n-particle quantum statistics on graphs

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Quantum statistics

Single particle space configuration space X.

Two particle statistics - alternative approaches:

- Quantize $X^{\times 2}$ and restrict Hilbert space to the symmetric or anti-symmetric subspace.

$$\psi(x_1, x_2) = \pm\psi(x_2, x_1) \tag{1}$$

Bose-Einstein/Fermi-Dirac statistics.

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  Bose-Einstein/Fermi-Dirac statistics.

- (Leinaas and Myrheim '77)
  Treat particles as indistinguishable, $\psi(x_1, x_2) \equiv \psi(x_2, x_1)$.
  Quantize two particle configuration space.

Definition

Configuration space of n indistinguishable particles in X,

$$C_n(X) = (X^{\times n} - \Delta_n)/S_n$$

where $\Delta_n = \{x_1, \ldots, x_n | x_i = x_j \text{ for some } i \neq j\}$.

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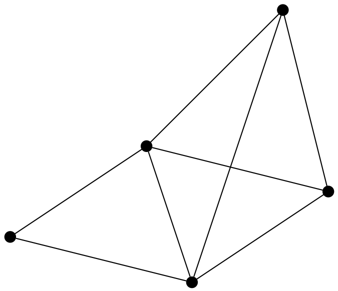
  Any single phase $\mathrm{e}^{\mathrm{i}\theta}$ can be associated to every primitive exchange path – anyon statistics.
- $H_1(C_n(\mathbb{R})) = 1$
  particles cannot be exchanged.

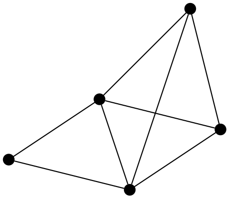
What happens on a graph where the underlying space has arbitrarily complex topology?



Graph connectivity

- Given a connected graph Γ a *k-cut* is a set of k vertices whose removal makes Γ disconnected.
- Γ is *k-connected* if the minimal cut is size k.
- **Theorem** (Menger) For a k-connected graph there exist at least k independent paths between every pair of vertices.

Example:



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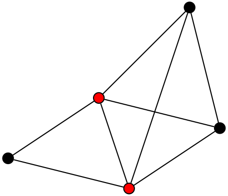

Two cut

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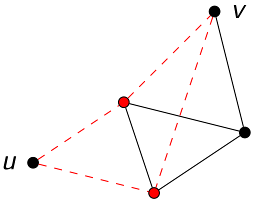

Two independent paths joining u and v.

Features of graph statistics

On 3-connected graphs statistics only depend on whether the graph is planar (Anyons) or non-planar (Bosons/Fermions).

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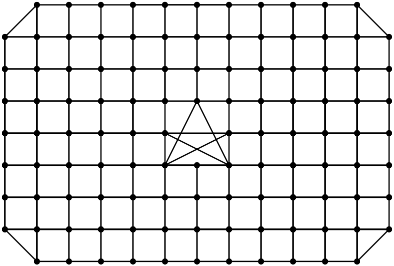

A two dimensional lattice with a small section that is non-planar is locally planar but has Bose/Fermi statistics.

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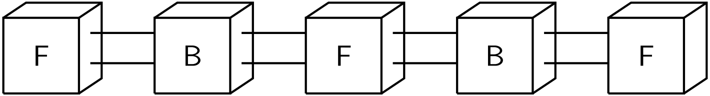

For example, one could construct a chain of 3-connected non-planar components where particles behave with alternating Bose/Fermi statistics.

On 1-connected graphs the statistics *depends* on the no. of particles n.

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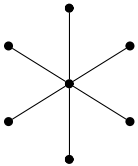
Example, star with E edges.



no. of anyon phases

$$\binom{n+E-2}{E-1}(E-2)-\binom{n+E-2}{E-2}+1\ .$$

1st homology group of graph

By the structure theorem for finitely generated modules (for a suitably subdivided graph Γ)

$$H_1(C_n(\Gamma)) = \mathbb{Z}^k \oplus \mathbb{Z}_{n_1} \oplus \ldots \oplus \mathbb{Z}_{n_l}\,, \tag{2}$$

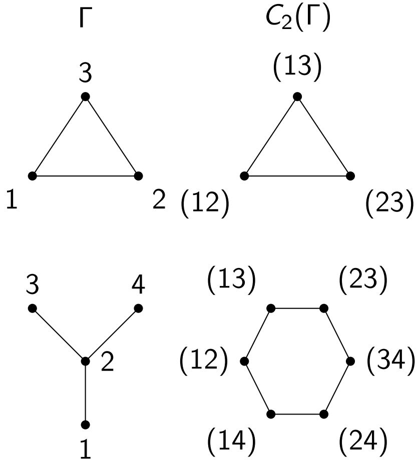
where $n_i | n_{i+1}$.

So $H_1(C_n(\Gamma))$ is determined by k free (anyon) phases $\{\phi_1, \ldots, \phi_k\}$ and l discrete phases $\{\psi_1, \ldots, \psi_l\}$ such that for each $i \in \{1, \ldots l\}$

$$n_i \psi_i = 0 \bmod 2\pi,\ \ n_i \in \mathbb{N} \ \text{ and } n_i | n_{i+1}\,. \tag{3}$$

Basic cases

For 2 particles.



Exchange of 2 particles around loop c; one free phase ϕ_{c2}.

Exchange of 2 particles at Y-junction; one free phase ϕ_Y.

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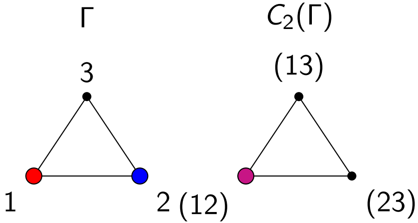

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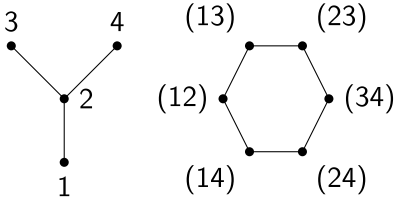

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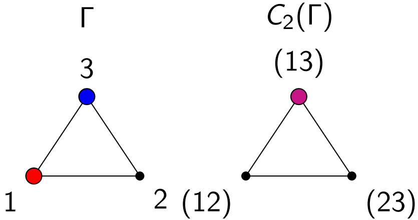

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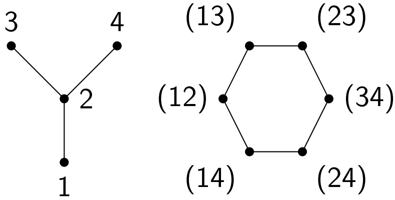

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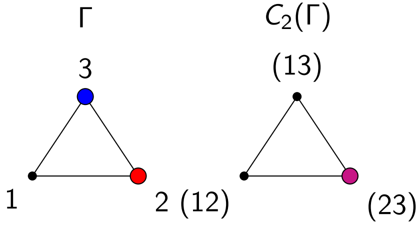

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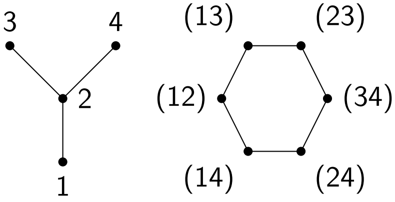

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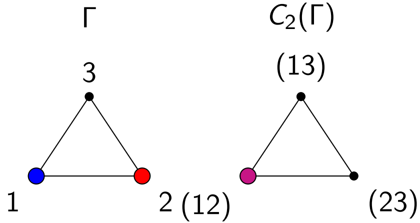

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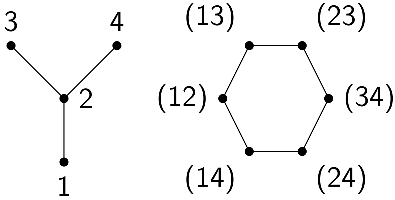

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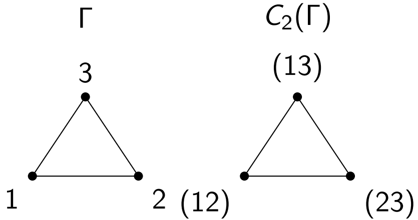

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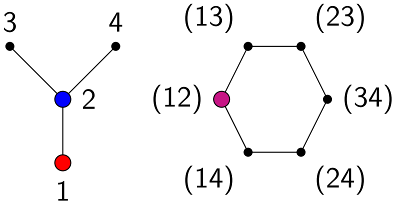

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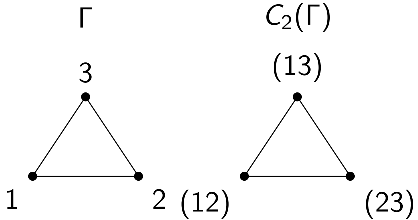

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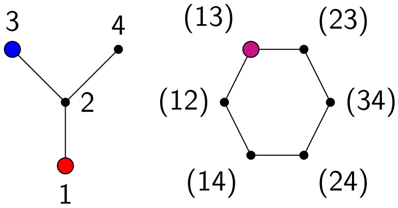

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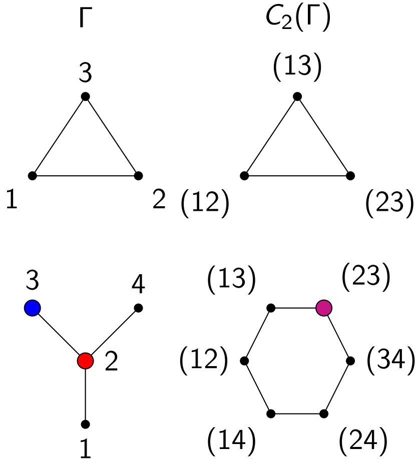

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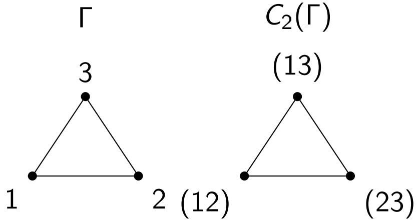

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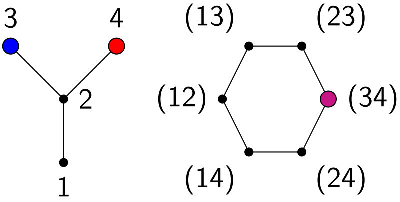

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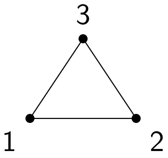
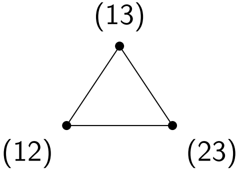


$C_2(\Gamma)$



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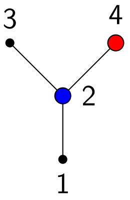

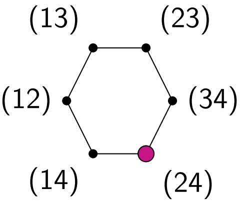

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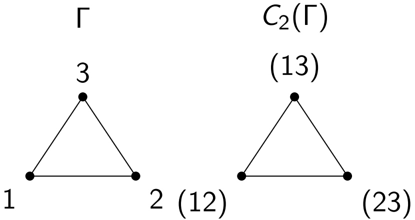

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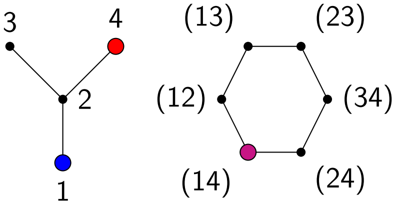

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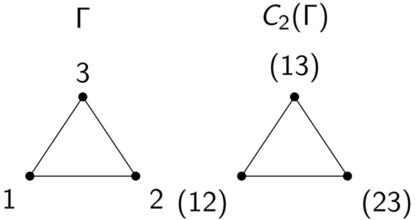

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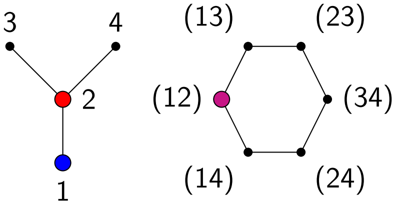

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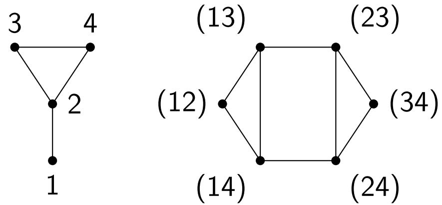
Lasso graph



Identify three 2-particle cycles:

(i) Rotate both particles around loop c; phase $\phi_{c,2}$.

(ii) Exchange particles on Y-subgraph; phase ϕ_Y.

(iii) Rotate one particle around loop c other particle at vertex 1; $(1,2) \to (1,3) \to (1,4) \to (1,2)$, phase $\phi^1_{c,1}$.

Relation from contactable 2-cell $\phi_{c,2} = \phi^1_{c,1} + \phi_Y$.

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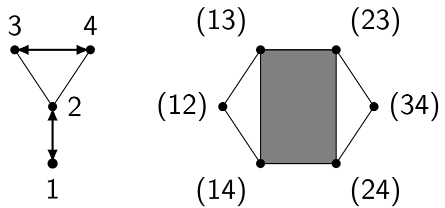

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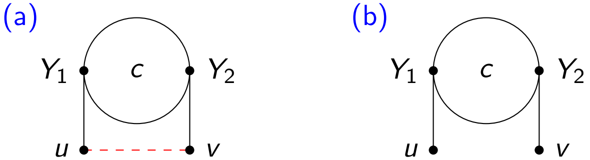
Let c be a loop. What is the relation between $\phi^{u}_{c,1}$ and $\phi^{v}_{c,1}$?

(a) u and v joined by path disjoint with c.
$\phi^{u}_{c,1} = \phi^{v}_{c,1}$ as exchange cycles homotopy equivalent.

(b) u and v *only* joined by paths through c.
Two lasso graphs so $\phi_{c,2} = \phi^{u}_{c,1} + \phi_{Y_1}$ & $\phi_{c,2} = \phi^{v}_{c,1} + \phi_{Y_2}$.
Hence $\phi^{u}_{c,1} - \phi^{v}_{c,1} = \phi_{Y_2} - \phi_{Y_1}$.



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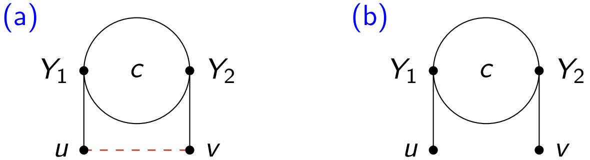
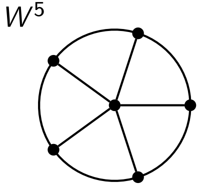


- Relations between phases involving c encoded in phases ϕ_Y.
$H_1(C_2(\Gamma)) = \mathbb{Z}^{\beta_1(\Gamma)} \oplus A$, where A determined by Y-cycles.
- In (a) we have a $\mathcal{B}$ subgraph & using (b) also $\phi_{Y_1} = \phi_{Y_2}$.

3-connected graphs

The prototypical 3-connected graph is a *wheel* W^k.



Theorem (Wheel theorem)

Let Γ be a simple 3-connected graph different from a wheel. Then for some edge $e \in \Gamma$ either $\Gamma \setminus e$ or Γ / e is simple and 3-connected.

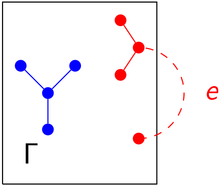
- $\Gamma \setminus e$ is Γ with the edge e removed.
- Γ / e is Γ with e contracted to identify its vertices.

Lemma

For 3-connected simple graphs all phases ϕ_Y are equal up to a sign.

Sketch proof. The lemma holds on K_4 (minimal wheel). By wheel thm we only need to show that adding an edge or expanding a vertex any new phases ϕ_Y are the same as the original phase.

Adding an edge: $\Gamma \cup e$



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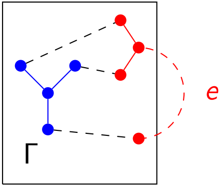

Using 3-connectedness identify independent paths in Γ to make $\mathcal{B}$. Then $\phi_Y = \phi_Y$.

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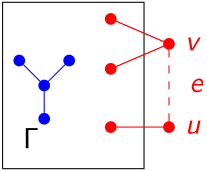
Vertex expansion: Split vertex of degree > 3 into two vertices u and v joined by a new edge e.



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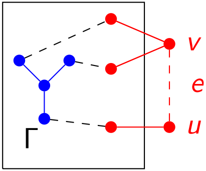

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Theorem

For a 3-connected simple graph, $H_1(C_2(\Gamma)) = \mathbb{Z}^{\beta_1(\Gamma)} \oplus A$*, where* $A = \mathbb{Z}_2$ *for non-planar graphs and* $A = \mathbb{Z}$ *for planar graphs.*

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Proof.

- For K_5 and $K_{3,3}$ every phase $\phi_Y = 0$ or π. By Kuratowski's theorem a non-planar graph contains a subgraph which is isomorphic to K_5 or $K_{3,3}$.

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Proof.

- For K_5 and $K_{3,3}$ every phase $\phi_Y = 0$ or π. By Kuratowski's theorem a non-planar graph contains a subgraph which is isomorphic to K_5 or $K_{3,3}$.
- For planar graphs the anyon phase can be introduced by drawing the graph in the plane and integrating the anyon vector potential $\frac{\alpha}{2\pi}\hat{z} \times \frac{r_1 - r_2}{|r_1 - r_2|^2}$ along the edges of the two-particle graph, where r_1 and r_2 are the positions of the particles.

Summary

- Full classification of abelian quantum statistics on graphs by decomposing graph in 1-, 2- and 3-connected components.
- Physical insight into dependance of statistics on graph connectivity.
- Interesting new features of graph statistics.
- Statistics incorporated in gauge potential.

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