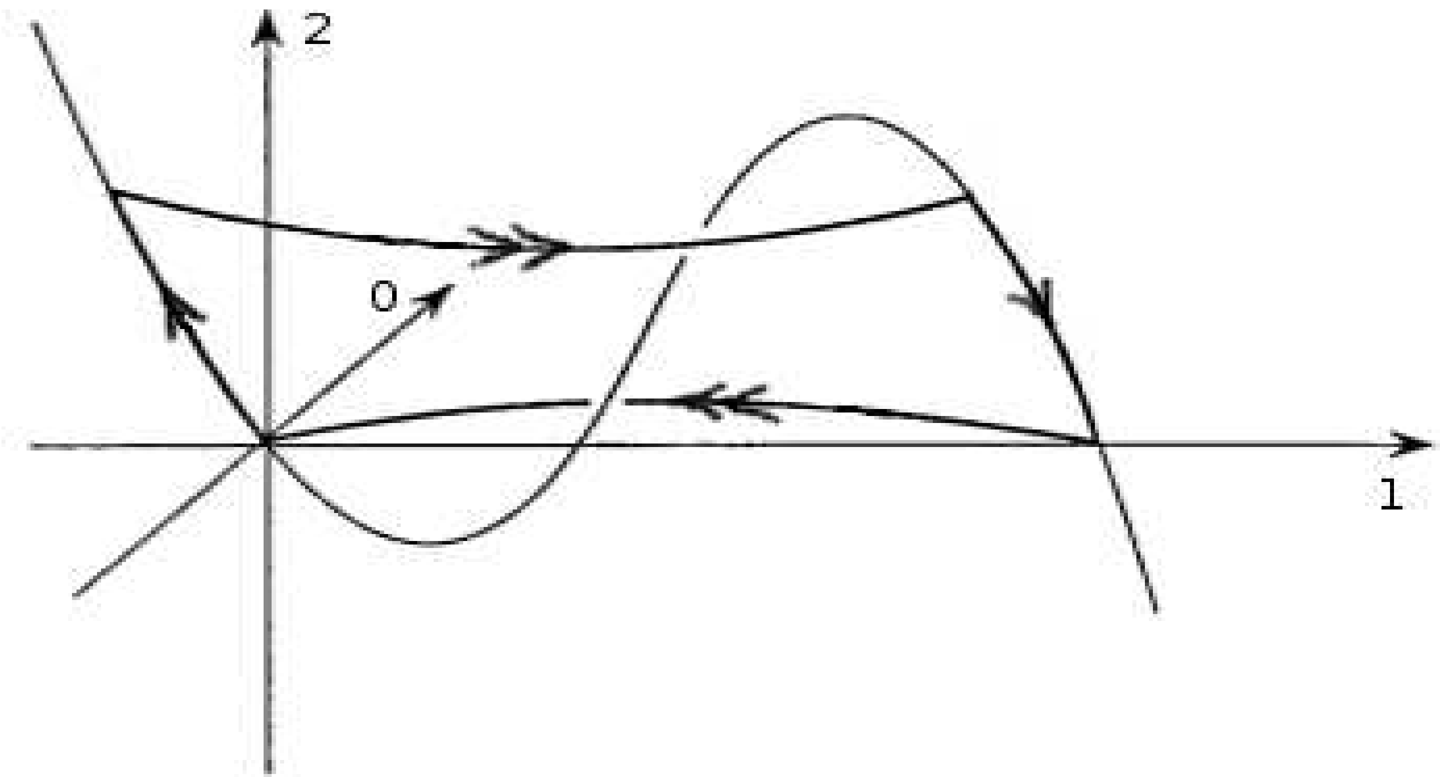
We consider both on the circle $\mathbb{S}_\ell = \mathbb{R}/(\ell\,\mathbb{Z})$ with circumference $\ell = 128$, and the real line $\mathbb{R}$.
A pulse traveling with velocity c is a solution $w_j(x,t) = \phi_j(x - ct)$.
The equation for such a pulse can be written as

$$\phi' = X(\phi)\,, \qquad \phi = \begin{bmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \end{bmatrix}, \qquad X(\phi) = \begin{bmatrix} -c\phi_0 - f(\phi_1) + \phi_2 \\ \phi_0 \\ -c^{-1}\epsilon(\phi_1 - \gamma\phi_2) \end{bmatrix}.$$

In the case of a pulse on $\mathbb{R}$, one also imposes the conditions $\phi(\pm\infty) = 0$.
So a pulse ϕ corresponds to a homoclinic orbit for X.

For $\epsilon = 0$ and any $c > 0$ we have $X(\phi) = 0 \iff \phi_0 = 0\,,\ \phi_2 = f(\phi_1)\,,$
with $DX(\phi)$ having no positive eigenvalue at a fixed point ϕ when $f'(\phi_1) < 0$.
For a specific velocity $c > 0$ there are “connecting orbits” as shown below.

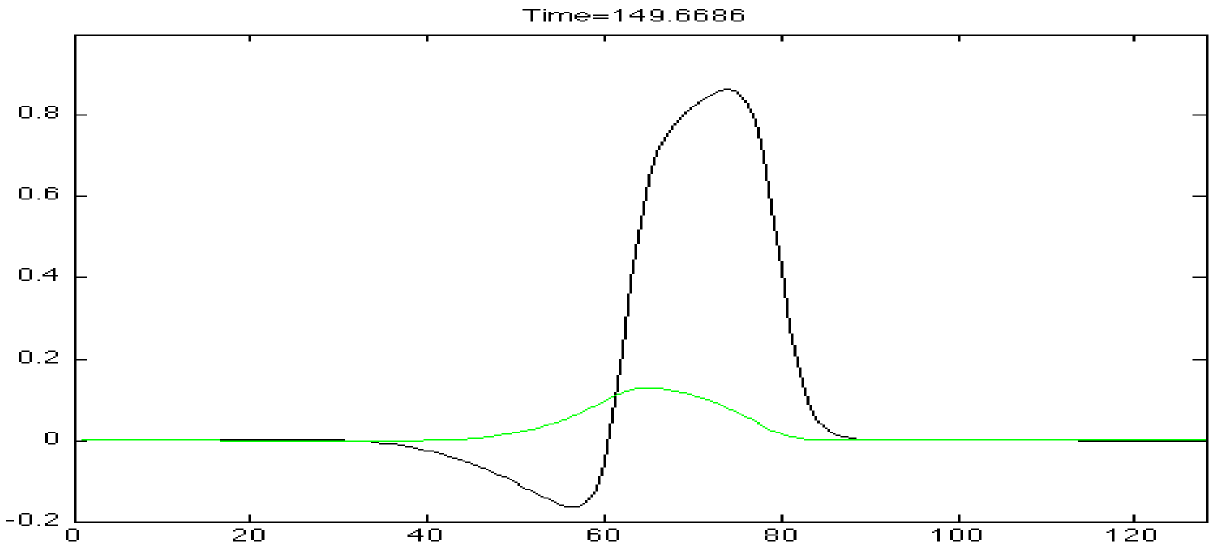
For $\varepsilon \ll 1$:



Existence of a fast pulse for some $c > 0$ by [S. Hastings 1976; G. Carpenter 1977; . . .] and several others.
Stability of the pulse by [C.K.R.T. Jones 1984, E. Yanagida 1985] using results from [J.W. Evans I-IV 1972-1975]

Consider now more "standard" model values $\epsilon = \frac{1}{100}\,,\qquad \gamma = 5\,,\qquad a = \frac{1}{10}\,,$ and $\ell = 128$ in the periodic case.

Theorem 1. *The FHN equation on $\mathbb{R} \times \mathbb{S}_\ell$ has a real analytic and exponentially stable traveling pulse solution with velocity $c = 0.470336308\ldots$*



Theorem 2. *The FHN equation on $\mathbb{R} \times \mathbb{R}$ has a traveling pulse solution with $c = 0.470336270\ldots$ This solution is real analytic, decreases exponentially at infinity, and is exponentially stable.*

<u>Space $\mathcal{C}$</u>: w_1 and w_2 are bounded and uniformly continuous in x. Using the sup-norm.
<u>Domain $\mathcal{C}'$</u>: $w_1, \partial_x w_1, \partial_x^2 w_1, w_2, \partial_x w_2$ belong to $\mathcal{C}$.

We use the notation $\underline{w} = [w_1\ w_2]^\top$ and $w_j : t \mapsto w_j(t)$ and $w_j(t) : x \mapsto w_j(x,t)$.
A pulse solution $\underline{\phi}$ is **exponentially stable** if for any nearby solution $\underline{w} \in \mathcal{C}'$ of FHN, $\underline{w}(t)$ converges exponentially to some translate of $\underline{\phi}$, as $t \to \infty$.
The exponential rate is fixed, and other constants only depend on norms.

Steps of the proof.

(1) Determine the pulse and its velocity by
  (P) formulating and solving an appropriate fixed point problem.
  (H) finding c where the stable manifold $\mathcal{W}_c^s$ of the origin intersects (in fact includes) the unstable manifold $\mathcal{W}_c^u$.

(2) Get full exponential stability from linear exponential stability.

(3) Relate linear exponential stability to the spectrum of L_ϕ and show that the relevant part of the spectrum is discrete.

(4) Prove bounds that exclude eigenvalues outside a manageable region Ω containing 0.

(5) Show that L_ϕ has non nonzero eigenvalue in Ω by
  (P) using perturbation theory about a simpler operator, in a simpler space.
  (H_1) using that such eigenvalues are related to the zeros of the Evans function and
  (H_2) estimating the Evans function along $\partial\Omega$.

The fact ($5H_1$) was proved in [J.W. Evans IV] for general “nerve axon equations”.
Steps (2H) and (3H) are proved in [J.W. Evans I-III] but we need (2) and (3).

Existence of the periodic pulse.

Rescale from periodicity ℓ to periodicity 2π. Let $\eta = \ell/(2\pi)$.
Consider a Banach space $\mathcal{F}$ of functions that are analytic on a strip.
Rewrite the pulse equation as an equation for $\varphi = \phi_1(\eta\boldsymbol{.})$ alone,

$$g = \mathcal{N}_c(g)\,, \qquad g = \mathrm{I}_0\varphi = \varphi - \text{average}(\varphi)\,,$$

where

$$\mathcal{N}_c(g) = \eta^2\big(D^2 - \kappa^2\mathrm{I}\big)^{-1}(\mathrm{I} - \kappa D^{-1})\mathrm{I}_0\Big(-f(\varphi) + \epsilon\gamma g + \epsilon c^{-1}\eta D^{-1}\big[\gamma f(\varphi) - \varphi\big]\Big)\,.$$

"Eliminate" the eigenvalue 1 using a projection P of rank 1.

$$\boldsymbol{\mathcal{N}_c'(g) = (\mathrm{I} - P)\mathcal{N}_c(g)\,, \qquad P\mathcal{N}_c'(g) = 0\,, \qquad g \in \mathrm{I}_0\mathcal{F}\,.}$$

The fixed point problem for $\mathcal{N}_c'$ is nonsingular. Now use a quasi-Newton map

$$\mathcal{M}_c(h) = h + \mathcal{N}_c'(p_0 + Ah) - (p_0 + Ah)\,, \qquad h \in \mathrm{I}_0\mathcal{F}\,,$$

with p_0 an approximate fixed point of $\mathcal{N}_c'$ and A an approximation to $\big[\mathrm{I} - D\mathcal{N}_c'(p_0)\big]^{-1}$.

Lemma 3. *For some $c_0 = 0.4703363082\ldots$ and $K, r, \varepsilon > 0$ satisfying $\varepsilon + Kr < r$,*

$$\|\mathcal{M}_c(0)\| < \varepsilon\,, \qquad \|D\mathcal{M}_c(h)\| < K\,, \qquad c \in I\,, \quad h \in B_r(0)\,.$$

where $I = [c_0 - 2^{-60}, c_0 + 2^{-60}]$. Furthermore $c \mapsto P\mathcal{N}_c'(p_0 + Ah)$ changes sign on I for every $h \in B_r(0)$.

Existence of the homoclinic pulse. Solve

$$\phi' = DX(0)\phi + B(\phi_1)\,, \qquad DX(0) = \begin{bmatrix} -c & a & 1 \\ 1 & 0 & 0 \\ 0 & -c^{-1}\epsilon & c^{-1}\epsilon\gamma \end{bmatrix},$$

with $B(0) = 0$ and $DB(0) = 0$.

For the local stable manifold write $\phi^s(y) = \Phi^s(e^{\mu_0 y})$, and

$$\Phi^s(r) = \ell^s(r) + Z^s(r)\,, \qquad \ell^s(r) = r\mathtt{U}_0\,, \qquad Z^s(r) = \mathcal{O}(r^2)\,,$$

where $\mathtt{U}_0$ is the eigenvector of $DX(0)$ for the eigenvalue μ_0 that has a negative real part.

$$\boldsymbol{Z^s = \left[\partial_y - DX(0)\right]^{-1} B(\ell_1^s + Z_1^s)\,, \qquad \partial_y = \mu_0 r \partial_r\,.}$$

This equation can be solved "order by order" in powers of r.
Prolongation from $y = \frac{5}{2}$ backwards in time to $y = -43$ is done via a simple Taylor integrator.

For the local unstable manifold write $\phi^u(y) = \Phi^u(Re^{\nu_0 y}, Re^{\bar\nu_0 y})$, for some $R > 0$, and

$$\Phi^u(s) = \ell^u(s) + Z^u(s)\,, \qquad \ell^u(s) = s_1\mathtt{V}_0 + s_2\bar{\mathtt{V}}_0\,, \quad Z^u(s) = \mathcal{O}(|s|^2)\,,$$

where $\mathtt{V}_0$ and $\bar{\mathtt{V}}_0$ are eigenvectors of $DX(0)$ for the eigenvalues ν_0 and $\bar\nu_0$, respectively.

$$\boldsymbol{Z^u = \left[\partial_y - DX(0)\right]^{-1} B(\ell_1^u + Z_1^u)\,, \qquad \partial_y = \nu_0 s_1 \partial_{s_1} + \bar\nu_0 s_2 \partial_{s_2}\,.}$$

This equation can be solved "order by order" in powers of s_1 and s_2.

Recall that everything depends on the velocity parameter c.

For the local unstable manifold we use a space $\mathcal{A}$ where

$$Z_j^u(c,s) = \sum_{k+m\geq 2} Z_{j,k,m}^u(c) s_1^k s_2^m \,, \qquad \|Z_j^u\| = \sum_{k+m\geq 2} \|Z_{j,k,m}^u\| \rho^{k+m} \,.$$

The coefficients $Z_{j,k,m}^u$ belong to a space $\mathcal{B}$ of functions that are analytic on a disk $|c - c_0| < \varrho$. Similarly for the functions $c \mapsto Z_j^s(c, -43)$.

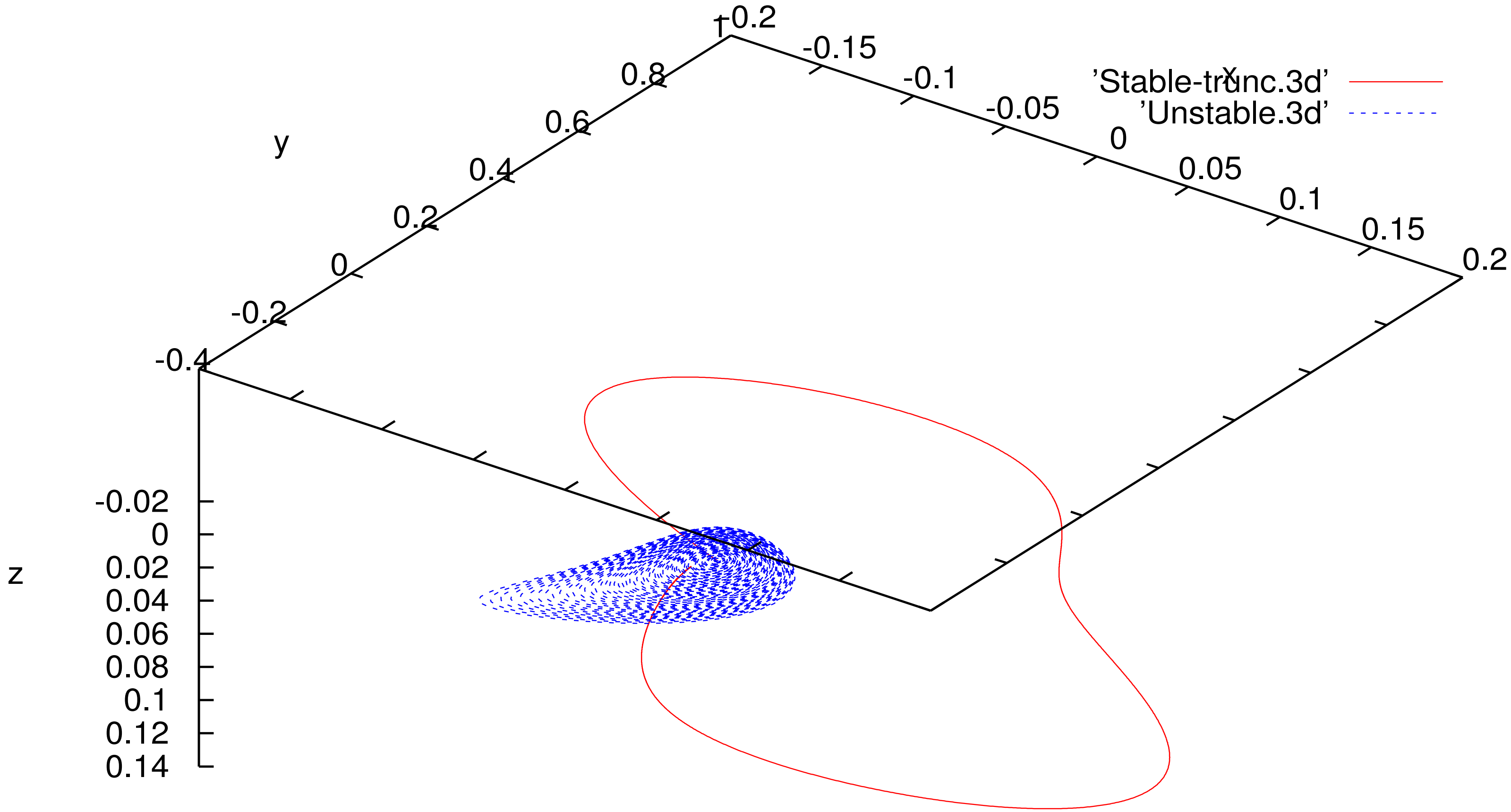
The two families of manifolds intersect if the difference

$$\Upsilon(c,\sigma,\tau) = \Phi^u(c, \sigma + i\tau, \sigma - i\tau) - \phi^s(c, -43)$$

vanishes for some real values of c, σ, and τ. Let $\rho = 2^{-5}$ and $\varrho = 2^{-96}$.

Lemma 4. *For some $c_0 = 0.4703362702\ldots$ the function Υ is well defined and differentiable on the domain $|\sigma + i\tau| < \rho$ and $|c - c_0| < \varrho$. In this domain there exists a cube where Υ has a unique zero, and $|c - c_0| < 2^{-172}$ for all points in this cube.*

Stable and unstable manifolds.



Reduction to **linear stability**.

For convenience use moving coordinates $y = x - tc$ and $w_j(x,t) = u_j(y,t)$, so

$$\partial_t \underline{u} = \begin{bmatrix} \partial_y^2 u_1 + c\partial_y u_1 + f(u_1) - u_2 \\ c\partial_y u_2 + \epsilon[u_1 - \gamma u_2] \end{bmatrix}, \qquad \underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

The linearization about a traveling pulse $u_j(y,t) = \phi_j(y)$ is

$$\partial_t \underline{v} = L_\phi \underline{v}, \qquad L_\phi \underline{v} = \begin{bmatrix} \partial_y^2 + c\partial_y + f'(\phi_1) & -1 \\ \epsilon & c\partial_y - \epsilon\gamma \end{bmatrix} \underline{v}.$$

Notice that $\underline{\phi}'$ is a stationary point: $L_\phi \underline{\phi}' = 0$.
Write $\underline{v}(0) \mapsto \underline{v}(t)$ as e^{tL_ϕ}.

$\underline{\phi}'$ is said to be **exponentially stable** if there exists a continuous linear functional $p : \mathcal{C} \to \mathbb{R}$, and two constants $C, \omega > 0$, such that $\left\| e^{tL_\phi}\underline{v} - p(\underline{v})\underline{\phi}' \right\| \le Ce^{-t\omega}\|\underline{v}\|$ for all $\underline{v} \in \mathcal{C}'$ and all $t \ge 0$.

Lemma 5. *If $\underline{\phi}'$ is exponentially stable for the linear system then $\underline{\phi}$ is exponentially stable for the full system.*

Our proof is naturally not far from [J.W. Evans I].
It applies both to the periodic and the homoclinic case (and is short).

Reduction to an **eigenvalue problem**.

Split $L_\phi : \mathcal{C}' \to \mathcal{C}$ as in

$$L_\phi = L_0 + F\,, \qquad L_0 = \begin{bmatrix} D^2 + cD - \theta & -1 \\ \epsilon & cD - \epsilon\gamma \end{bmatrix}, \quad F = \begin{bmatrix} f'(\phi_1) + \theta & 0 \\ 0 & 0 \end{bmatrix},$$

where $\theta = -f'(0) = -a$, unless specified otherwise.

Proposition 6. *L_0 generates a strongly continuous semigroup on $\mathcal{C}$ that satisfies $\left\|e^{tL_0}\right\| \leq Ce^{-t\epsilon\gamma}$ for some $C > 0$ and all $t \geq 0$.*

Proposition 7. *F is compact relative to L_0. And Fe^{tL_0} is compact for $t > 0$.*

Proposition 8. *The difference $e^{t(L_0+F)} - e^{tL_0}$ is compact for all $t \geq 0$.*

Superficially, this follows from *bounded∗compact=compact* and

$$e^{t(L_0+F)} - e^{tL_0} = \int_0^t e^{(t-s)(L_0+F)} F e^{sL_0}\, ds\,.$$

But need something like [J. Voigt 1992] for the strong operator topology.

Consider half-planes $\boldsymbol{H_\alpha = \left\{z \in \mathbb{C} : \mathrm{Re}\,(z) > -\alpha\right\}}$.

Proposition 9. *Assume that L_ϕ has no spectrum in H_α except for a simple eigenvalue 0. Denote by $P_\perp$ the spectral projection associated with the spectrum of L_ϕ in $\mathbb{C}\setminus\{0\}$. Then for every $\omega < \alpha$ there exists a constant $C_\omega > 0$ such that $\left\|e^{tL_\phi} P_\perp\right\| \leq C_\omega^{-t\omega}$ for all $t \geq 0$.*

For **eigenvalue bounds** use the Hilbert space $\mathcal{H}$,

$$\langle u, v\rangle = \epsilon^{1/2}\int u_1(y)\overline{v_1(y)}\,dy + \epsilon^{-1/2}\int u_2(y)\overline{v_2(y)}\,dy\,.$$

Fact 10. *The spectrum of $L_0 : \mathcal{H}' \to \mathcal{H}$ consists of . . .*

$$\lambda^-(p) = -p^2 + icp - \theta - \frac{1}{2}\Big[\sqrt{(p^2+\theta-\epsilon\gamma)^2 - 4\epsilon} - (p^2+\theta-\epsilon\gamma)\Big]\,,$$
$$\lambda^+(p) = icp - \epsilon\gamma + \frac{1}{2}\Big[\sqrt{(p^2+\theta-\epsilon\gamma)^2 - 4\epsilon} - (p^2+\theta-\epsilon\gamma)\Big]\,.$$

Proposition 11. *If λ is an eigenvalue of $L_\phi : \mathcal{H}' \to \mathcal{H}$ then*

$$\mathbf{Re}\,(\boldsymbol{\lambda}) \le \boldsymbol{\Lambda}\,, \qquad \boldsymbol{\Lambda} \stackrel{\text{def}}{=} \sup_r \boldsymbol{f'(r)} = \tfrac{91}{300}\,.$$

Proposition 12. *For every $\delta > 0$ there exists $\omega > 0$ such that the following holds. If λ is any eigenvalue of $L_\phi : \mathcal{H}' \to \mathcal{H}$, then either $\mathrm{Re}\,(\lambda) < -\omega$ or else*

$$\big|\mathbf{Im}\,(\boldsymbol{\lambda})\big| \le \sqrt{\boldsymbol{c}^2 + \boldsymbol{\gamma}^{-1}}\,\boldsymbol{\Lambda}^{1/2} + \boldsymbol{\delta}\,.$$

Using our bounds on c we have $\sqrt{c^2+\gamma^{-1}}\,\Lambda^{1/2} < \Theta \stackrel{\text{def}}{=} 0.35745\,.$

Estimating **eigenvalues in the periodic case**.

$\mathcal{V}$ Space of 2π-periodic functions that are analytic on a strip; contains all relevant eigenvectors of $L_\phi : \mathcal{C}' \to \mathcal{C}$.

P Projection onto low "Fourier modes".

M, U Suitable Fourier-multiplier operators.

$$\boldsymbol{M^{-1} L_\phi M = \mathcal{L}_1 , \qquad \mathcal{L}_s = \mathcal{L}_0 + s\mathcal{K} , \qquad \mathcal{L}_0 = M^{-1} L_0 M + PFP .}$$

Control first $z\mathrm{I} - \mathcal{L}_0$ and then

$$z\mathrm{I} - \mathcal{L}_s = \bigl[\mathrm{I} - s\mathcal{K}(z\mathrm{I} - \mathcal{L}_0)^{-1}\bigr](z\mathrm{I} - \mathcal{L}_0) , \qquad z \in \Gamma .$$

Estimate the spectral radius of the "green operator", using

$$U\bigl[\mathcal{K}(z\mathrm{I} - \mathcal{L}_0)^{-1}\bigr]U^{-1} = \bigl(U\mathcal{K}U\bigr)\bigl[U^{-1}(z\mathrm{I} - \mathcal{L}_0)^{-1}U^{-1}\bigr] .$$

Lemma 13. *$\mathcal{L}_0 : \mathcal{V} \to \mathcal{V}$ has no spectrum in Ω except for a simple eigenvalue. Furthermore, the following bound holds for all $z \in \Gamma$:*

$$\|U\mathcal{K}U\| < \frac{1}{2500} , \qquad \bigl\|U^{-1}(z\mathrm{I} - \mathcal{L}_0)^{-1}U^{-1}\bigr\| < 2500 .$$

Estimating **eigenvalues in the homoclinic case** à la [J.W. Evans IV].

Write $L_\phi \underline{\mathrm{u}} = \lambda \underline{\mathrm{u}}$ as

$$u' = A_{\phi_1}(\lambda)u\,, \qquad A_\varphi(z) = \begin{bmatrix} -c & -f'(\varphi)+z & 1 \\ 1 & 0 & 0 \\ 0 & -c^{-1}\epsilon & c^{-1}(\epsilon\gamma+z) \end{bmatrix}, \qquad z \in H_\alpha\,.$$

Need $u \in \mathcal{C}$ and thus bounded; in fact vanishing at $\pm\infty$ since
The matrix $A_0(z)$ is hyperbolic. One eigenvalue μ_z has negative real part.

First solve

$$\boldsymbol{u' = A_{\phi_1}(z)u\,, \qquad \lim_{y\to+\infty} u_z(y)e^{-y\mu_z} = \mathtt{U}_z\,, \qquad A_0(z)\mathtt{U}_z = \mu_z \mathtt{U}_z\,.}$$

As $y \to -\infty$ the normalized $u_z(y)$ must approach the unstable subspace of $A_0(z)$ which is perpendicular to the stable subspace of $A_0(z)^\top$.

Propagate this condition from $-\infty$ to y by solving the <u>adjoint equation</u>

$$\boldsymbol{v' = -A_{\phi_1}(z)^\top v\,, \qquad \lim_{y\to-\infty} v_z(y)e^{y\mu_z} = \mathtt{V}_z\,, \qquad A_0(z)^\top \mathtt{V}_z = \mu_z \mathtt{V}_z\,.}$$

Then

$$[v_z^\top u_z]' = [v_z']^\top u_z + v_z^\top u_z' = [-A_{\phi_1}(z)^\top v_z]^\top + v_z^\top A_{\phi_1}(z)u_z' = 0\,.$$

So

$$\boldsymbol{\Delta(z) = v_z(y)^\top u_z(y)} \qquad \text{(Evans function)}$$

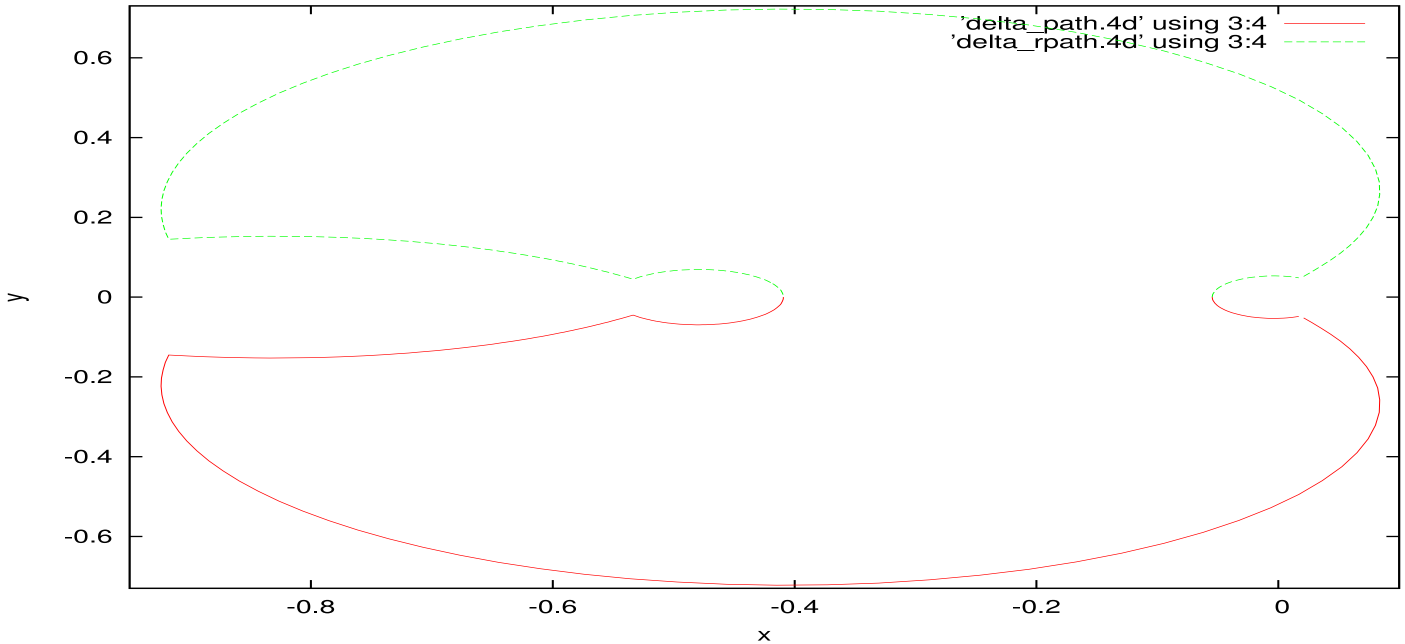
is independent of y and vanishes precisely when z is an eigenvalue of L_ϕ.

Theorem. [J.W. Evans IV]. *Δ is analytic in H_α and has a zero of order m at λ if and only if λ is an eigenvalue of L_ϕ with algebraic multiplicity m.*

Let $r = \frac{2485}{8192}$ and $\theta = \frac{5857}{16384}$. Denote by R the closed rectangle in $\mathbb{C}$ with corners at $\pm i\theta$ and $r \pm i\theta$. Let D be the closed disk in $\mathbb{C}$, centered at the origin, with radius $\frac{1}{32}$.

Lemma 14. *Δ has a simple zero at 0 and no other zeros in D.*
The restriction of Δ to the boundary of $R \setminus D$ takes no real values in the interval $[0, \infty)$.

Values of $2^{-31}\Delta$ along the boundary of $R \setminus D$.



Some details: periodic case

For 2π-periodic functions on the strip $|\mathrm{Im}\,(x)| < \rho$ we use the Banach algebra $\mathcal{F}$,

$$h(x) = \sum_{k=0}^{\infty} h_k \cos(kx) + \sum_{k=1}^{\infty} h_{-k} \sin(kx)\,, \qquad \|h\| = \sum_{k=-\infty}^{\infty} |h_k| \cosh(\rho k)\,.$$

Convenient for products, antiderivatives; and for estimating operator norms:
Let $(e_1, e_2, \ldots)$ be an enumeration of the Fourier modes $c_k \cos(k\boldsymbol{.})$ and $s_k \sin(k\boldsymbol{.})$, with c_k and s_k chosen in such a way that $\|e_j\| = 1$ for all j. Then

$$\|A\| = \sup_j \|Ae_j\| \quad \le \quad \max\bigl\{\|Ae_1\|, \|Ae_2\|, \ldots, \|Ae_{n-1}\|, \|AE_n\|\bigr\}\,,$$

where $E_n = \{e_n, e_{n+1}, \ldots\}$.
This is used with $A = D\mathcal{M}_c(h)$ for Lemma 3 and $A = U\mathcal{K}U$ for Lemma 13.

To estimate $U^{-1}(z\mathrm{I} - L)^{-1}U^{-1}$ along Γ for the low-mode (matrix) part $L = PL_\phi P$ of $\mathcal{L}_0$ we cover Γ with disks $|z - z_j| < \delta_j$ with centers $z_j \in \Gamma$.
The resolvent matrices $R_j = (z_j\mathrm{I} - L)^{-1}$ are computed explicitly (with error estimates, of course) and shown to satisfy $\delta_j \|R_j\| < 1$. This bound implies that

$$z\mathrm{I} - L = \bigl[\mathrm{I} + (z - z_j)R_j\bigr](z_j\mathrm{I} - L)$$

is invertible whenever $|z - z_j| < \delta_j$. Its inverse is bounded by ...

Some details: homoclinic case

The equations for u_z, v_z are integrated the same way as those for ϕ^s, ϕ^u. Write

$$u_z(y) = e^{-(\mu_0-\mu_z)y}\, \boldsymbol{U}_z(e^{\mu_0 y})\,, \qquad v_z(y) = e^{-(\nu_0+\bar\nu_0+\mu_z)y}\, \boldsymbol{V}_z\big(Re^{\nu_0 y}, Re^{\bar\nu_0 y}\big)\,,$$

and

$$\boldsymbol{U}_z(s) = r\,\mathtt{U}_z + \mathfrak{Z}^s(z,r)\,, \qquad \mathfrak{Z}^s_j(z,r) = \sum_{n\ge 2} \mathfrak{Z}^s_{j,n}(z) r^n\,,$$

$$\boldsymbol{V}_z(s) = s_1 s_2 \mathtt{V}_z + \mathfrak{Z}^u(z,s)\,, \qquad \mathfrak{Z}^u_j(z,s) = \sum_{k+m\ge 3} \mathfrak{Z}^u_{j,k,m}(z) s_1^k s_2^m\,.$$

The resulting equations for $\mathfrak{Z}^s$ and $\mathfrak{Z}^u$ can again be solved order by order .
The coefficients $\mathfrak{Z}^s_{j,n}$ and $\mathfrak{Z}^u_{j,k,m}$ belong to a space $\mathcal{B}$ of analytic functions on $|z-z_0| < \varrho$.

As in the case of ϕ^s, the resulting curve v_z is prolonged backwards in time from $y = \frac{5}{2}$ to $y = -43$.
The integration uses Taylor expansions that can be computed order by order .
At the end we evaluate $\Delta(z) = v_z(-43)^\top u_z(-43)$.

Let $(X_k, \|.\|_k)$ be Banach spaces for $k = 0, 1, 2, \ldots$ and let $(X, \|.\|)$ be the Banach space of all functions $x : \mathbb{N} \to \bigcup_k X_k$ with $x(k) \in X_k$ for all k and $\|x\| = \sum_k \|x(k)\|_k$ finite. Denote by P_n the projection on X defined by setting $(P_n x)(k) = x(k)$ for $k \le n$ and $(P_n x)(k) = 0$ for $k > n$.

Proposition 15. (order by order) *Let Y_0 be a closed bounded subset of X such that $P_n Y_0 \subset Y_0$ for all n, and $P_0 Y_0 = \{y_0\}$ for some $y_0 \in X$. Let $F : Y_0 \to Y_0$ be continuous, having the property that $P_{n+1}F = P_{n+1}FP_n$ for all n. Then F has a unique fixed point $y \in Y_0$, and $P_n y = P_n F^m(y_0)$ whenever $n \le m$.*

$\mathbb{S}$	an algebra (commutative Banach algebra with unit). Subspaces $\mathbb{S}_i$.
$\mathbb{F}$	any algebra of functions $f : \mathcal{D} \to \mathbb{S}$ including the constant functions.
$\mathcal{R}(\mathbb{S})$	`Scalars`: the representable subsets of $\mathbb{S}$. Includes an element `Undefined`.
$\mathcal{R}(\mathbb{S}, \mathbb{F})$	same, but any ball $\{s \in \mathbb{S}_i : \|s\| \le r\}$ replaced by $\{f \in \mathbb{F}_i : \|f\| \le r\}$.

Sum : $\mathcal{R}(\mathbb{S}) \times \mathcal{R}(\mathbb{S}) \to \mathcal{R}(\mathbb{S})$ such that $s_1 \in S_1$ and $s_2 \in S_2$ implies $s_1 + s_2 \in \mathrm{Sum}(S_1, S_2)$.

Consider a disk $\mathcal{D}_r = \{z \in \mathbb{C} : |z| < r\}$ for some representable $r > 0$.
`Taylor1`s: define a space $\mathbb{T}_r$ of analytic functions $f : \mathcal{D}_r \to \mathbb{S}$,

$$f(z) = \sum_{n=0}^{\infty} c_n z^n , \qquad \|f\|_{\mathbb{T}} = \sum_{n=0}^{\infty} \|c_n\|_{\mathbb{S}}\, r^n , \qquad c_n \in \mathbb{S} .$$

$\mathcal{R}(\mathbb{T}_r)$ consists of all sets

$$\mathtt{F} : z \mapsto \sum_{n=0}^{d} \mathtt{C}(n) z^n , \qquad \mathtt{C}(0 \,..\, k-1) \in \mathcal{R}(\mathbb{S}) , \quad \mathtt{C}(k \,..\, d) \in \mathcal{R}(\mathbb{S}, \mathbb{T}_r) .$$

```
function Sum(F1,F2: Taylor1) return Taylor1 is
  F3: Taylor1;
begin
  F3.R := Min(F1.R,F2.R);
  F3.J := Min(F1.K,F2.K);
  for N in 0 .. D loop
    F3.C(N) := Sum(F1.C(N),F2.C(N));
  end loop;
  return F3;
end Sum;
```

Similarly `Neg`, `Diff`, `Prod`, `Inv`, `Sqrt`, `Exp`, `Log`, `Cos`, `Sin`, `ArcCos`, `ArcSin`, `Norm`, `Includes`, `IsZero`, `Assign`, ...
Similarly for `Taylor2` and `Fourier1`. All can be used again as `Scalars` !