Global Existence of Smooth Solutions to a Cross-Diffusion System

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SKT cross-diffusion system

- Let $\Omega \subset \mathbb{R}^n$ be open, smooth, bounded and $n \geq 2$. Consider the *Shigesada-Kawasaki-Teramoto* system of equations

$$\begin{cases} u_t = \Delta[(d_1 + a_{11}u + a_{12}v)u] + u(a_1 - b_1 u - c_1 v), \Omega \times (0, \infty), \\ v_t = \Delta[(d_2 + a_{21}u + a_{22}v)v] + v(a_2 - b_2 u - c_2 v), \Omega \times (0, \infty), \end{cases}$$

with homogenous Newman boundary conditions and

$$u(\cdot, 0) = u_0(\cdot) \geq 0, \quad v(\cdot, 0) = v_0(\cdot) \geq 0 \quad \text{in } \Omega.$$

- This system models the segregation phenomena of two competing species.
- u and v denote the population densities of two species.
- $d_k, a_k, b_k, c_k > 0$ and $a_{ij} \geq 0$ are constants;
- a_{11}, a_{22} are self-diffusion coefficients and a_{12}, a_{21} are cross-diffusion coefficients.

Divergence form

The PDE of the SKT system can be written in the divergence form:

$$U_t = \nabla \cdot [J(U)\nabla U] + F(U),$$

where

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \quad J(U) = \begin{pmatrix} d_1 + 2a_{11}u + a_{12}v & a_{12}v \\ a_{21}u & d_2 + a_{21}u + 2a_{22}v \end{pmatrix},$$

and

$$F(U) = \begin{pmatrix} u(a_1 - b_1 u - c_1 v) \\ v(a_2 - b_2 u - c_2 v) \end{pmatrix}.$$

Local well-posedness: H. Amann Theorem

Theorem (H. Amann, 1990)

Let $p_0 > n$ and $U_0 \in W^{1,p_0}(\Omega)^2$ with non-negative entry. Then, there exists maximal existence time $t_{\max} > 0$ such that the SKT system

$$\begin{cases} U_t &= \nabla \cdot [J(U)\nabla U] + F(U), & \Omega \times (0,\infty), \\ \frac{\partial U}{\partial \vec{\nu}} &= 0, & \partial\Omega \times (0,\infty), \\ U(\cdot,0) &= U_0, & \Omega, \end{cases}$$

has unique, local non-negative solution $U = (u,v)^T$ with

$$U \;\in\; C([0,t_{\max}); W^{1,p_0}(\Omega)^2) \;\cap\; C^\infty(\overline{\Omega} \times (0,t_{\max}))^2.$$

Moreover, if $t_{\max} < \infty$ then

$$\lim_{t \to t_{\max}^-} \left\| U(\cdot,t) \right\|_{W^{1,p_0}(\Omega) \times W^{1,p_0}(\Omega)} = \infty.$$

Global or finite time blow-up solution?

- The solution for the STK system when J is a FULL 2×2 matrix, i.e.

$$J(U) = \begin{pmatrix} d_1 + 2a_{11}u + a_{12}v & a_{12}v \\ a_{21}u & d_2 + a_{21}u + 2a_{22}v \end{pmatrix},$$

  exists globally in time or has finite time blow up? **Vastly Unknown**.
- We restrict our study on the case when $a_{21} = 0$, that is

$$J(U) = \begin{pmatrix} d_1 + 2a_{11}u + a_{12}v & a_{12}v \\ 0 & d_2 + 2a_{22}v \end{pmatrix}.$$

  Let us call the SKT system with this J: Triangular SKT System.

Triangular SKT: Known results

- The Triangular SKT System, i.e.

$$\begin{cases} U_t & = \nabla \cdot [J(U)\nabla U] + F(U), & \Omega \times (0,\infty), \\ \frac{\partial U}{\partial \vec{\nu}} & = 0, & \partial\Omega \times (0,\infty), \\ U(\cdot,0) & = U_0, & \Omega, \end{cases}$$

  with

$$J(U) = \begin{pmatrix} d_1 + 2a_{11}u + a_{12}v & a_{12}v \\ 0 & d_2 + 2a_{22}v \end{pmatrix},$$

  has global solution when $n \leq 9$.

- Y. Lou, W.-M. Ni and J. Wu (1998): $n = 2$.
- D. Le, L. Nguyen, T. Nguyen (2003); Y. Choi, R. Lui, Y. Yamada (2004): $n \leq 5$.
- T. P. (2008): $n \leq 9$.
- Many other results: Restrictive conditions on the coefficients.

Today main result

Theorem (L. Hoang, T. Nguyen and T. P. – 2013)

Let $\Omega \subset \mathbb{R}^n$ be open, bounded for any $n \geq 2$, and let $U_0 \in [W^{1,p_0}(\Omega)]^2$ with $p_0 > n$. Then, the solution $U = (u, v)^T$ of the Triangular SKT system

$$\begin{cases} U_t & = \nabla \cdot [J(U)\nabla U] + F(U), & \Omega \times (0,\infty), \\ \frac{\partial U}{\partial \vec{\nu}} & = 0, & \partial\Omega \times (0,\infty), \\ U(\cdot, 0) & = U_0, & \Omega, \end{cases}$$

where

$$J(U) = \begin{pmatrix} d_1 + 2a_{11}u + a_{12}v & a_{12}v \\ 0 & d_2 + 2a_{22}v \end{pmatrix}$$

exists uniquely, globally in time and

$$U \in \left[C([0,\infty); W^{1,p_0}(\Omega))\right]^2 \cap \left[C^\infty(\overline{\Omega} \times (0,\infty))\right]^2.$$

Ideas of the proof

- Let $T > 0$ be the maximal time existence and assume $T < \infty$, we prove by contradiction that

$$\lim_{t\to T^-}\left[\left\|u(\cdot,t)\right\|_{W^{1,p_0}(\Omega)} + \left\|v(\cdot,t)\right\|_{W^{1,p_0}(\Omega)}\right] < \infty.$$

- Sufficient to establish the bound ($0 < \epsilon \ll 1$)

$$\|\nabla v\|_{L^p(\Omega\times(\epsilon,T))} + \|u\|_{L^p(\Omega\times(\epsilon,T))} \leq C(T), \quad p > n+2.$$

- Important known estimates:
  (i) Maximum Principle (Lou-Ni-Wu, 2003): The PDE of v is

$$v_t = \nabla\cdot[(d_2 + 2a_{22}v)\nabla v] + v(a_2 - b_2 u - c_2 v).$$

  Therefore, $0 \leq v \leq \max\left\{\max_{\overline{\Omega}} v_0, \frac{a_2}{c_2}\right\}$. However, M.P. is not available for u, b/c

$$u_t = \nabla\cdot[(d_1 + 2a_{11}u + a_{12}v)\nabla u + a_{12}u\nabla v] + u(a_1 - b_1 u - c_1 v).$$

  (ii) T. P. (2008): $\|\nabla v\|_{L^4(\Omega\times(0,T))} \leq C(T)$.

Key iteration lemma

- The PDE of u:

$$u_t = \nabla \cdot [(d_1 + 2a_{11}u + a_{12}v)\nabla u + a_{12}u\nabla v] + u(a_1 - b_1 u - c_1 v).$$

Lemma

Let $p > 2$ and assume that

$$\|\nabla v\|_{L^p(\Omega_T)} \leq C(p, T).$$

Then for each $q \in \left[p, \frac{p(n+1)}{(n+2-p)_+}\right]$ with $q \neq \infty$, we have

$$\|u\|_{L^q(\Omega_T)} \leq C(p, q, T).$$

- Main question: If $u \in L^q(\Omega_T)$, can we derive the estimate

$$\|\nabla v\|_{L^q(\Omega_T)} \leq C(q, p, T) \quad ?$$

Regularity problem

- The PDE of v:

$$v_t = \nabla \cdot [(d_2 + 2a_{22}v)\nabla v] + v(a_2 - c_2 v) - b_2 uv, \quad \text{in } \Omega \times (0, T).$$

- Goal: To establish

$$\|\nabla v\|_{L^p(\Omega\times(0,T))} \leq C\Big[1 + \|u\|_{L^p(\Omega\times(0,T))}\Big].$$

- Difficulties:
  - (i) Main term $(d_2 + 2a_{22}v)$ depends on solution. Therefore, its oscillation is not small.
  - (ii) The equation is not invariant under either of the scalings

$$v(x,t) \to \frac{v(x,t)}{\lambda} \quad \text{or} \quad v(x,t) \to \frac{v(\theta x, \theta^2 t)}{\theta}, \quad \lambda, \theta > 0.$$

  - (iii) The equation is not invariant under the change of coordinates.

Equations with double scaling parameters

- Denote $\Omega_T = \Omega \times (0, T)$, we study the equation:

$$\begin{cases} w_t = \nabla \cdot [(1 + \lambda \alpha w)\mathbf{A}\nabla w] + \theta^2 w(1 - \lambda w) - \lambda \theta c w & \text{in } \Omega_T, \\ \dfrac{\partial w}{\partial \vec{\nu}} = 0 & \text{on } \partial\Omega \times (0, T), \\ w(\cdot, 0) = w_0(\cdot) & \text{in } \Omega. \end{cases}$$

- Here, $\theta, \lambda > 0$ and $\alpha \geq 0$ are constants,
- $c(x, t)$ is a nonnegative measurable function,
- $\mathbf{A} = (a_{ij}) : \Omega_T \to \mathcal{M}^{n \times n}$ is symmetric, measurable and $\exists \Lambda > 0$ such that

$$\Lambda^{-1}|\xi|^2 \leq \xi^T \mathbf{A}(x, t)\xi \leq \Lambda|\xi|^2 \quad \text{for a.e. } (x, t) \in \Omega_T \text{ and for all } \xi \in \mathbb{R}^n.$$

Calderón - Zygmund type estimates

Theorem (L. Hoang, T. Nguyen and T. P., 2013)

Let $p > 2$. Then there exists a number $\delta = \delta(p, \Lambda, n, \alpha) > 0$ such that if Ω is a Lipschitz domain with the Lipschitz constant $\leq \delta$ and $[\mathbf{A}]_{BMO(\Omega_T)} \leq \delta$, then for any weak solution w of

$$\begin{cases} w_t = \nabla \cdot [(1 + \lambda \alpha w)\mathbf{A}\nabla w] + \theta^2 w(1 - \lambda w) - \lambda \theta c w & \text{in } \Omega_T, \\ \dfrac{\partial w}{\partial \vec{v}} = 0 & \text{on } \partial\Omega \times (0, T), \\ w(\cdot, 0) = w_0(\cdot) & \text{in } \Omega. \end{cases}$$

satisfying $0 \leq w \leq \lambda^{-1}$ in Ω_T, we have

$$\int_{\Omega \times [\bar{t}, T]} |\nabla w|^p \, dxdt \leq C \left\{ \left(\frac{\theta}{\lambda} \vee \|w\|_{L^2(\Omega_T)} \right)^p + \int_{\Omega_T} |c|^p \, dxdt \right\}$$

for every $\bar{t} \in (0, T)$. Here $C > 0$ is a constant depending only on Ω, $\bar{t}$, p, Λ, α and n and independent of θ, λ.

Main steps in the proof (interior estimates)

- PERTURBATION TECHNIQUE(Caffarelli–Peral): Comparing the solution of

$$w_t = \nabla \cdot [(1 + \lambda\alpha w)\mathbf{A}\nabla w] + \theta^2 w(1 - \lambda w) - \lambda\theta c w \quad \text{in} \quad Q_6 \quad (1)$$

  with that of the reference equation

$$h_t = \nabla \cdot [(1 + \lambda\alpha h)\bar{\mathbf{A}}_{B_4}(t)\nabla h] + \theta^2 h(1 - \lambda h) \quad \text{in} \quad Q_4, \qquad (2)$$

  where $\bar{\mathbf{A}}_{B_4}(t)$ is the average of $\mathbf{A}(\cdot, t)$ over B_4, that is,

$$\bar{\mathbf{A}}_{B_4}(t) := \frac{1}{|B_4|}\int_{B_4} \mathbf{A}(x, t)dx.$$

- Notice that h is a weak solution of (2) iff $\bar{h} := \lambda h$ is a weak solution of

$$\bar{h}_t = \nabla \cdot [(1 + \alpha\bar{h})\bar{\mathbf{A}}_{B_4}(t)\,\nabla\bar{h}] + \theta^2\bar{h}(1 - \bar{h}) \quad \text{in} \quad Q_4.$$

Lemma

Let $\bar{h}$ be a weak solution of

$$\bar{h}_t = \nabla \cdot [(1 + \alpha \bar{h})\bar{\mathbf{A}}_{B_4}(t)\nabla \bar{h}] + \theta^2 \bar{h}(1 - \bar{h}) \quad \textit{in} \quad Q_4$$

satisfying $0 \leq \bar{h} \leq 1$ *in* Q_4*. Then*

$$\|\nabla \bar{h}\|^2_{L^\infty(Q_3)} \leq C(n, \Lambda, \alpha)\frac{1}{|Q_4|}\int_{Q_4} |\nabla \bar{h}|^2 \, dxdt.$$

Key Ideas: De Giorgi - Nash - Moser.

First approximation lemma

Lemma

For any $\varepsilon > 0$, *there exists* $\delta = \delta(\varepsilon, n, \Lambda, \alpha) > 0$ *such that if*

$$\int_{Q_4} \left[|\mathbf{A}(x,t) - \bar{\mathbf{A}}_{B_4}(t)|^2 + |c(x,t)|^2 \right] dxdt \leq \delta,$$

and w *is a weak solution of* (1) *in* Q_5 *satisfying*

$$0 \leq w \leq \lambda^{-1} \quad \text{and} \quad \int_{Q_4} |\nabla w|^2 \, dxdt \leq 1,$$

and h *is the weak solution of* (2) *with* $h = w$ *on* $\partial_p Q_4$ *and* $0 \leq h \leq \lambda^{-1}$ *in* Q_4, *then*

$$\int_{Q_4} |w - h|^2 \, dxdt \leq \varepsilon.$$

Key Ideas: Compactness argument + energy estimates.

Second approximation lemma

Lemma

For any $\varepsilon > 0$, *there exists* $\delta = \delta(\varepsilon, n, \Lambda, \alpha) > 0$ *such that for all* $0 < r \leq 1$, *if*

$$\frac{1}{|Q_{4r}|} \int_{Q_{4r}} \left[|\mathbf{A}(x,t) - \bar{\mathbf{A}}_{B_{4r}}(t)|^2 + |c(x,t)|^2 \right] dxdt \leq \delta,$$

then for any weak solution w *of* (1) *in* Q_{5r} *satisfying*

$$0 \leq w \leq \lambda^{-1} \text{ in } Q_{4r}, \quad \text{and} \quad \frac{1}{|Q_{4r}|} \int_{Q_{4r}} |\nabla w|^2 \, dxdt \leq 1,$$

and weak solution h *of* (2) *in* Q_{4r} *sastifying* $h = w$ *on* $\partial_p Q_{4r}$ *and* $0 \leq h \leq \lambda^{-1}$, *we have*

$$\frac{1}{|Q_{4r}|} \int_{Q_{4r}} |w - h|^2 \, dxdt \leq \varepsilon r^2 \quad \text{and} \quad \frac{1}{|Q_{4r}|} \int_{Q_{2r}} |\nabla w - \nabla h|^2 \, dxdt \leq \varepsilon.$$

Lemma

Assume $c \in L^2(Q_6)$. $\exists N = N(n, \Lambda, \alpha) > 1$ *such that for any* $\varepsilon > 0$, *we can find* $\delta = \delta(\varepsilon, n, \Lambda) > 0$ *such that if*

$$\sup_{0<\rho\leq 4} \sup_{(y,s)\in Q_1} \frac{1}{|Q_\rho(y,s)|} \int_{Q_\rho(y,s)} |\mathbf{A}(x,t) - \bar{\mathbf{A}}_{B_\rho(y)}(t)|^2 \, dxdt \leq \delta,$$

then for any weak solution w *of* (1) *satisfying*

$$0 \leq w \leq \lambda^{-1} \quad \textit{in} \quad Q_5, \quad \textit{and} \quad \left|\{Q_1 : \mathcal{M}_{Q_5}(|\nabla w|^2) > N\}\right| \leq \varepsilon |Q_1|,$$

we have

$$\left|\{Q_1 : \mathcal{M}_{Q_5}(|\nabla w|^2) > N\}\right|$$
$$\leq (10)^{n+2} \varepsilon \left\{ \left|\{Q_1 : \mathcal{M}_{Q_5}(|\nabla w|^2) > 1\}\right| + \left|\{Q_1 : \mathcal{M}_{Q_5}(c^2) > \delta\}\right| \right\}.$$

THANK YOU