The Strichartz inequality for orthonormal functions

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Introduction – The Schrödinger equation

By spectral theory the solution $e^{-itH}\psi$ of the **time-dependent Schrödinger equation**

$$i\frac{\partial}{\partial t}\Psi = H\Psi\,, \qquad \Psi|_{t=0} = \psi$$

with H self-adjoint satisfies $\|e^{-itH}\psi\| = \|\psi\|$ for all $t \in \mathbb{R}$.
Here we are interested in the phenomenon of **dispersion**.

Example: $H = -\Delta$ in $L^2(\mathbb{R}^d)$ and $\psi(x) = (\pi\sigma^2)^{-d/4} e^{ip\cdot x} e^{-x^2/2\sigma^2}$. Then

$$\left|\left(e^{it\Delta}\psi\right)(x)\right|^2 = \left(\frac{\sigma^2}{\pi(\sigma^4+4t^2)}\right)^{d/2} e^{-\sigma^2(x-2tp)^2/(\sigma^4+4t^2)}\,.$$

Dispersion is quantified by **Strichartz inequalities**. Simplest form:

$$\int_{\mathbb{R}}\int_{\mathbb{R}^d} \left|\left(e^{it\Delta}\psi\right)(x)\right|^{2(d+2)/d} dx\,dt \le C_d \left(\int_{\mathbb{R}^d} |\psi(x)|^2\,dx\right)^{(d+2)/d}\,.$$

Due to **Strichartz** (1977); see also **Lindblad–Sogge**, **Ginibre–Velo**, **Keel–Tao**, **Foschi**, ...

Goal – A Strichartz inequality for orthonormal functions

Is there an inequality for

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} \left(\sum_j \left| \left(e^{it\Delta} \psi_j \right)(x) \right|^2 \right)^{(d+2)/d} dx\, dt$$

with ψ_j orthonormal in $L^2(\mathbb{R}^d)$?

Obvious answer: By triangle inequality (without using orthogonality!)

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} \left(\sum_{j=1}^N \left| \left(e^{it\Delta} \psi_j \right)(x) \right|^2 \right)^{(d+2)/d} dx\, dt \leq C_d \; N^{(d+2)/d}$$

Can we do better than that?

Main result: **Yes, we can!**

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} \left(\sum_{j=1}^N \left| \left(e^{it\Delta} \psi_j \right)(x) \right|^2 \right)^{(d+2)/d} dx\, dt \leq C_d' \; N^{\mathbf{(d+1)/d}}$$

And this is best possible!

Compare with Lieb–Thirring inequalities

The **Sobolev interpolation inequality** says that for $\gamma \geq 1$ (and more)

$$\int_{\mathbb{R}^d} |\nabla\psi|^2 \, dx \geq S_{d,\gamma} \left(\int_{\mathbb{R}^d} |\psi|^2 \, dx \right)^{-\frac{\gamma-1}{d/2}} \left(\int_{\mathbb{R}^d} |\psi|^{\frac{2(\gamma+d/2)}{\gamma+d/2-1}} \, dx \right)^{\frac{\gamma+d/2-1}{d/2}} .$$

This was generalized by **Lieb–Thirring** (1976) to **orthonormal functions** ψ_j

$$\sum_{j=1}^N \int_{\mathbb{R}^d} |\nabla\psi_j|^2 \, dx \geq K_{d,\gamma} N^{-\frac{\gamma-1}{d/2}} \left(\int_{\mathbb{R}^d} \left(\sum_{j=1}^N |\psi_j|^2 \right)^{\frac{\gamma+d/2}{\gamma+d/2-1}} dx \right)^{\frac{\gamma+d/2-1}{d/2}} .$$

This is better than $N^{-\frac{\gamma}{d/2}}$ (from triangle inequality) and **optimal** in the semi-classical limit. Case $\gamma = 1$ is used in the Lieb–Thirring proof of **stability of matter**.

Slightly more precise version: for any operator $\Gamma \geq 0$ on $L^2(\mathbb{R}^d)$,

$$\mathrm{Tr}(-\Delta)\Gamma \geq K_{d,\gamma} \left(\mathrm{Tr}\, \Gamma^{\frac{\gamma}{\gamma-1}} \right)^{-\frac{\gamma-1}{d/2}} \left(\int_{\mathbb{R}^d} \Gamma(x,x)^{\frac{\gamma+d/2}{\gamma+d/2-1}} \, dx \right)^{\frac{\gamma+d/2-1}{d/2}} .$$

‘Semi-classical’ intuition behind Strichartz

Why is $\iint \left(\sum_{j=1}^N \left|\left(e^{it\Delta}\psi_j\right)(x)\right|^2\right)^{(2+d)/d} dx\,dt \le C_d'\, N^{\mathbf{(d+1)/d}}$ **best possible?**

Heuristics: At $t=0$ consider N electrons in a box of size L with const. density $\rho = L^{-d}N$. For $|t|\geq T$ the electrons have (approximately) disjoint supports and therefore

$$\iint_{|t|\geq T} \left(\sum_{j=1}^N \left|\left(e^{it\Delta}\psi_j\right)(x)\right|^2\right)^{(2+d)/d} dx\,dt \approx N \ll N^{(d+1)/d}.$$

We think of T as the typical time it takes an electron to move a distance comparable with the size of the system. By **Thomas–Fermi theory** the expected momentum per particle is $\approx \rho^{1/d}$ and therefore, if the electrons move **ballistically** $T\approx L\rho^{-1/d}$. Thus,

$$\iint_{|t|\leq T} \left(\sum_{j=1}^N \left|\left(e^{it\Delta}\psi_j\right)(x)\right|^2\right)^{(2+d)/d} dx\,dt \approx T L^d \rho^{(2+d)/d} \approx N^{(d+1)/d}.$$

The main result

Theorem 1. *Let* $d \geq 1$ *and assume that* $1 < p, q < \infty$ *satisfy*

$$1 < q \leq 1 + \frac{2}{d} \quad \text{and} \quad \frac{2}{p} + \frac{d}{q} = d\,.$$

Then, for any **orthonormal** ψ_j *and any* $n_j \in \mathbb{C}$

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}^d} \left| \sum_j n_j \left| (e^{it\Delta}\psi_j)(x) \right|^2 \right|^q dx \right)^{\frac{p}{q}} dt \leq C_{d,q}^p \left(\sum_j |n_j|^{\frac{2q}{q+1}} \right)^{\frac{p(q+1)}{2q}} . \qquad (1)$$

that is, with the notations $\gamma(t) = e^{it\Delta}\gamma e^{-it\Delta}$ *and* $\rho_\gamma(x) = \gamma(x,x)$,

$$\left\| \rho_{\gamma(t)} \right\|_{L^p_t(\mathbb{R}, L^q_x(\mathbb{R}^d))} \leq C_{d,q} \left\| \gamma \right\|_{\mathfrak{S}^{\frac{2q}{q+1}}} .$$

This is best possible in the sense that

$$\sup_\gamma \frac{\left\| \rho_{\gamma(t)} \right\|_{L^p_t(\mathbb{R}, L^q_x(\mathbb{R}^d))}}{\left\| \gamma \right\|_{\mathfrak{S}^r}} = \infty \qquad \text{if } r > \frac{2q}{q+1}\,.$$

Remarks

Recall

$$\left\|\rho_{\gamma(t)}\right\|_{L^p_t(\mathbb{R},L^q_x(\mathbb{R}^d))} \le C_{d,q} \left\|\gamma\right\|_{\mathfrak{S}^{\frac{2q}{q+1}}} \qquad \text{if } 1<q\le 1+\frac{2}{d}\,.$$

Remarks. (1) The inequality with the **trace norm** $\|\gamma\|_{\mathfrak{S}^1}$ on the right side is known, even for the full range $1\le p,q\le\infty$ with $(p,q,d)\neq(1,\infty,2)$ (plus scaling condition).
(2) This implies an **inhomogeneous Strichartz inequality**: if

$$i\dot\gamma(t) = [-\Delta,\gamma(t)] + iR(t)\,, \qquad \gamma(t_0)=0\,,$$

with $R(t)$ self-adjoint, then for q as in our theorem

$$\left\|\rho_{\gamma(t)}\right\|_{L^p_t(\mathbb{R},L^q_x(\mathbb{R}^d))} \le C \left\| \int_{\mathbb{R}} e^{-is\Delta}|R(s)|e^{is\Delta}\,ds \right\|_{\mathfrak{S}^{\frac{2q}{q+1}}} \,.$$

(3) We prove that the inequality **fails** for $q\ge(d+1)/(d-1)$. How about the range $1+2/d<q<(d+1)/(d-1)$?

A NEW RESULT

The following solves the **endpoint case**. This is joint work with J. Sabin.

Theorem 2. *Let* $d \geq 1$, $q = (d+1)/(d-1)$ *and* $p = (d+1)/d$. *Then, with the notations* $\gamma(t) = e^{it\Delta}\gamma e^{-it\Delta}$ *and* $\rho_\gamma(x) = \gamma(x,x)$,

$$\left\| \rho_{\gamma(t)} \right\|_{L^p_t(\mathbb{R}, L^q_x(\mathbb{R}^d))} \leq C'_d \left\| \gamma \right\|_{\mathfrak{S}^{\frac{2q}{q+1}, 1}} .$$

Note the **Lorentz-1 norm** (dual of weak norm) on the right side!
Via real interpolation, we get the full result.

Corollary 3. *Let* $d \geq 1$ *and assume that* $1 < p, q < \infty$ *satisfy*

$$1 < q < \frac{d+1}{d-1} \quad \text{and} \quad \frac{2}{p} + \frac{d}{q} = d \,.$$

Then,

$$\left\| \rho_{\gamma(t)} \right\|_{L^p_t(\mathbb{R}, L^q_x(\mathbb{R}^d))} \leq C_{d,q} \left\| \gamma \right\|_{\mathfrak{S}^{\frac{2q}{q+1}}} .$$

The dual formulation

Using **Hölder's inequality** (for operators and for functions) and the fact that

$$\iint_{\mathbb{R}\times\mathbb{R}^d} V(t,x)\rho_{\gamma(t)}(x)\,dx\,dt = \operatorname{Tr}\gamma\int_{\mathbb{R}} e^{-it\Delta}V(t,\cdot)e^{it\Delta}\,dt$$

we see that Theorem 1 is **equivalent** to

Theorem 4. *Let $d \geq 1$ and assume that $1 < p', q' < \infty$ satisfy*

$$1 + \frac{d}{2} \leq q' < \infty \quad \textit{and} \quad \frac{2}{p'} + \frac{d}{q'} = 2.$$

Then, with the same constant as in Theorem 1,

$$\left\| \int_{\mathbb{R}} e^{-it\Delta}V(t,\cdot)e^{it\Delta}\,dt \right\|_{\mathfrak{S}^{2q'}} \leq C_{d,q}\, \|V\|_{L_t^{p'}(\mathbb{R},L_x^{q'}(\mathbb{R}^d))}\,.$$

By **interpolation** it suffices to prove this for $q' = p' = 1 + d/2$.

Proof of Theorem 2

For $V \geq 0$,

$$\left\| \int_{\mathbb{R}} e^{-it\Delta} V(t,\cdot) e^{it\Delta}\, dt \right\|_{\mathfrak{S}^{d+2}}^{d+2} = \mathrm{Tr} \left(\int_{\mathbb{R}} e^{-it\Delta} V(t,\cdot) e^{it\Delta}\, dt \right)^{d+2}$$

$$= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \mathrm{Tr}\, V(t_1, x + 2t_1 p) \cdots V(t_{d+2}, x + 2t_{d+2} p)\, dt_{d+2} \cdots dt_1$$

Here we use the **notation**

$$f(x + 2tp) = e^{-it\Delta} f(x) e^{it\Delta}\,.$$

Lemma 5 (**Generalized Kato–Simon–Seiler ineq.**). *For* $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ *and* $r \geq 2$,

$$\| f(\alpha x + \beta p)\, g(\gamma x + \delta p) \|_{\mathfrak{S}^r} \leq \frac{\|f\|_{L^r(\mathbb{R}^d)}\, \|g\|_{L^r(\mathbb{R}^d)}}{(2\pi)^{\frac{d}{r}} |\alpha\delta - \beta\gamma|^{\frac{d}{r}}}\,.$$

Thus,

$$\left| \mathrm{Tr} \left(V(t_1, x + 2t_1 p) \cdots V(t_{d+2}, x + 2t_{d+2} p) \right) \right| \leq \frac{\|V(t_1,\cdot)\|_{L_x^{1+d/2}} \cdots \|V(t_{d+2},\cdot)\|_{L_x^{1+d/2}}}{(4\pi)^d\, |t_1 - t_2|^{\frac{d}{d+2}} \cdots |t_{d+2} - t_1|^{\frac{d}{d+2}}}$$

Proof of Theorem 2, cont'd

We have shown that

$$\left\| \int_{\mathbb{R}} e^{-it\Delta} V(t,\cdot) e^{it\Delta}\, dt \right\|_{\mathfrak{S}^{d+2}}^{d+2} \leq \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \frac{\|V(t_1,\cdot)\|_{L_x^{1+d/2}} \cdots \|V(t_{d+2},\cdot)\|_{L_x^{1+d/2}}}{(4\pi)^d\, |t_1 - t_2|^{\frac{d}{d+2}} \cdots |t_{d+2} - t_1|^{\frac{d}{d+2}}} dt_{d+2} \cdots dt_1$$

Lemma 6 (**Multi-linear HLS inequality; Christ, Beckner**). *Assume that* $(\beta_{ij})_{1\leq i,j\leq N}$ *and* $(r_k)_{1\leq k\leq N}$ *are real-numbers such that*

$$\beta_{ii} = 0, \quad 0 \leq \beta_{ij} = \beta_{ji} < 1, \quad r_k > 1, \quad \sum_{k=1}^N \frac{1}{r_k} > 1, \quad \sum_{i=1}^N \beta_{ik} = \frac{2(r_k - 1)}{r_k}.$$

Then
$$\left| \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \frac{f_1(t_1) \cdots f_N(t_N)}{\prod_{i<j} |t_i - t_j|^{\beta_{ij}}} dt_N \cdots dt_1 \right| \leq C \prod_{k=1}^N \|f_k\|_{L^{r_k}(\mathbb{R})} .$$

For us, $N = d+2$, $\beta_{ij} = \delta_{j,i+1} d/(d+2)$ and $r_k = 1 + d/2$ and thus

$$\left\| \int_{\mathbb{R}} e^{-it\Delta} V(t,\cdot) e^{it\Delta}\, dt \right\|_{\mathfrak{S}^{d+2}}^{d+2} \leq C \|V\|_{L_{t,x}^{1+d/2}}^{d+2} . \qquad \square$$

An application

Consider the **unitary propagator** $U_V(t,t_0)$ satisfying

$$i\frac{\partial}{\partial t}U_V(t,t_0) = \big(-\Delta + V(t,x)\big)U_V(t,t_0)\,, \qquad U_V(t_0,t_0) = 1\,,$$

and the **wave operator**

$$\mathcal{W}_V(t,t_0) := U_0(t_0,t)U_V(t,t_0) = e^{i(t_0-t)\Delta}U_V(t,t_0)\,. \tag{2}$$

The wave operator can be formally expanded in a **Dyson series**.

Theorem 7. *Let $d \geq 1$ and assume that $1 < p', q' < \infty$ satisfy*

$$1 + \frac{d}{2} \leq q' < \infty \quad \text{and} \quad \frac{2}{p'} + \frac{d}{q'} = 2\,.$$

If $V \in L_t^{p'}(\mathbb{R}, L_x^{q'}(\mathbb{R}^d))$, then $\lim_{t\to\pm\infty}\mathcal{W}_V(t,t_0) - 1 \in \mathfrak{S}^{2q'}$ and the Dyson series converges in $\mathfrak{S}^{2q'}$.

Improves parts of results of **Howland**, **Yajima**, **Jensen**, ...

THANK YOU FOR YOUR ATTENTION!