Aluthge and Mean transforms of bounded linear operators

Jasang Yoon (The Univ. of Texas-Pan American)

Outline of this talk

$\boxed{1}$ Introduction

$\boxed{2}$ Basic results of the AT of 1-vari. operators

$\boxed{3}$ " " MT " "

$\boxed{4}$ The AT of a commuting pair

$\boxed{5}$ Open problems

$\boxed{1}$ Introduction

- $\mathcal{H}$: complex Hilbert space
- $B(\mathcal{H})$: Algebra of bounded operators on $\mathcal{H}$
- For $T \in B(\mathcal{H})$, the polar decomposition of T is

non-negative operator

$$T = U|T|$$

$$|T| = \sqrt{T^*T}$$

partial isometry

$$U|_{(\ker U)^\perp} : \text{isometry}$$

- Examples

- The polar form (decomposition) of a nonzero complex number:

$$z = r \cdot e^{i\theta} = \underbrace{e^{i\theta}}_{\text{the complex sign of } z} \underbrace{|z|}_{\text{the absolute value of } z}$$

- The polar decomposition of a square complex matrix $T = \underbrace{U}_{\text{Unitary matrix}} \underbrace{|T|}_{\text{Hermitian matrix}}$

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}}_{\text{unitary matrix}} \underbrace{\begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}}_{\text{Hermitian matrix}}$$

- The Aluthge transform of $T \in B(\mathcal{H})$ is the operator $\tilde{T} := |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$
- This transform was first studied by Aluthge in [IEOT, 1990] in order to study <u>p-hyponormal</u> & <u>log-hypo. oper.</u>

$(T^*T)^p \geq (TT^*)^p$ for $0 < p \leq 1$

$\log(T^*T) \geq \log(TT^*)$ & T: invertible

- The AT has received much attention in recent years. One reason is its connection with the invariant subspace problem

- Invariant subspace problem
  (1932, J. Von Neumann)

Let X be a Banach space of $\dim X \geq 2$
& $T \in B(X)$.

Does T have a nontrivial $(\neq \{0\}, X)$ invariant subspace?

$\boxed{2}$ Basic results of the AT

- Jung, Ko and Pearcy proved in [IEOT, 2000] that for $T \in B(\mathcal{H})$, T has a nontrivial invariant subspace $\Leftrightarrow$ $\widetilde{T}$ does.

- Jung, Ko & Pearcy also proved in [IEOT, 01] that the spectrum of $\widetilde{T}$ equals that of T.

- Lee, Lee and Yoon proved in [IEOT, 12] that AT need not preserve the k-hyponormality for $k \geq 2$.

- It is well known that T has a nontrivial invariant subspace if T is normal.
- It is also known that the <u>iterated AT</u> of an operator is closer to being a normal operator.

$$\widetilde{T}^{(0)} := T$$
$$\widetilde{T}^{(1)} := \widetilde{T}$$
$$\widetilde{T}^{(2)} := \widetilde{\left(\widetilde{T}\right)}$$
$$\vdots$$
$$\widetilde{T}^{(n+1)} := \widetilde{\left(\widetilde{T}^{(n)}\right)}$$

- For a weighted shift

$$W_\alpha \equiv \text{shift}(\alpha_0, \alpha_1, \alpha_2, \cdots) : \ell^2(\mathbb{Z}_+) \longrightarrow \ell^2(\mathbb{Z}_+),$$

$$W_\alpha = U_+ D_\alpha = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \alpha_0 & & & \\ & \alpha_1 & & 0 \\ & 0 & \alpha_2 & \\ & & & \ddots \end{pmatrix},$$

where $\alpha_i > 0$.

- $\widetilde{W}_\alpha = D_\alpha^{\frac{1}{2}} U_+ D_\alpha^{\frac{1}{2}} = \text{shift}\big(\underbrace{\sqrt{\alpha_0\alpha_1}}_{\text{geometric mean}}, \sqrt{\alpha_1\alpha_2}, \sqrt{\alpha_2\alpha_3}, \cdots\big)$

$$\Big(\because\ \widetilde{W}_\alpha(e_n) = D_\alpha^{\frac{1}{2}} U_+ D_\alpha^{\frac{1}{2}}(e_n) = D_\alpha^{\frac{1}{2}} U_+ \big(\sqrt{\alpha_n}\, e_n\big)$$

$$= \sqrt{\alpha_n} \cdot D_\alpha^{\frac{1}{2}} U_+(e_n) = \sqrt{\alpha_n} \cdot D_\alpha^{\frac{1}{2}}(e_{n+1})$$

$$= \sqrt{\alpha_n}\big(\sqrt{\alpha_{n+1}}\, e_{n+1}\big) = \sqrt{\alpha_n \alpha_{n+1}}\, e_{n+1} \Big)$$

- In the view of the practical use, it is so hard to find the AT $\tilde{T}$ of T,

$$\tilde{T} = \sqrt[4]{T^*T}\, U \sqrt[4]{T^*T}, \quad T = U\sqrt{T^*T} \in B(\mathcal{H}),$$

because it involves the term $\underbrace{\sqrt[4]{T^*T}}_{?}$.

- Motivation of the Mean transform $\hat{T}$ of $T \in B(\mathcal{H})$.

Which transform converts W_α to shift $\left(\frac{\alpha_0+\alpha_1}{2}, \frac{\alpha_1+\alpha_2}{2}, \cdots\right)$?

- If $T = U|T|$ is the polar decomposition of T, then we define the Mean transform $\widehat{T}$ of $T \in B(\mathcal{H})$ by

$$\widehat{T} = \frac{1}{2}\left(U|T| + |T|U\right),$$

where $|T|U =: \widetilde{T}^D$ is called Duggal transform.

- For $W_\alpha \equiv \text{shift}(\alpha_0, \alpha_1, \cdots)$,

$$\widehat{W_\alpha} = \text{shift}\left(\frac{\alpha_0+\alpha_1}{2}, \frac{\alpha_1+\alpha_2}{2}, \cdots\right).$$

- We get the MT easily, if we know the polar decomposition of operators.

- We say that

- $T \in B(\mathcal{H})$ is normal if $T^*T = TT^*$.
- $T \in B(\mathcal{H})$ is subnormal if $T = N|_{\mathcal{H}}$,

  where N is normal & $N(\mathcal{H}) \subseteq \mathcal{H}$.
- $T \in B(\mathcal{H})$ is hyponormal if $T^*T \geq TT^*$.
- $T \in B(\mathcal{H})$ is k-hyponormal if $(I, T, T^2, \ldots, T^k)$

  is jointly hyponormal
- $T \in B(\mathcal{H})$ is 2-hypo. if $\begin{pmatrix} I & T^* & T^{*2} \\ T & T^*T & T^{*2}T \\ T^2 & T^*T^2 & T^{*2}T^2 \end{pmatrix} \geq 0$.
- $T \in B(\mathcal{H})$ is subnormal

  iff T is ∞-hyponormal.

$\boxed{3}$ Basic results of the MT

In [IEOT, 13], we show the followings:

(i) The spectrum of $\widehat{T}$ is not equal to that of T.

Ex: $T := \begin{pmatrix} 0 & P \\ 0 & 0 \end{pmatrix} \in B(\mathcal{H} \oplus \mathcal{H})$, where $P \in B(\mathcal{H})$ is a positive operator.

Then $\sigma(T) = \{0\} \neq \{\pm \frac{\sigma(P)}{2}\} = \sigma(\widehat{T})$.

(ii) The mean transform map $T \longrightarrow \widehat{T}$ is $(\|\cdot\|, SOT)$ - continuous on $B(\mathcal{H})$.

That is, $\|\widehat{T_n}(x) - \widehat{T_0}(x)\| \xrightarrow[n\to\infty]{} 0$ for each $x \in \mathcal{H}$

if $\|T_n - T_0\| \xrightarrow[n\to\infty]{} 0$.

(iii) If $T \in B(\mathcal{H})$ is hyponormal, then the MT $\widehat{T}$ is also hyponormal.

But the converse of it is not true in general.

Ex: If $W_\alpha \equiv \text{shift}\left(\frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \frac{3}{2}, \cdots\right)$, then W_α is clearly not hypo. but $\widehat{T} = U_+$ is sub.

(iv) For $W_\alpha \equiv \text{shift}(\alpha_0, \alpha_1, \alpha_2, \cdots)$ with $\alpha_0 = \alpha_1 = \frac{1}{2}$, let $\widehat{W_\alpha} \equiv B_+^2 \equiv \text{shift}\left(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots\right)$. Then we have:

(a) $W_\alpha \equiv \text{shift}\left(\frac{1}{2}, \frac{1}{2}, \frac{5}{6}, \frac{2}{3}, \frac{44}{75}, \frac{11}{15}, \cdots\right)$ is not hypo.

(b) $\widetilde{W_\alpha}$ is hypo. (c) $\widetilde{W_\alpha}$ is not 2-hypo.

(d) $\widehat{W_\alpha}$ is subnormal.

$\boxed{4}$ The AT of a commuting pair (T_1, T_2)

- 1^{st} plausible definition

$$\tilde{\mathbf{T}} := \widetilde{(T_1, T_2)} := \left(\sqrt[4]{T_1^*T_1}\, U \sqrt[4]{T_1^*T_1},\ \sqrt[4]{T_2^*T_2}\, U \sqrt[4]{T_2^*T_2}\right)$$

But we can't find a common partial isometry U which satisfies, simultaneously, the following:

$$T_1 = U\sqrt{T_1^*T_1} \quad \& \quad T_2 = U\sqrt{T_2^*T_2}$$

Thus, we don't consider this one.

- 2^{nd} plausible definition

- $\widetilde{\mathcal{U}} := (\sqrt{P}U_1\sqrt{P}, \sqrt{P}U_2\sqrt{P})$, where $P := \sqrt{T_1^*T_1 + T_2^*T_2}$.

Then a polar decomposition of the pair (T_1, T_2) is $(T_1, T_2) = (U_1P, U_2P)$.

- $\widetilde{T_1} = \sqrt{P}U_1\sqrt{P}$ is not the AT of T_1, but (T_1, T_2) is a (joint) partial isometry, that is, $U_1^*U_1 + U_2^*U_2\big|_{(\ker(U_1)\cap\ker(U_2))^\perp} = 1$.

- $T_1 \leftrightarrow T_2 \Rightarrow \sqrt{P}U_1\sqrt{P} \leftrightarrow \sqrt{P}U_2\sqrt{P}$

- $\exists$ (T_1, T_2) s.t T_1 : hypo, but $\sqrt{P}U_1\sqrt{P}$ is not hypo.

$\therefore$ we don't consider this one.

- A suitable definition of the AT for a commuting pair $\mathbf{T} \equiv (T_1, T_2)$ is
$$\widetilde{\mathbf{T}} \equiv (\widetilde{T}_1, \widetilde{T}_2) := \left(|T_1|^{\frac{1}{2}} U_1 |T_1|^{\frac{1}{2}}, |T_2|^{\frac{1}{2}} U_2 |T_2|^{\frac{1}{2}} \right).$$
definition

- Based on the definition of AT $\widetilde{\mathbf{T}}$, we study the following two problems.
- Prob 1. For $k \geq 1$, if $W_{(\alpha,\beta)} \equiv (T_1, T_2)$ is k-hypo., does it follow that AT $\widetilde{W}_{(\alpha,\beta)}$ k-hypo?
- Prob 2. Does Taylor spectrum of $\widetilde{W}_{(\alpha,\beta)}$ equal that of $W_{(\alpha,\beta)}$?

2-variable weighted shift

- 2-variable weighted shift $W_{(\alpha,\beta)} \equiv (T_1, T_2)$ $: \ell^2(\mathbb{Z}_+^2) \longrightarrow \ell^2(\mathbb{Z}_+^2)$ such that

$$T_1(e_{(k_1,k_2)}) := \alpha_{(k_1,k_2)} e_{(k_1,k_2)+\varepsilon_1}$$

& $$T_2(e_{(k_1,k_2)}) := \beta_{(k_1,k_2)} e_{(k_1,k_2)+\varepsilon_2},$$

where $\varepsilon_1 = (1,0)$, $\varepsilon_2 = (0,2)$,

$\{e_{(k_1,k_2)} : (k_1,k_2) \in \mathbb{Z}_+^2\}$: ONB for $\ell^2(\mathbb{Z}_+^2)$,

$\alpha_{(k_1,k_2)} > 0$, $\beta_{(k_1,k_2)} > 0$.

- $\mathfrak{C}_0$: the class of commuting pairs of operators on a Hilbert space $\mathcal{H}$.
- $\mathfrak{C}_k$: the class of k-hypo. pairs in $\mathfrak{C}_0$ $(k \geq 1)$.

- $\mathfrak{C}_\infty$: the class of subnormal pairs in $\mathfrak{C}_0$.
- We have that $\mathfrak{C}_\infty \subseteq \cdots \subseteq \mathfrak{C}_k \subseteq \cdots \subseteq \mathfrak{C}_1 \subseteq \mathfrak{C}_0$.
- Weight diagram of $W_{(\alpha,\beta)} \equiv (T_1, T_2)$

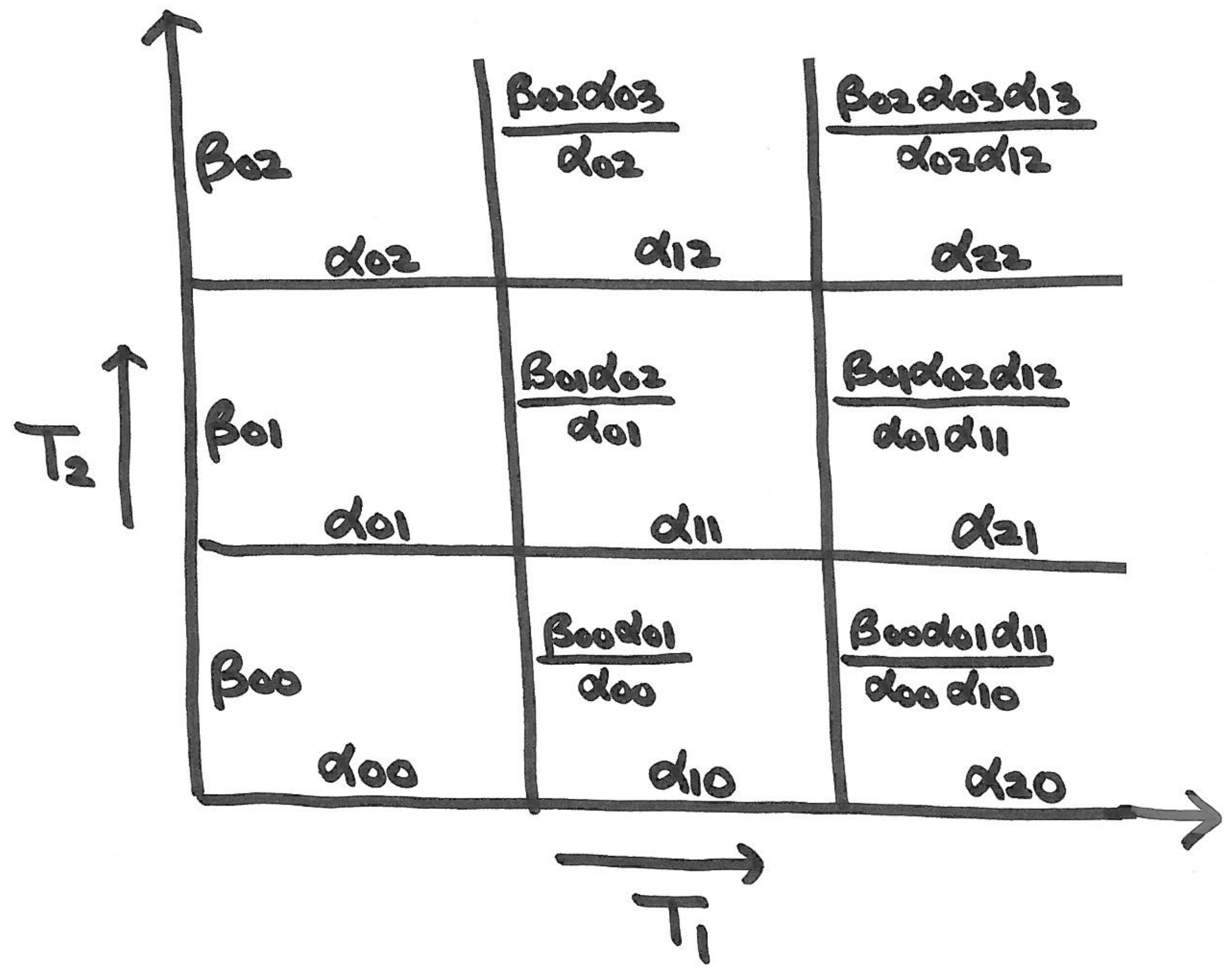


Figure 1

- Weight diagram of $\tilde{\omega}_{(\alpha,\beta)} = (|T_1|^{\frac{1}{2}} U_1 |T_1|^{\frac{1}{2}}, |T_2|^{\frac{1}{2}} U_2 |T_2|^{\frac{1}{2}})$
$= (\tilde{T}_1, \tilde{T}_2)$

$\sqrt{\beta_{02}\beta_{03}}$ $\sqrt{\alpha_{02}\alpha_{12}}$	$\sqrt{\frac{\beta_{02}\beta_{03}\alpha_{04}}{\alpha_{02}}}$ $\sqrt{\alpha_{12}\alpha_{22}}$	$\sqrt{\alpha_{22}\alpha_{32}}$
$\sqrt{\beta_{01}\beta_{02}}$ $\sqrt{\alpha_{01}\alpha_{11}}$	$\sqrt{\frac{\beta_{01}\beta_{02}\alpha_{03}}{\alpha_{01}}}$ $\sqrt{\alpha_{11}\alpha_{21}}$	$\sqrt{\frac{\beta_{01}\beta_{02}\alpha_{03}\alpha_{13}}{\alpha_{01}\alpha_{11}}}$ $\sqrt{\alpha_{21}\alpha_{31}}$
$\sqrt{\beta_{00}\beta_{01}}$ $\sqrt{\alpha_{00}\alpha_{10}}$	$\sqrt{\frac{\beta_{00}\beta_{01}\alpha_{02}}{\alpha_{00}}}$ $\sqrt{\alpha_{10}\alpha_{20}}$	$\sqrt{\frac{\beta_{00}\beta_{01}\alpha_{02}\alpha_{12}}{\alpha_{00}\alpha_{10}}}$ $\sqrt{\alpha_{20}\alpha_{30}}$

$\tilde{T}_2 \uparrow$

$\tilde{T}_1 \rightarrow$

- $\mathbf{T} \equiv (T_1, T_2)$ is (jointly) hyponormal

if
$$\begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] \end{pmatrix} \geq 0, \quad \text{where } [S,T] = ST - TS$$

- $\mathbf{T}$ is (jointly) k-hypo $\quad (k \geq 1)$

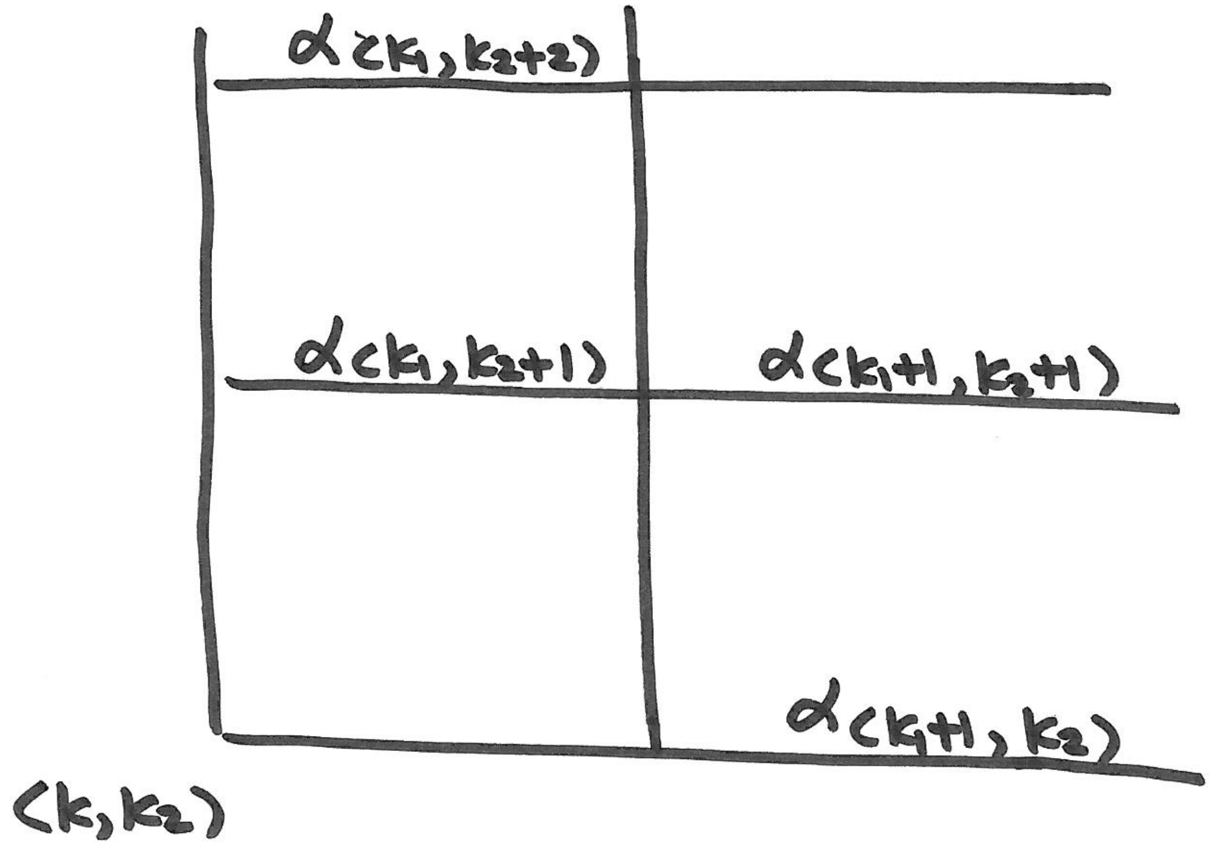
if $(T_1, T_2, T_1^2, T_2T_1, T_2^2, \cdots, T_1^k, T_2T_1^{k-1}, \cdots, T_2^k)$ is (jointly) hypo.

$$\Longleftrightarrow \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \cdots & [(T_2^k)^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \cdots & [(T_2^k)^*, T_2] \\ [T_1^*, T_1^2] & [T_2^*, T_1^2] & \cdots & [(T_2^k)^*, T_1^2] \\ \vdots & \vdots & \ddots & \vdots \\ [T_1^*, T_2^k] & [T_2^*, T_2^k] & \cdots & [(T_2^k)^*, T_2^k] \end{pmatrix} \geq 0$$

- Thm 1 :

For $\omega_{(\alpha,\beta)} \in \mathbb{C}_0$, we have that for $k_1, k_2 \geq 0$

$$\tilde{\omega}_{(\alpha,\beta)} \in \mathbb{C}_0 \Leftrightarrow \alpha_{(k_1,k_2+1)}\alpha_{(k_1+1,k_2+1)} = \alpha_{(k_1+1,k_2)}\alpha_{(k_1,k_2+2)}$$



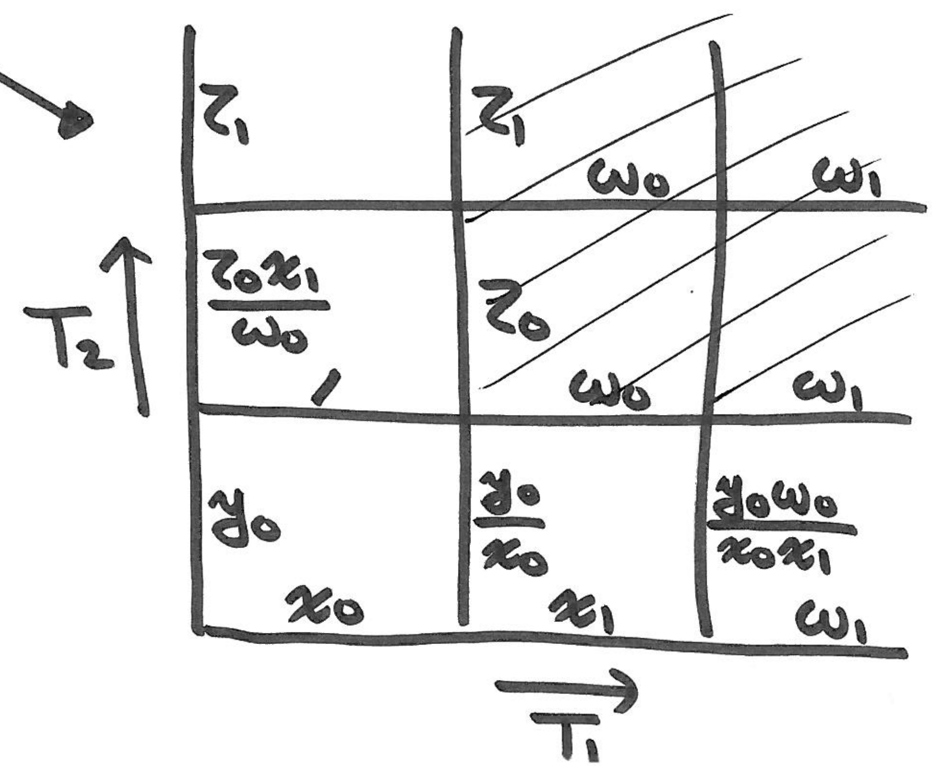
- Thm 2 :

$\exists\ \omega_{(\alpha,\beta)} \in \mathbb{C}_1$ s.t $\tilde{\omega}_{(\alpha,\beta)} \notin \mathbb{C}_1$

- Thm 3 :

$\sigma_T(\Delta) = \sigma_T(\tilde{\Delta})$, if $\omega_{(\alpha,\beta)}$ is given

by Figure 2



- A cochain complex (called the Koszul complex) for $\mathbb{T} \equiv (T_1, T_2)$, denoted by $K(\mathbb{T}, X)$ is

Banach space

$$K(\mathbb{T}, X):\ 0 \longrightarrow X \otimes \Lambda^0 \xrightarrow{D^0_{\mathbb{T}}} X \otimes \Lambda^1 \xrightarrow{D^1_{\mathbb{T}}}$$

$$\cdots \xrightarrow{D^{n-1}_{\mathbb{T}}} X \otimes \Lambda^n \xrightarrow{D^n_{\mathbb{T}}} 0\ ,$$

where $D^i_{\mathbb{T}}$ is the restriction of $D_{\mathbb{T}}$ to $X \otimes \Lambda^i$

$D_{\mathbb{T}} = \sum_{i=1}^{2} T_i \otimes E_i$

creation operator

$\Lambda^0 = \langle \underset{}{\overset{1}{\overset{\shortparallel}{e_0}}} \rangle \cong \mathbb{C}$

$\Lambda^1 = \langle e_1 \rangle \oplus \cdots \oplus \langle e_n \rangle \cong \mathbb{C}^n$

$\Lambda^2 = \langle e_1 \wedge e_2 \rangle \oplus \cdots \oplus \langle e_{n-1} \wedge e_n \rangle$

$\vdots$

- If $\mathrm{Ran}\, D^{i}_{\mathbb{T}} = \ker D^{i+1}_{\mathbb{T}}$, then the Koszul complex is said to exact.

- $\sigma_T(\mathbb{T}) := \{ (\lambda_1, \lambda_2) \in \mathbb{C}^2 \mid K((T_1 - \lambda_1, T_2 - \lambda_2), \mathcal{H})$ is not exact $\}$

[5] Open problems

- This work is only a start on the theory of the MT (resp. AT of a commuting pair) of bounded operators

① If T is log-hyponormal, is $\widehat{T}$ log-hypo.?

the MT

② For $0<p<1$, if T is p-hyponormal, does it follow that the MT $\widehat{T}$ of T is also p-hypo. ?

③ Is the mean transform map $T \longrightarrow \widehat{T}$ $(\|\cdot\|, \|\cdot\|)$ - continuous on $B(\mathcal{H})$?

That is, $\|\widehat{T_n} - \widehat{T_0}\| \xrightarrow[n\to\infty]{?} 0$

if $\|T_n - T_0\| \xrightarrow[n\to\infty]{} 0$

④ T has a nontrivial invariant subspace

$\overset{?}{\Longleftrightarrow}$ $\widehat{T}$ does

5 $\sigma_T(\omega_{(\alpha,\beta)}) \overset{?}{=} \sigma_T(\widetilde{\omega}_{(\alpha,\beta)})$

the AT of $\omega_{(\alpha,\beta)}$

6 $\sigma_T((T_1, T_2)) \overset{?}{=} \sigma_T(\widetilde{(T_1, T_2)})$

7 If $\omega_{(\alpha,\beta)}$ is subnormal, does it follow that $\widetilde{\omega}_{(\alpha,\beta)}$ is subnormal ?

8 (T_1, T_2) has a common nontrivial subspace

$\Longleftrightarrow$ $\widetilde{(T_1, T_2)}$ has a common nontrivial sub.

Thank you !!