Stability of Eigenvalues of Quantum Graphs with Respect to Magnetic Perturbation

Tracy Weyand

Texas A&M University
College Station, TX 77843-3368

www.math.tamu.edu/~tweyand

tweyand@math.tamu.edu

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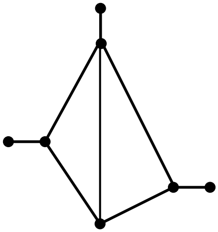
Metric Graphs

$\Gamma = \{V, E, L\}$ *Compact*



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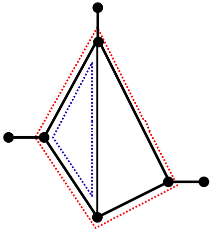

Functions: $\widetilde{H}^2(\Gamma) = \oplus_{e \in E} H^2(e)$

1^{st} Betti $\# = |E| - |V| + 1$

Metric Graph + Differential Operator

Schrödinger Operator

$$H^0(\Gamma) : f \mapsto -\frac{d^2}{dx^2} f(x) + q(x) f(x), \quad f \in \widetilde{H}^2(\Gamma, \mathbb{C})$$

$$\begin{cases} f(x) \text{ is continuous at } v, \\ \sum_{e \in E_v} \frac{d}{dx_e} f(x)\Big|_v = \chi_v f(v), \quad \chi_v \in \mathbb{R} \end{cases}$$

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Magnetic Schrödinger Operator

$$H^A(\Gamma) : f \mapsto -\left(\frac{d}{dx} - iA(x)\right)^2 f(x) + q(x) f(x), \quad f \in \widetilde{H}^2(\Gamma, \mathbb{C})$$

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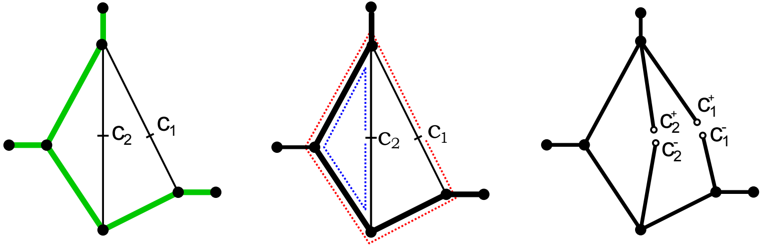
Magnetic Flux



$$\alpha_j = \int_{c_j^-}^{c_j^+} A(x)dx \mod 2\pi$$

Magnetic Flux: $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_\beta)$

Unitarily Equivalent Operators

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$$H^\alpha(\Gamma) : f \mapsto -\frac{d^2}{dx^2} f(x) + q(x)f(x), \quad f \in \widetilde{H^2}(T, \mathbb{C})$$

$$\begin{cases} f(x) \text{ is continuous at } v \\ \sum_{e \in E_v} \frac{df}{dx_e}(v) = \chi_v f(v) \quad \text{for } v \in \Gamma \\ f(c_j^-) = e^{i\alpha_j} f(c_j^+) \\ f'(c_j^-) = -e^{i\alpha_j} f'(c_j^+) \end{cases}$$

Now we consider $\lambda_n(\boldsymbol{\alpha})$ as a function of $\boldsymbol{\alpha}$.

Nodal Surplus

$\phi_n = \#$ of zeros of the n^{th} eigenfunction
$\nu_n = \#$ of subgraphs formed by removing the ϕ_n zeros from Γ

Nodal Surplus: $\phi_n - (n-1)$

Nodal Deficiency: $n - \nu_n$

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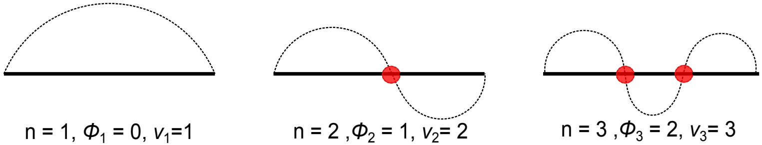

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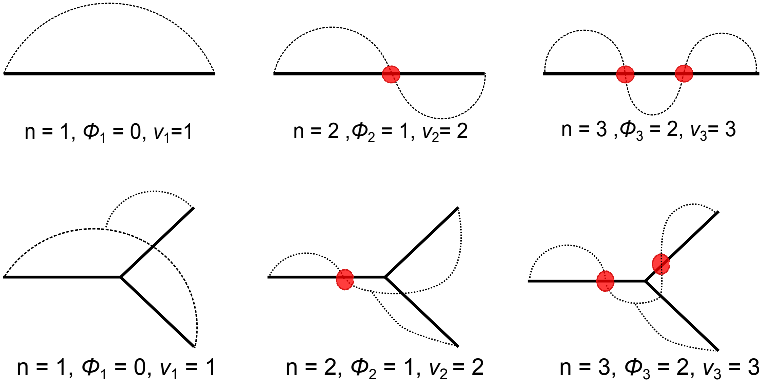

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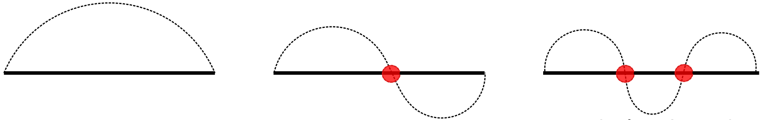
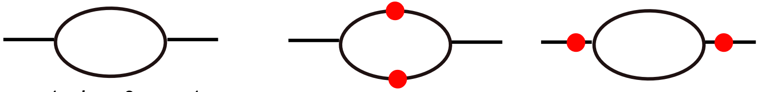


$n = 1$, $\Phi_1 = 0$, $v_1 = 1$ — $n = 2$, $\Phi_2 = 1$, $v_2 = 2$ — $n = 3$, $\Phi_3 = 2$, $v_3 = 3$



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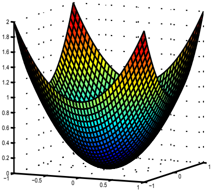
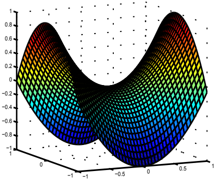
Morse Index $=$ $\#$ of negative eigenvalues of the Hessian matrix

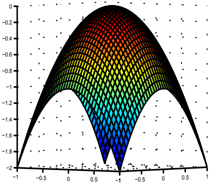
$$H_{i,j} = \frac{d^2\lambda_n(\boldsymbol{\alpha})}{d\alpha_i d\alpha_j}$$



Morse Index = 0



Morse Index = 1



Morse Index = 2

Main Result

Theorem (Berkolaiko & Weyand, 2013)

Let λ_n be a simple eigenvalue of H^0 whose eigenfunction has ϕ internal zeros.

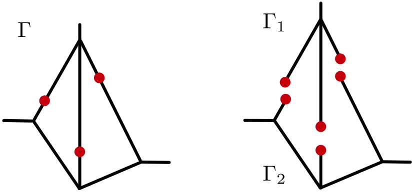
Consider the eigenvalues $\lambda_n(\boldsymbol{\alpha})$ of $H^{\boldsymbol{\alpha}}$ as a function of $\boldsymbol{\alpha}$:

- $\boldsymbol{\alpha} = (0, 0, \ldots, 0)$ *is a non-degenerate critical point of* $\lambda_n(\boldsymbol{\alpha})$ *and*
- *the Morse index of this critical point is equal to* $\phi - (n-1)$

Partitions

Proper m-Partition: Set of m points, none of which lie on vertices

Partition Subgraphs: Subgraphs Γ_j formed by applying Dirichlet conditions at the m-partition points



Corollary

$$\Lambda(P) := \max_j \lambda_1(\Gamma_j)$$

Equipartition: All partition subgraphs have the same first eigenvalue

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Corollary (Berkolaiko & Weyand, 2013)

Consider Λ on the set of equipartitions:

- *the ϕ-equipartition formed from the zeros of the n^{th} eigenfunction is a non-degenerate critical point of Λ and*
- *the Morse index of this critical point is equal to $n - \nu$.*

Note: This strengthens the result of Band, Berkolaiko, Raz, and Smilansky ('12)

Can one "hear" the shape of a graph?

Given only eigenvalues, can one reconstruct the graph?

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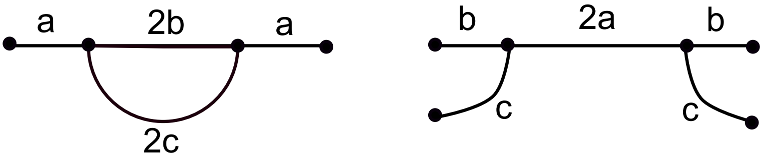
Given only eigenvalues, can one reconstruct the graph?

No, isospectral quantum graphs exist (Sunada, '85).

Cannot Determine: (Band and Parzanchevski, '10)

- # of edges and vertices
- # of independent cycles ($\beta = |E| - |V| + 1$)



Only a Tree is a Tree

On a tree, $\phi_n = n - 1 \ \forall n$.

Theorem (Band, 2013)

If $\phi_n = n - 1 \ \forall n$, then the graph is a tree.

References

- R. BAND, *The nodal count* $\{0, 1, 2, 3, \ldots\}$ *is a tree.* preprint `arXiv:1212.6710 [math-ph]`, 2012.
- R. BAND, G. BERKOLAIKO, H. RAZ, AND U. SMILANSKY, *The number of nodal domains on quantum graphs as a stability index of graph partitions*, Comm. Math. Phys., 311 (2012), pp. 815-838.
- G. BERKOLAIKO AND T. WEYAND, *Stability of eigenvalues of quantum graphs with respect to magnetic perturbation and the nodal count of the eigenfunctions*, Philosophical Transactions of the Royal Society A, accepted `arXiv:1212.4475 [math-ph]`, 2012.

Contact Information

Tracy Weyand

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