Low Regularity Uniqueness of Solutions to the Gross-Pitaevskii Hierarchy

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(based on joint works with Y. Hong and Z. Xie)

November 22, 2014

The Gross-Pitaevskii Hierarchy

- The cubic GP hierarchy is a infinite sequence of coupled partial differential equations for marginal density matrices $\gamma^{(k)} : \mathbb{R} \times \mathbb{R}^{dk} \times \mathbb{R}^{dk} \to \mathbb{C}$, with $k \in \mathbb{N}$, given by

$$i\partial_t \gamma^{(k)}(t, \underline{x}_k, \underline{x}'_k) = \sum_{j=1}^{k} (-\Delta_{x_j} + \Delta_{x'_j}) \gamma^{(k)}(t, x_k, x'_k)$$
$$+ \sum_{j=1}^{k} \int (\delta(x_j - x_{k+1}) - \delta(x'_j - x_{k+1}))$$
$$\times \gamma^{(k+1)}(t, \underline{x}_k, x_{k+1} \underline{x}'_k, x_{k+1})\, dx_{k+1}$$
$$\gamma^{(k)}(0, \underline{x}_k, \underline{x}'_k) = \gamma_0^{(k)}(\underline{x}_k, \underline{x}'_k).$$

- The cubic GP hierarchy arises in the derivation of the cubic nonlinear Schrödinger equation (NLS) from the N-body Schrödinger equation. Proving uniqueness of solutions to the GP hierarchy is a key step is the derivation process.

Equivalence of the GP Hierarchy to the NLS

- It is straightforward to show that if $\phi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ satisfies the cubic NLS

$$i\partial_t \phi = -\Delta \phi + |\phi|^2 \phi,$$

with initial data $\phi|_{t=0} = \phi_0$, then

$$\gamma^{(k)}(t, \underline{x}_k, \underline{x}'_k) := \prod_{j=1}^{k} \phi(t, x_j) \overline{\phi(t, x'_j)}$$

satisfies the cubic GP hierarchy with initial data

$$\gamma_0^{(k)}(\underline{x}_k, \underline{x}'_k) := \prod_{j=1}^{k} \phi_0(x_j) \overline{\phi_0(x'_j)}. \tag{1}$$

- In this sense, the GP hierarchy is a generalization of the NLS.
- Instead of (1), we will from now on write

$$\gamma_0^{(k)} := (|\phi_0\rangle\langle\phi_0|)^{\otimes k}.$$

Sobolev Spaces for Marginal Density Matrices

- For $\alpha \geq 0$, We define $\mathfrak{H}^\alpha$ by

$$\mathfrak{H}^\alpha := \big\{ (\gamma^{(k)})_{k\in\mathbb{N}} \big| \mathrm{Tr}(|S^{(k,\alpha)}\gamma^{(k)}|) < M^{2k} \text{ for some constant } M < \infty \big\}$$

where

$$S^{(k,\alpha)} := \prod_{j=1}^{k} (1-\Delta_{x_j})^{\alpha/2}(1-\Delta_{x_j'})^{\alpha/2}.$$

- Observe that in the factorized case $\gamma^{(k)} = |\phi\rangle\langle\phi|^{\otimes k}$, we have

$$\mathrm{Tr}(|S^{(k,\alpha)}\gamma^{(k)}|) = \|\phi\|_{H^\alpha}^{2k} < M^{2k}$$

iff $\|\phi\|_{H^\alpha} < M$.

The Quantum de Finetti Theorem

Due to Hudson-Moody (1976/77), Stormer (1969), Lewin-Nam-Rougerie (2013).

Theorem

Let $\mathcal{H}$ be any separable Hilbert space, and let $\mathcal{H}^k = \otimes_{sym}^k \mathcal{H}$ denote the corresponding bosonic k-particle space. Assume that γ_N^N is an arbitrary sequence of mixed states on $\mathcal{H}^N$, $N \in \mathbb{N}$, satisfying $\mathrm{Tr}_{\mathcal{H}^N}(\gamma_N^{(N)}) = 1$, and assume that its k-particle marginals have weak- limits*

$$\gamma_N^{(k)} := \mathrm{Tr}_{k+1,\dots,N}(\gamma_N^{(N)}) \rightharpoonup \gamma^{(k)} \text{ as } N \to \infty$$

in trace class on $\mathcal{H}^k$ for all $k \geq 1$. Then, there exists a unique Borel probability measure μ on the unit ball $\mathcal{B} \in \mathcal{H}$, invariant under multiplication of $\phi \in \mathcal{H}$ by complex numbers of modulus one, such that

$$\gamma^k = \int d\mu(\phi)(|\phi\rangle\langle\phi|)^{\otimes k} \quad , \quad \forall k \in \mathbb{N}.$$

Consequences of the Quantum de Finetti Theorem

- Given existence of solutions to the NLS, existence of solutions to the GP hierarchy follows immediately from the quantum de Finetti theorem, given that the initial data comes from a weak star limit. Given initial data

$$\gamma^k(0) = \int d\mu(\phi)(|\phi\rangle\langle\phi|)^{\otimes k},$$

  a solution to the GP hierarchy is given by

$$\gamma^k(t) = \int d\mu(\phi)(|S_t\phi\rangle\langle S_t\phi|)^{\otimes k},$$

  where S_t is the NLS time evolution operator.
- Does the quantum de Finetti theorem help us prove uniqueness of solutions to the GP hierarchy?

Low Reqularity Uniqueness Theorem Statement

Theorem

Let

$$\begin{cases} s \geq \frac{d}{6} & \text{if } d = 1, 2, \\ s > s_c & \text{if } d \geq 3, \end{cases}$$

where $s_c = \frac{d-2}{2}$. If $\{\gamma^{(k)}(t)\}_{k\in\mathbb{N}}$ is a mild solution in $L^\infty_{t\in[0,T)}\mathfrak{H}^s$ to the cubic GP hierarchy with initial data $\{\gamma^{(k)}(0)\}_{k\in\mathbb{N}}$, which is either admissible or a limiting hierarchy for each t, then it is the only such solution for the given initial data.

Uniqueness Proof Outline

1. The GP hierarchy is linear, so we will use the fact that the difference between two solutions γ_1 and γ_2 with the same initial data is a solution $\gamma = \gamma_1 - \gamma_2$ with zero initial data.
2. We use the Duhamel integral formula n times to expand the solution $\gamma^{(k)}$.
3. We use a combinatorial argument of Erdös, Schlein, and Yau to control the number of terms in our expansion.
4. We express our solution as an integral of factorized states, using the quantum de Finetti theorem.
5. We prove estimates on each factor to show that as the number of Duhamel interations n goes to infinity, our bound on the solution γ goes to zero, and so γ is the zero solution.

The Duhamel Expansion

- First, we define

$$U^{(k)}(t) := e^{-it(\sum_{j=1}^{k}(-\Delta_{x_j}+\Delta_{x_j'}))}$$

$$B^{+}_{j,k+1}\gamma^{(k+1)}(t,\underline{x}_k,\underline{x}'_k) := \int \delta(x_j - x_{k+1}) \times \gamma^{(k+1)}(t,\underline{x}_k,x_{k+1}\underline{x}'_k,x_{k+1})\,dx_{k+1}$$

$$B^{-}_{j,k+1}\gamma^{(k+1)}(t,\underline{x}_k,\underline{x}'_k) := \int \delta(x'_j - x_{k+1}) \times \gamma^{(k+1)}(t,\underline{x}_k,x_{k+1}\underline{x}'_k,x_{k+1})\,dx_{k+1}$$

$$B_{k+1} := \sum_{j=1}^{k} B^{+}_{j,k+1} - B^{-}_{j,k+1}.$$

- Then the k^{th} equation in the GP hierarchy takes the form

$$i\partial_t\gamma^{(k)} = \sum_{j=1}^{k}(-\Delta_{x_j}+\Delta_{x_j'})\gamma^{(k)} + B_{k+1}\gamma^{(k+1)}$$

The Duhamel Expansion

- Then, after two iterations of the Duhamel formula, we have

$$\begin{aligned}\gamma^{(k)}(t) &= \int_0^t U^{(k)}(t-t_1)B_{k+1}\gamma^{(k+1)}(t_1)\,dt_1 \\ &= \int_0^t U(t-t_1)B_{k+1}\int_0^{t_1} U^{(k+1)}(t_2-t_1)B_{k+2}\gamma^{(k+2)}(t_2)\,dt_1\,dt_2.\end{aligned}$$

- Since B_ℓ has $2\ell-2$ terms, n applications of the Duhamel formula yields a number of terms of order $n!$ in n.
- Klainerman and Machedon's formulation of board game argument of Erdös, Schlein, and Yau allows us to group these terms into approximately C^n many groups.

Application of the Quantum de Finetti Theorem

- Next, we express the last term using the quantum de Finetti theorem, and find that

$$\begin{aligned}\gamma^{(k)}(t) &= \int_0^t U^{(k)}(t-t_1)B_{k+1}\gamma^{(k+1)}(t_1)\,dt_1 \\ &= \int_0^t U(t-t_1)B_{k+1}\int_0^{t_1} U^{(k+1)}(t_2-t_1)B_{k+2} \\ &\qquad\qquad \times \int d\tilde{\mu}_{t_2}(\phi)(|\phi\rangle\langle\phi|)^{\otimes k+2}\,dt_1\,dt_2,\end{aligned}$$

  where $\tilde{\mu}_{t_2}$ is the difference between the two measures associted to the two solutions γ_1 and γ_2.
- The product $(|\phi\rangle\langle\phi|)^{\otimes k+2}$ has $k+2$ factors. Since each contraction operator B_ℓ contracts two factors, we end up with a product of k one-particle density matrices.
- Similarly, if we apply the Duhamel formula n times, we end up with a product of k one-particle density matrices. This was a key observation by Chen-Hainzl-Pavlović-Seiringer.
- To keep track of these factors, we the binary tree graph organizational structure developed by CHPS, which, in the interest of time, we omit in this presentation.

Key Estimates used by CHPS

- The next step in the proof of uniqueness is to find a bound on each factor so that the product goes to zero as n goes to infinity.
- In Chen-Hainzl-Pavlović-Seiringer, the authors work in $L_t^\infty \mathfrak{H}^1$. To bound each factor, they use the estimates

$$\begin{aligned} &\|(e^{it\Delta} f_1)(x)\overline{(e^{it\Delta} f_2)(x)}(e^{it\Delta} f_3)(x)\|_{L_t^2(\mathbb{R})L_x^2(\mathbb{R}^3)} \\ &\leq C\|f_1\|_{H_x^1}\|f_2\|_{H_x^1}\|f_3\|_{L_x^2} \end{aligned}$$

and

$$\begin{aligned} &\|\nabla_x((e^{it\Delta} f_1)(x)\overline{(e^{it\Delta} f_2)(x)}(e^{it\Delta} f_3)(x))\|_{L_t^2(\mathbb{R})L_x^2(\mathbb{R}^3)} \\ &\leq C\|f_1\|_{H_x^1}\|f_2\|_{H_x^1}\|f_3\|_{H_x^1}, \end{aligned}$$

which are established via Strichartz estimates and Sobolev embedding.

Our Key Estimates

- We work in $L^\infty_t \mathfrak{H}^s$, where $s \geq \frac{d}{6}$ if $d = 1, 2$ and $s > s_c = \frac{d-2}{2}$ if $d \geq 3$.
- To push the regularity s down to this level, we needed different estimates.
- Since we are only proving uniquness, it is sufficient to show that the $L^\infty_t \mathfrak{H}^{-d}$ norm of the solution with zero initial data is zero, even though $s > -d$.
- We also make use of dispersive estimates, and prove that, for $d \geq 3$ and $\epsilon > 0$,

$$\begin{aligned}&\|(e^{it\Delta}f_1)(x)\overline{(e^{it\Delta}f_2)(x)}(e^{it\Delta}f_3)(x)\|_{L^1_{t\in[0,T)}W_x^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} \\ &\lesssim T^\epsilon \|f_1\|_{W_x^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} \|f_2\|_{H^{s_\epsilon}} \|f_3\|_{H^{s_\epsilon}}\end{aligned}$$

and

$$\begin{aligned}&\|(e^{it\Delta}f_1)(x)\overline{(e^{it\Delta}f_2)(x)}(e^{it\Delta}f_3)(x)\|_{L^1_{t\in[0,T)}H_x^{s_\epsilon}} \\ &\lesssim T^\epsilon \|f_1\|_{H^{s_\epsilon}} \|f_2\|_{H^{s_\epsilon}} \|f_3\|_{H^{s_\epsilon}},\end{aligned}$$

where $s_\epsilon = s_c + \epsilon = \frac{d-2}{2} + \epsilon$ and $r_\epsilon = \frac{2d}{d+2(1-\epsilon)}$.

Application of the Key Estimates

- Note that to achieve lower regularity, we work in $W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}$, where CHPS work in L^2.
- In $W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}$, the linear propagators $e^{it\Delta}$ in the Duhamel expansion are no longer isometries, so we have to carefully rearrange them in our implementation of the binary tree graph organizational structure used by CHPS.
- After bounding all factors in the expression with n interations of the Duhamel formula, we find that

$$\mathrm{Tr}(|S^{(k,-d)}\gamma^{(k)}(t)|) \leq \frac{M^{2k}2^{k-1}T}{CT^\epsilon}(4CT^\epsilon M^2)^n$$
$$\to 0 \text{ as } n \to \infty$$

  for $T < (4CM^2)^{-1/\epsilon}$, where we recall that, by definition, since $\gamma \in \mathfrak{H}^\alpha$, we have that $0 \leq M < \infty$.
- Thus $\gamma^{(k)}(t) = 0$ for $t \in [0, T]$, and so we have established uniqueness of solutions to the cubic GP hiearchy in $L_t^\infty \mathfrak{H}^\alpha$.

Summary of the Uniqueness Proof

1. The GP hierarchy is linear, so we began with the fact that the difference between two solutions γ_1 and γ_2 with the same initial data is a solution $\gamma = \gamma_1 - \gamma_2$ with zero initial data.
2. We used the Duhamel integral formula n times to expand the solution $\gamma^{(k)}$.
3. We used a combinatorial argument of Erdös, Schlein, and Yau to control the number of terms in our expansion from roughly $n!$ to C^n.
4. We expressed our solution as an integral of factorized states, using the quantum de Finetti theorem.
5. We proved estimates on each factor to show that as the number of Duhamel interations n goes to infinity, our bound on the solution γ goes to zero, and so γ is the zero solution.

Summary of How We Lowered the Regularity in Step 5

- We used a negative order Sobolev space.
  - This is sufficient for proving uniqueness, since we only need to show that $\gamma = 0$.
- We used dispersive estimates instead of Strichartz estimates.
  - Strichartz estimates are too strong, since they provide a bound on an integral over all $t \in \mathbb{R}$. We only need an estimate local in time.
- The dispersive estimates put put us in an L^p space where $p \neq 2$, where the linear propagator $e^{it\Delta}$ is not an isometry. We had to rearrange the linear propagators in our proof.

Uniqueness in the Quintic Case

- One can also consider the *quintic* GP hierarchy.
- In the quintic case, low regularity uniquness is especially important because it is necessary in order to derive the quintic NLS from the N-body Schrödinger equation with three-body interactions.
- Since current bounds on solutions to the N-body Schrödinger equation with three-body interactions are with respect to the $\mathfrak{H}^1$ norm, it is necessary to prove uniqueness of solutions to the quintic GP hierarchy in $\mathfrak{H}^1$, where it is scaling critical.
- By extending the binary tree graph argument to an argument that uses trees with three branches at each vertex, it is possible to prove uniqueness of small solutions to the quintic GP hierarchy in $L^\infty_t \mathfrak{H}^1$ (Hong, Taliaferro, Xie, 2014).

Thank You!

References

1. T. Chen, C. Hainzl, N. Pavlović, and R. Seiringer. Unconditional Uniqueness for the Cubic Gross-Pitaevskii Hierarchy via Quantum de Finetti. *Comm. Pure Appl. Math. To appear*, 2014.
2. L. Erdös, B. Schlein, and H.T. Yau. Derivation of the cubic non-linear Schrödinger equation from quantum dynamics of many-body systems. *Invent. Math. 167, no. 3, 515-614,* 2007.
3. Y. Hong, K. Taliaferro, and X. Xie. Unconditional Uniqueness of the Cubic Gross-Pitaevskii Hierarchy with Low Regularity. *Submitted*, 2014.
4. Y. Hong, K. Taliaferro, and X. Xie. Uniqueness of Solutions to the 3D Quintic Gross-Pitaevskii Hierarchy. *Preprint*, 2014.
5. S. Klainerman and M. Machedon. On the uniqueness of solutions to the Gross-Pitaevskii hierarchy. *Comm. Math. Phys., 279(1):169-185*, 2008.