Capacities in nonlinear PDEs with power nonlinearities

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Introduction: The two model equations

Lane-Emden type:

$$-\Delta_p u = u^q + \mu, \qquad u \geq 0.$$

$$F_k[-u] = u^q + \mu, \qquad u \geq 0.$$

Stationary Navier-Stokes:

$$\begin{cases} -\Delta U + U \cdot \nabla U + \nabla P & = & F, \\ \mathrm{div}\, U & = & 0. \end{cases}$$

$$U = (U_1, U_2, \ldots, U_n), \qquad F = (F_1, F_2, \ldots, F_n).$$

- Here μ is a non-negative measure, and $q > 0$.
- $\Delta_p u := \mathrm{div}(|\nabla u|^{p-2} \nabla u)$ is the p-Laplacian, $p > 1$.

- $F_k[u]$, $k = 1, 2, \ldots, n$, is the k-Hessian of u defined by

$$F_k[u] = \sum k \times k \text{ principal minors of } D^2 u.$$

That is

$$F_k[u] = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}$$

where $\lambda_1, \ldots, \lambda_n$ are eigenvalues of $D^2 u$. In particular,

$$F_1[u] = \Delta u, \qquad F_n[u] = \det(D^2 u).$$

Note that

$$\det(\lambda I_n - D^2 u) = \sum_{k=0}^{n} F_k[-u] \lambda^{n-k}.$$

Capacities

- Bessel capacity: Let $\alpha > 0$ and $s > 1$.

$$\mathrm{Cap}_{\alpha,\, s}(K) := \inf\left\{ \|f\|_{L^s}^s : f \geq 0, \mathbf{G}_\alpha * f \geq 1 \text{ on } K \right\}$$

$$\simeq \inf\{\|u\|^s_{W^{\alpha,s}(\mathbb{R}^n)} : u \in C_0^\infty(\mathbb{R}^n), u \geq 1 \text{ on } K\}.$$

Here

$$\mathbf{G}_\alpha = \mathcal{F}^{-1}[(1+|\xi|^2)^{\frac{-\alpha}{2}}] \quad \text{(Bessel kernel)}.$$

- Riesz capacity: Let $0 < \alpha < n$ and $s > 1$.

$$\mathrm{cap}_{\alpha,\, s}(K) := \inf\left\{ \|f\|_{L^s}^s : f \geq 0, \mathbf{I}_\alpha * f \geq 1 \text{ on } K \right\},$$

where

$$\mathbf{I}_\alpha * f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

- Locally we also have the equivalence: for $\alpha s < n$

$$\mathrm{Cap}_{\alpha,\, s}(K) \simeq \mathrm{cap}_{\alpha,\, s}(K), \qquad \forall K \subset B_1.$$

Capacities

• Capacity of a ball: $\mathrm{Cap}_{\alpha,s}(B_r) \simeq |B_r|^{1-\alpha s/n}$, $\alpha s < n$, $0 < r \leq 1$.
• Capacity of a general compact set: $\mathrm{Cap}_{\alpha,s}(K) \gtrsim |K|^{1-\alpha s/n}$. This follows from Sobolev Embedding Theorem.

Capacities play an important role in analysis and PDEs. For example, they are used to study:

• pointwise behaviors of Sobolev functions (Luzin type theorem).
• removable singularities of solutions to PDEs.
• Dirichlet problems on arbitrary domains (Wiener's criterion), etc.

We are particularly interested in the following remarkable use of capacity on trace inequalities:

Theorem (Maz'ya-Adams-Dahlberg)

Let $\nu \in M^+(\mathbb{R}^n)$, $\alpha > 0$, *and* $1 < s < \infty$. *Then*

$$\int_{\mathbb{R}^n} |u|^s d\nu \lesssim \|u\|^s_{W^{\alpha,s}(\mathbb{R}^n)}, \quad \forall u \in C_0^\infty(\mathbb{R}^n).$$

$$\Updownarrow$$

$$\int_{\mathbb{R}^n} (\mathbf{G}_\alpha * f)^s d\nu \lesssim \int_{\mathbb{R}^n} f^s dx, \quad \forall f \in L^s(\mathbb{R}^n), f \geq 0.$$

$$\Updownarrow$$

$$\nu(K) \lesssim \mathrm{Cap}_{\alpha,s}(K), \quad \forall K \subset \mathbb{R}^n.$$

Remark: A similar result holds for $\mathbf{I}_\alpha$ and the Riesz capacity $\mathrm{cap}_{\alpha,s}$.

Quasilinear Lane-Emden type equations

Theorem (P.-Verbitsky, Ann. Math. 2008)

Let $q > p - 1$. *Suppose that* $\mathrm{supp}\mu \Subset \Omega$ *with* $\mu \geq 0$. *If the equation*

$$\begin{cases} -\Delta_p u = u^q + \mu \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega \end{cases} \tag{1}$$

has a solution then

$$\mu(K) \leq C\, \mathrm{Cap}_{p, \frac{q}{q-p+1}}(K), \qquad \forall K \subset \Omega. \tag{2}$$

• *Conversely,* $\exists\ C_0 = C_0(n, p, q) > 0$ *such that if* (2) *holds with* $C \leq C_0$ *then* (1) *has a solution.*

For $p = 2$: Adams-Pierre (1991).

• Necessary condition: $\mu \leq C\,\mathcal{H}_\infty^{n-\frac{pq}{q-p+1}}$ (Hausdorff content). But this is far from being sufficient.

• Simple sufficient condition: $\mu = f \in L^{\frac{n(q-p+1)}{pq},\infty}(\Omega)$. This gives the answer to a problem posed Bidaut-Veron in 2002.

• Fefferman-Phong sufficient condition: Let $\mu = fdx$. For some $\epsilon > 0$

$$\int_B f^{1+\epsilon}dx \leq C|B|^{1-\frac{(1+\epsilon)pq}{n(q-p+1)}}, \qquad \forall \,\text{balls } B.$$

Here one checks only over balls, but a small bump $\epsilon > 0$ on f is needed.

Removable Singularities for $-\Delta_p u = u^q$

Theorem (P.-Verbitsky, 2008)

Let $E \subset \Omega$ be compact. Then

$$\mathrm{Cap}_{p,\, \frac{q}{q-p+1}}(E) = 0$$

is necessary and sufficient in order that:

$$\begin{cases} u \in L^q_{\mathrm{loc}}(\Omega \setminus E), \quad u \geq 0, \\ -\Delta_p u = u^q \quad \text{in} \quad \mathcal{D}'(\Omega \setminus E). \end{cases}$$

$$\Downarrow$$

$$\begin{cases} u \in L^q_{\mathrm{loc}}(\Omega), \quad u \geq 0, \\ -\Delta_p u = u^q \quad \text{in} \quad \mathcal{D}'(\Omega). \end{cases}$$

Remark: No information of u near E is needed.

Hessian Lane-Emden type equations

Theorem (P.-Verbitsky, Ann. Math. 2008)

Let $q > k$. Suppose $\mathrm{supp}\mu \Subset \Omega$, Ω *is uniformly* $(k-1)$*-convex.*

$$\begin{cases} F_k[-u] = u^q + \mu \text{ in } \Omega, \\ u \geq 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

$$\Updownarrow$$

$$\mu(K) \leq C\,\mathrm{Cap}_{2k,\, \frac{q}{q-k}}(K).$$

$(k-1)$-convex domain Ω in $\mathbb{R}^n$: $H_j(\partial\Omega) > 0, j = 1, \ldots, k-1$; H_j denotes the j-mean curvature of $\partial\Omega$.

Removable Singularities: A closed set E is removable for $F_k[-u] = u^q$ iff $\mathrm{Cap}_{2k,\, \frac{q}{q-k}}(E) = 0$.

Relation to semilinear equation: It is also known that

$$\mu(K) \leq C \operatorname{cap}_{p, \frac{q}{q-p+1}}(K)$$

$$\Updownarrow$$

$$u = \mathbf{I}_p * (u^{q/(p-1)}) + \mathbf{I}_p * \mu \qquad \text{in } \mathbb{R}^n.$$

As $\mathbf{I}_p = (-\Delta)^{-p/2}$, in some sense we have the equivalence

$$-\Delta_p u = u^q + \mu \quad \Longleftrightarrow \quad (-\Delta)^{p/2} u = u^{q/(p-1)} + \mu$$

Likewise, one has

$$F_k[-u] = u^q + \mu \quad \Longleftrightarrow \quad (-\Delta)^k u = u^{q/k} + \mu$$

Stationary Navier-Stokes equations

First, the Cauchy problem for non-stationary N-S equations:

$$u_t + (u\cdot\nabla)u + \nabla p = \Delta u, \quad \operatorname{div} u = 0, \quad u(x,0) = u_0(x).$$

$$u = u(x,t) = (u_1, u_2, \ldots u_n).$$

Time-global existence with *small* initial data:

- T. Kato: $u_0 \in L^n$.
- T. Kato, Cannone, Federbush, Y. Meyer, M. Taylor:

$$u_0 \in L^{n,\infty}, \qquad u_0 \in \mathcal{M}^{p,p}, 1 \leq p \leq n.$$

The Morrey space $\mathcal{M}^{p,p}$ is defined by the norm

$$\|f\|_{\mathcal{M}^{p,p}} = \sup_{B_R} \left(R^{p-n} \int_{B_R} |f|^p dx \right)^{\frac{1}{p}}.$$

- Koch-Tataru: $u_0 \in BMO^{-1}$.

$$\|f\|_{BMO^{-1}} = \sup_{B_R} \left(\frac{1}{|B_R|} \int_{B_R} \int_0^{R^2} |e^{t\Delta} f(y)|^2 dt dy \right)^{\frac{1}{2}}.$$

$$(f = \operatorname{div} \vec{F}, \quad \vec{F} \in BMO^n).$$

- Bourgain-Pavlović: Ill-posedness in $B^{-1}_{\infty,\infty}$.

$$\|f\|_{B^{-1}_{\infty,\infty}} = \sup_{t>0} t^{\frac{1}{2}} \left\| e^{t\Delta} f(\cdot) \right\|_{L^\infty}.$$

One has the continuous emdeddings: $1 \le p \le n$

$$L^n \subset L^{n,\infty} \subset \mathcal{M}^{p,p} \subset BMO^{-1} \subset B^{-1}_{\infty,\infty}.$$

Critical spaces:

$$\|f\|_E = \|\lambda f(\lambda \cdot)\|_E, \qquad \forall \lambda > 0.$$

Stationary Navier-Stokes:

$$-\Delta U + U\cdot\nabla U + \nabla P = F, \qquad \operatorname{div} U = 0.$$

$$U = (U_1, U_2, \ldots, U_n), \qquad F = (F_1, F_2, \ldots, F_n).$$

It is invariant under the scaling

$$(U, P, F) \mapsto (U_\lambda, P_\lambda, F_\lambda),$$

where

$$U_\lambda = \lambda U(\lambda\cdot), \quad P_\lambda = \lambda^2 P(\lambda\cdot), \quad F_\lambda = \lambda^3 F(\lambda\cdot) \quad \forall\lambda > 0.$$

Integral form:

$$U = \Delta^{-1}\mathbb{P}\nabla\cdot(U\otimes U) - \Delta^{-1}\mathbb{P}F, \tag{3}$$

where

$$\mathbb{P} := Id - \nabla\Delta^{-1}\nabla\cdot$$

is the Laray projection onto the divergence-free vector fields.

The role of $\mathcal{M}^{2,2}$: largest Banach space $E \subset L^2_{\mathrm{loc}}(\mathbb{R}^n)$ that is invariant under translation and that $\|\lambda U(\lambda \cdot)\|_E = \|U\|_E$.
Thus one is tempted to look for solutions in $\mathcal{M}^{2,2}$ under the smallness condition

$$\left\|(-\Delta)^{-1}F\right\|_{\mathcal{M}^{2,2}} \leq \epsilon.$$

However, it seems impossible to prove such existence results under this condition as for $U \in \mathcal{M}^{2,2}$ the matrix $U \otimes U$ would belong to $\mathcal{M}^{1,2}$, but unfortunately the first order Riesz potentials of functions in $\mathcal{M}^{1,2}$ may not even belong to $L^2_{\mathrm{loc}}(\mathbb{R}^n)$.

The space $\mathcal{V}^{1,2}$:

$$\mathcal{V}^{1,2}(\mathbb{R}^n) := \{u \in L^2_{\mathrm{loc}}(\mathbb{R}^n) : \|u\|_{\mathcal{V}^{1,2}(\mathbb{R}^n)} < +\infty\},$$

where

$$\|u\|_{\mathcal{V}^{1,2}(\mathbb{R}^n)} = \sup_{K \subset \mathbb{R}^n} \left[\frac{\int_K u^2 dx}{\mathrm{cap}_{1,2}(K)}\right]^{\frac{1}{2}}.$$

Embeddings:

$$\mathcal{M}^{2+\epsilon, 2+\epsilon} \subset \mathcal{V}^{1,2} \subset \mathcal{M}^{2,2}, \qquad \forall \epsilon > 0.$$

Theorem (Phan-P., Adv. Math. 2013)

There exists a sufficiently small number $\delta_0 > 0$ such that if $\left\|(-\Delta)^{-1}F\right\|_{\mathcal{V}^{1,2}} < \delta_0$, then the equation (3) has unique solution U satisfying

$$||U||_{\mathcal{V}^{1,2}} \leq C \left\|(-\Delta)^{-1}F\right\|_{\mathcal{V}^{1,2}}.$$

Kozono-Yamazaki, 1995: Existence in smaller spaces $\mathcal{M}^{2+\epsilon, 2+\epsilon}$.

Key bilinear estimate: Let

$$B(U, V) = \Delta^{-1}\mathbb{P}\nabla \cdot (U \otimes V).$$

One has

$$B : \mathcal{V}^{1,2} \times \mathcal{V}^{1,2} \to \mathcal{V}^{1,2}$$

with

$$\|B(U, V)\|_{\mathcal{V}^{1,2}} \leq C \|U\|_{\mathcal{V}^{1,2}} \|V\|_{\mathcal{V}^{1,2}}.$$

Stationary Navier-Stokes equations

Stability results:
Let $U \in \mathcal{V}^{1,2}$ be the solution of (3) with external force F satisfying

$$\left\|(-\Delta)^{-1} F\right\|_{\mathcal{V}^{1,2}} < \delta_0.$$

Consider the Cauchy problem

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \Delta u + F, & \text{in } \mathbb{R}^n \times [0, \infty), \\ \nabla \cdot u = 0, & \text{in } \mathbb{R}^n \times [0, \infty), \\ u(0) = u_0, & \text{in } \mathbb{R}^n, \end{cases} \tag{4}$$

where $u_0 \in \mathcal{V}^{1,2}$ with $\mathrm{div}\, u_0 = 0$.
Goal: Show that for u_0 near U there exists a unique time-global solution $u(t)$ of (4) such that as time $t \to \infty$ we have $u(t) \to U$ in some sense.

Stationary Navier-Stokes equations

Theorem (Phan-P., Adv. Math. 2013)

Let $\sigma_0 \in (1/2, 1)$. *There exists a number* $0 < \delta_1 \leq \delta_0$ *such that for* $||(-\Delta)^{-1}F||_{\mathcal{V}^{1,2}} < \delta_1$, *the following results hold:*
There is a positive number $\epsilon_0 > 0$ *such that for every* u_0 *satisfying* $||u_0 - U||_{\mathcal{V}^{1,2}} < \epsilon_0$, *there exists uniquely a time-global solution* $u(x,t)$ *of* (4) *with the* **initial condition** *being understood as*

$$\sup_{t>0} t^{\alpha/2} \|(-\Delta)^{\frac{\alpha}{2}}[u(\cdot,t) - u_0]\|_{\mathcal{V}^{1,2}} \leq C||u_0 - U||_{\mathcal{V}^{1,2}}$$

for all $\alpha \in [-1, 0]$. *Moreover, for every* $\sigma \in [0, \sigma_0]$, *the solution* u *enjoys the* **time-decay estimate**

$$\|(-\Delta)^{\frac{\sigma}{2}}[u(\cdot,t) - U]\|_{\mathcal{V}^{1,2}} \leq C\, t^{\frac{-\sigma}{2}} ||u_0 - U||_{\mathcal{V}^{1,2}}. \quad (5)$$

- Kozono-Yamazaki, 1995: Stability in (smaller) Morrey spaces.