Magnetic Vortices, Vortex Lattices and Automorphic Functions

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Ginzburg-Landau Equations

Equilibrium states of superconductors (macroscopically) and of the $U(1)$ Higgs model of particle physics are described by the Ginzburg-Landau equations:

$$-\Delta_A \Psi = \kappa^2 (1 - |\Psi|^2)\Psi$$
$$\mathrm{curl}^2 A = \mathrm{Im}(\bar{\Psi} \nabla_A \Psi)$$

where $(\Psi, A) : \mathbb{R}^d \to \mathbb{C} \times \mathbb{R}^d$, $d = 2, 3$, $\nabla_A = \nabla - iA$, $\Delta_A = \nabla_A^2$, the covariant derivative and covariant Laplacian, respectively, and κ is the Ginzburg-Landau material constant.

Origin of Ginzburg-Landau Equations

Superconductivity. $\Psi : \mathbb{R}^d \to \mathbb{C}$ is called the *order parameter*; $|\Psi|^2$ gives the density of (Cooper pairs of) superconducting electrons. $A : \mathbb{R}^d \to \mathbb{R}^d$ is the magnetic potential. $\mathrm{Im}(\bar{\Psi}\nabla_A\Psi)$ is the superconducting current.

Particle physics. Ψ and A are the Higgs and $U(1)$ gauge (electro-magnetic) fields, respectively. (Part of Weinberg - Salam model of electro-weak interactions/ a standard model.)

Geometrically, A is a connection on the principal $U(1)$- bundle $\mathbb{R}^2 \times U(1)$, and Ψ, a section of the associated bundle.

Similar equations appear in superfluidity and fractional quantum Hall effect.

Quantization of Flux

From now on we let $d = 2$. Finite energy states (Ψ, A) are classified by the topological degree

$$\deg(\Psi) := \deg\left(\frac{\Psi}{|\Psi|}\bigg|_{|x|=R}\right),$$

where $R \gg 1$. For each such state we have the quantization of magnetic flux:

$$\int_{\mathbb{R}^2} B = 2\pi \deg(\Psi) \in 2\pi\mathbb{Z},$$

where $B := \operatorname{curl} A$ is the magnetic field associated with the vector potential A.

Type I and II Superconductors

Two types of superconductors:

$\kappa < 1/\sqrt{2}$: Type I superconductors, exhibit first-order phase transitions from the non-superconducting state to the superconducting state (essentially, all pure metals);

$\kappa > 1/\sqrt{2}$: Type II superconductors, exhibit second-order phase transitions and the formation of vortex lattices (dirty metals and alloys).

For $\kappa = 1/\sqrt{2}$, Bogomolnyi has shown that the Ginzburg-Landau equations are equivalent to a pair of first-order equations. Using this Taubes described completely solutions of a given degree.

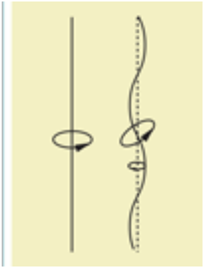
Vortices

"Radially symmetric" (more precisely, *equivariant*) solutions:

$$\Psi^{(n)}(x) = f^{(n)}(r)e^{in\theta} \quad \text{and} \quad A^{(n)}(x) = a^{(n)}(r)\nabla(n\theta),$$

where n = integer and (r, θ) = polar coordinates of $x \in \mathbb{R}^2$.

$\deg(\Psi^{(n)}) = n \in \mathbb{Z}$. (Berger-Chen)



$(\Psi^{(n)}, A^{(n)})$ = the *magnetic* n-*vortex* (superconductors) or *Nielsen-Olesen* or *Nambu string* (the particle physics).

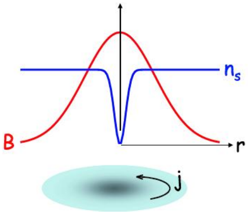
Vortex Profile

The profiles are exponentially localized:

$$|1 - f^{(n)}(r)| \leq ce^{-r/\xi}, \quad |1 - a^{(n)}(r)| \leq ce^{-r/\lambda},$$

Here ξ = *coherence length* and λ = *penetration depth*.

$$\kappa = \lambda/\xi.$$



The exponential decay is due to the Higgs mechanism of mass generation: massless $A \Rightarrow$ massive A, with $m_A = \lambda^{-1}$.

Stability/Instability of Vortices

Theorem

1. *For Type I superconductors all vortices are stable.*
2. *For Type II superconductors, the* ± 1*-vortices are stable, while the* n*-vortices with* $|n| \geq 2$*, are not.*

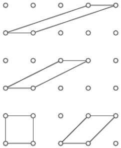
The statement of Theorem I was conjectured by Jaffe and Taubes on the basis of numerical observations (Jacobs and Rebbi, . . .).

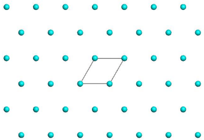
Abrikosov Vortex Lattice States

A pair (Ψ, A) for which all the physical characteristics

$$|\Psi|^2, \quad B(x) := \operatorname{curl} A(x), \quad J(x) := \operatorname{Im}(\bar{\Psi}\nabla_A \Psi)$$

are doubly periodic with respect to a lattice $\mathcal{L}$ is called the *Abrikosov (vortex) lattice state*.





Vortices and vortex lattices are equivariant solutions for different subgroups of the group of rigid motions (subgroups of rotations and lattice translations, respectively).

Existence of Abrikosov Lattices (High magnetic fields)

Let $H_{c2} = \kappa^2$ be the second critical magnetic field, at which the normal material becomes superconducting. Define

$$\kappa_c(\mathcal{L}) := \sqrt{\frac{1}{2}\left(1 - \frac{1}{\beta(\mathcal{L})}\right)} \; (< \tfrac{1}{\sqrt{2}}).$$

Theorem

For for every $\mathcal{L}$ and b satisfying $b|\Omega| = 2\pi$ and $\left|b - \kappa^2\right| \ll 1$ and

- *either $b < \kappa^2$ and $\kappa > \kappa_c(\mathcal{L})$ or $b > \kappa^2$ and $\kappa < \kappa_c(\mathcal{L})$,*

there exists an Abrikosov lattice solution, with one quantum of flux per cell and with average magnetic field per cell equal to b.

Theorem

If $\kappa > 1/\sqrt{2}$ (Type II superconductors), then the minimum of the average energy per cell is achieved for the triangular lattice.

Existence of Abrikosov Lattices (Weak MF)

- Similarly, near the first critical magnetic field, H_{c1} (at which the first vortex enters the superconducting sample), we have the following result

Theorem (Low magnetic fields)

For every $\mathcal{L}$, n and $b > H_{c1}$, satisfying $b|\Omega| = 2\pi$ (but close to H_{c1}), there exist non-trivial Abrikosov lattice solution, with n quanta of flux per cell and with average magnetic field per cell $= b$.

References

- Aver. magn. field $\approx H_{c2} = \kappa^2$.

Existence for $b < \kappa^2$ and $\kappa > \frac{1}{\sqrt{2}}$: Odeh, Barany - Golubitsky - Tursky, Dutour, Tzaneteas - IMS

Existence for $b < \kappa^2$ and $\kappa > \kappa_c(\mathcal{L})$ or $b > \kappa^2$ and $\kappa < \kappa_c(\mathcal{L})$: Tzaneteas - IMS
($\kappa_c(\mathcal{L})$ is a new threshold of the Ginzburg-Landau parameter)

Energy minim. by triangular lattices: Dutour, Tzaneteas - IMS, using results of Aftalion - Blanc - Nier, Nonnenmacher - Voros.

Finite domains: Almog, Aftalion - Serfaty.

- Aver. magn. field $\approx H_{c1}$.

Existence: Aydi - Sandier and others ($\kappa \to \infty$) and Tzaneteas - IMS (all κ's).

Time-Dependent Eqns. Superconductivity

In the leading approximation the evolution of a superconductor is described by the gradient-flow-type equations

$$\gamma(\partial_t + i\Phi)\Psi = \Delta_A \Psi + \kappa^2(1 - |\Psi|^2)\Psi$$
$$\sigma(\partial_t A - \nabla\Phi) = -\operatorname{curl}^2 A + Im(\bar{\Psi}\nabla_A \Psi),$$

$Re\gamma \geq 0$, the *time-dependent Ginzburg-Landau equations* or the *Gorkov-Eliashberg-Schmidt equations*. (Earlier versions: Bardeen and Stephen and Anderson, Luttinger and Werthamer.)

The last equation comes from two Maxwell equations, with $-\partial_t E$ neglected, (Ampère's and Faraday's laws) and the relations $J = J_s + J_n$, where $J_s = \text{Im}(\overline{\Psi}\nabla_A \Psi)$, and $J_n = \sigma E$.

Time-Dependent Eqns. $U(1)$ Higgs Model

The time-dependent $U(1)$ Higgs model is described by $U(1)-$Higgs (or Maxwell-Higgs) equations ($\Phi = 0$)

$$(\partial_t + i\Phi)^2\Psi = \Delta_A\Psi + \kappa^2(1 - |\Psi|^2)\Psi$$
$$(\partial_t A - \nabla\Phi)^2 A = -\operatorname{curl}^2 A + \operatorname{Im}(\bar{\Psi}\nabla_A\Psi),$$

coupled (covariant) wave equations describing the $U(1)$-gauge Higgs model of elementary particle physics.

In what follows we use the *temporal gauge* $\Phi = 0$.

Stability of Abrikosov Lattices

Let $(\Psi_\omega, A_\omega) =$ Abrikosov lattice solution specified by $\omega = (\mathcal{L}, b)$ and $\mathcal{E}_\Omega(\Psi, A) =$ *Ginzburg-Landau energy functional*

$$\mathcal{E}_\Omega(\Psi, A) := \frac{1}{2} \int_\Omega \left\{ |\nabla_A \Psi|^2 + (\operatorname{curl} A)^2 + \frac{\kappa^2}{2}(|\Psi|^2 - 1)^2 \right\}.$$

Finite-energy perturbations: perturbations satisfying,

$$\lim_{Q \to \mathbb{R}^2} \big(\mathcal{E}_Q(\Psi, A) - \mathcal{E}_Q(\Psi_\omega, A_\omega)\big) < \infty, \text{ for some } \omega.$$

Theorem (Tzaneteas - IMS)

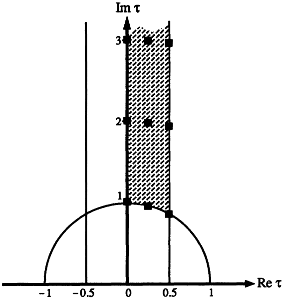
Let $b \approx H_{c2}$ (high magnetic fields).
There is $\gamma(\mathcal{L})$ s.t. the Abrikosov vortex lattice solutions are

(i) *asymptotically stable if $\kappa > \frac{1}{\sqrt{2}}$ and $\gamma(\mathcal{L}) > 0$;*

(ii) *unstable otherwise.*

Gamma Function

Let $\mathcal{L} = r(\mathbb{Z} + \tau\mathbb{Z})$, $r > 0$, $\tau \in \mathbb{C}$, $\operatorname{Im}\tau > 0$, and $\gamma(\tau) = \gamma(\mathcal{L})$. Then the function $\gamma(\tau)$ is invariant under modular group $SL(2, \mathbb{Z})$ and therefore can be reduced to the Poincaré strip, $\Pi^+/SL(2, \mathbb{Z})$,



Symmetries: $\gamma(-\bar{\tau}) = \gamma(\tau)$ and $\gamma(1 - \bar{\tau}) = \gamma(\tau)$
$\Rightarrow$ critical points at $\tau = e^{i\pi/2}$ and $\tau = e^{i\pi/3}$

Work in progress: Estimating $\gamma(\tau)$ and checking the critical points. So far we have $\gamma(e^{i\pi/3}) > 0$

Stability Definition

The stability is defined w.r.to distance to the infinite-dimensional manifold of $\mathcal{L}-$lattice solutions

$$\mathcal{M} = \{T_g^{sym} u_\omega : g \in G\},$$

where $T_g^{sym} = T_\gamma^{gauge} T_h^{trans} T_\rho^{rot},\ g = (\gamma, h, \rho)$, is the action of the symmetry group

$$G = H^2(\mathbb{R}^2; \mathbb{R}) \times \mathbb{R}^2 \times SO(2)$$

(semi-direct product) on Abrikosov vortex lattices $u_\omega = (\Psi_\omega, A_\omega)$. Here $T_\gamma^{\mathrm{gauge}}$, T_h^{trans} and T_ρ^{rot} are the *gauge transformations, translations and rotations*, i.e.

$$T_\gamma^{\mathrm{gauge}} :\ (\Psi(x),\ A(x)) \mapsto (e^{i\gamma(x)}\Psi(x),\ A(x) + \nabla\gamma(x)).$$

Central Step in Proof

Consider the hessian, $\mathcal{E}''(u_\omega)$, of Ginzburg-Landau energy functional $\mathcal{E}(\Psi, A)$ at a Abrikosov lattice solution $u_\omega = (\Psi_\omega, A_\omega)$.

(Recall that the Ginzburg-Landau equations are the Euler-Lagrange equations for $\mathcal{E}$.)

Signature of stability/instability is the sign of the lowest eigenvalue of $\mathcal{E}''(u_\omega)$

$\Longrightarrow$ estimate the lowest eigenvalue of $\mathcal{E}''(u_\omega)$ in transversal direction to $\mathcal{M}$.

Abrikosov Lattices and Equivariance

Recall: the *Abrikosov (vortex) lattice* is a pair (Ψ, A) for which all the physical characteristics

$$|\Psi|^2, \quad B(x) := \operatorname{curl} A(x), \quad J(x) := \operatorname{Im}(\bar{\Psi} \nabla_A \Psi)$$

are doubly periodic with respect to a lattice $\mathcal{L}$.

Theorem. (Ψ, A) is an Abrikosov lattice state if and only if it is an equivariant pair for the group of lattice translations for a lattice $\mathcal{L}$:

$$T_s^{\mathrm{transl}}(\Psi, A) = T_{\gamma_s}^{\mathrm{gauge}}(\Psi, A), \ \forall s \in \mathcal{L}, \tag{1}$$

where $\gamma_s : \mathbb{R}^2 \to \mathbb{R}$ is, in general, a multi-valued differentiable function, with differences of values at the same point $\in 2\pi\mathbb{Z}$.

$$(1) \Rightarrow \gamma_{s+t}(x) - \gamma_s(x+t) - \gamma_t(x) \in 2\pi\mathbb{Z}.$$

Magnetic Translations

The key point: $u_\omega = (\Psi_\omega, A_\omega)$ is equivariant $\Longrightarrow$ the Hessian $\mathcal{E}''(u_\omega)$ commutes with magnetic translations,

$$T_s = T^{\text{gauge}}_{\gamma_s} T^{\text{transl}}_s,$$

where, recall, $T^{\text{transl}}_s f(x) = f(x+s)$, and

$$T^{\text{gauge}}_\gamma : \ (\psi(x),\ a(x)) \mapsto (e^{i\gamma(x)}\psi(x),\ a(x) + \nabla\gamma(x));$$

and $\gamma_s : \mathbb{R}^2 \to \mathbb{R}$ is a multi-valued differentiable function, satisfying

$$\gamma_{s+t}(x) - \gamma_s(x+t) - \gamma_t(x) \in 2\pi\mathbb{Z}. \qquad (2)$$

$$(2) \quad \Rightarrow \quad T_{s+t} = T_s T_t.$$

($s \to T_s$ is a unitary repres. of $\mathcal{L}$ on $L^2(\mathbb{R}^2;\mathbb{C}) \times L^2(\mathbb{R}^2;\mathbb{R}^2)$.)

Direct Fibre Integral (Bloch Decomposition)

Since the Hessian operator $\mathcal{E}''(u_\omega)$ commutes with T_s, it can be decomposed into the fiber direct integral

$$U\mathcal{E}''(u_\omega)U^{-1} = \int_{\Omega^*}^{\oplus} L_k d\mu_k$$

where Ω^* is the fundamental cell of the reciprocal (dual) lattice, $U : L^2(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2) \to \mathscr{H} = \int_{\Omega^*}^{\oplus} \mathscr{H}_k d\mu_k$ is a unitary operator,

$$(Uv)_k(x) = \sum_{s \in \mathcal{L}} e^{-ik\cdot s} T_s v(x)$$

(decomposition into the Bloch waves, $v_k(x) = e^{ik\cdot x}\phi_k(x)$),
$\mathscr{H}_k := \{v \in L^2(\Omega, \mathbb{C} \times \mathbb{R}^2) : T_s v(x) = e^{ik\cdot s} v(x),\ s \in \text{basis}\}$,
L_k is the restriction of the operator $\mathcal{E}''(u_\omega)$ to $\mathscr{H}_k$.

ϑ-function

In the leading order in $\epsilon := \sqrt{\kappa^2 - b}$, the ground state energies of the fiber operators, L_k, are given by

$$\inf L_k = \gamma_k(\tau)\epsilon^2 + O(\epsilon^3),$$

where

$$\gamma_k(\tau) := 2\frac{\langle|\vartheta_k(\tau)|^2|\vartheta_0(\tau)|^2\rangle}{\langle|\vartheta_k(\tau)|^2\rangle\langle|\vartheta_0(\tau)|^2\rangle} + \cdots - \frac{\langle|\vartheta_0(\tau)|^4\rangle}{\langle|\vartheta_0(\tau)|^2\rangle^2}.$$

Here $\vartheta_k(z,\tau),\ k \in \Omega^*$, are the *modified theta functions*, i.e. entire functions satisfying ($\sqrt{\frac{2\pi}{\operatorname{Im}\tau}}i(a\tau + b) = k_1 + ik_2$)

$$\begin{cases} \vartheta_k(z+1,\tau) = e^{2\pi i a}\vartheta_k(z,\tau), \\ \vartheta_k(z+\tau,\tau) = e^{-2\pi i b}e^{-\pi i \tau z - 2\pi i z}\vartheta_k(z,\tau). \end{cases}$$

Conclusion of Sketch

The relations $\inf L_k = \gamma_k(\tau)\epsilon^2 + O(\epsilon^3)$ and

$$U\mathcal{E}''(u_\omega)U^{-1} = \int_{\Omega^*}^{\oplus} L_k d\mu_k$$

imply

$$\inf \mathcal{E}''(u_\omega) = \underbrace{\inf_{k\in\Omega^*} \gamma_k(\tau)}_{\gamma(\tau)} \epsilon^2 + O(\epsilon^3).$$

Hence the Abrikosov lattice is

- linearly stable if $\gamma(\tau) > 0$
- linearly unstable if $\gamma(\tau) < 0$.

Conclusions

In the context of superconductivity and particle physics, we described

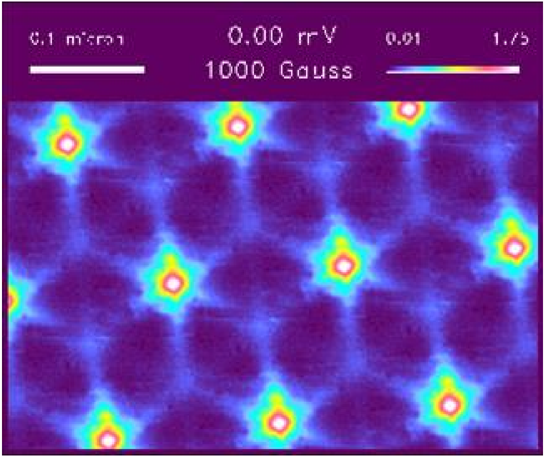
- existence and stability of magnetic vortices and vortex lattices
- a new threshold $\kappa_c(\tau)$ in the Ginzburg-Landau parameter appears in the problem of existence of vortex lattices
- while Abrikosov lattice energetics is governed by Abrikosov function $\beta(\tau)$, a new automorphic function $\gamma(\tau)$ emerges controlling stability of Abrikosov lattices.

We gave some indications how to prove the latter results. While the proof of existence leads to standard theta functions, the proof of stability leads to theta functions with characteristics.

Interesting extensions:

- unconventional/high T_c supercond.,
- Weinberg - Salam model of electro-weak interactions,
- microscopic/quantum theory.

Abrikosov Lattice. Experiment



Thank-you for your attention