Invariant measures and the soliton resolution conjecture

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The focusing nonlinear Schrödinger equation

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- The NLS is one of the most widely studied nonlinear dispersive equations. Has many applications.

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- Often, the function v is also called a soliton.

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- In certain situations, one needs to impose the additional condition that the solution does not blow up.
- Partially solved when $d = 1$ and $p = 3$, where the NLS is completely integrable. In higher dimensions, some progress in recent years. (See works of Kenig, Merle, Schlag, Tao,....)
- It is generally believed that proving a precise statement is "far out of the reach of current technology". See e.g. Terry Tao's blog entry on this topic, or Avy Soffer's ICM lecture notes.

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- That is, if $u(x,t)$ is a solution of the NLS, then $M(u(\cdot,t))$ and $H(u(\cdot,t))$ remain constant over time.

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  - In statistical physics parlance, this is the Canonical Ensemble.

Making sense of the Canonical Ensemble

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- Significant recent progress on canonical invariant measures for the NLS and other equations by many authors (Burq, Tzvetkov, Oh, Staffilani, Bulut,Thomann, Nahmod....).
- However, all in all, not much is known in $d \geq 3$. In fact, it is possible that the idea does not work at all in $d \geq 3$.

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- Instead of considering the Canonical Ensemble of Lebowitz, Rose & Speer, one may alternatively consider the Microcanonical Ensemble.
- The microcanonical ensemble, in this context, is the restriction of our fictitious Lebesgue measure on function space to the manifold of functions satisfying $M(u) = m$ and $H(u) = E$, where m and E are given.

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- Some physicists have briefly investigated this approach, with inconclusive results.
- A preliminary attempt was made in C. & Kirkpatrick (2010). Could not pass to the continuum limit.
- In recent work, I proved that it is indeed possible to take the discretized microcanonical ensemble to a continuum limit in such a way that very conclusive results can drawn about it in all dimensions. This is the topic of this talk.

Equivalence classes

- If u satisfies $M(u) = m$ and $H(u) = E$, so does the function

$$v(x) := \alpha_0 u(x + x_0)$$

for any $x_0 \in \mathbb{R}^d$ and $\alpha_0 \in \mathbb{C}$ with $|\alpha_0| = 1$.

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- Thus, it is reasonable to first quotient the function space by the equivalence relation $\sim$, where $u \sim v$ means that u and v are related in the above manner.
- We will generally talk about functions and equivalence classes as the same thing.

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  - For each $m > 0$, there is a unique $\lambda(m) > 0$ such that $Q_{\lambda(m)}$ is the ground state soliton of mass m.

Main result

Theorem (C., 2012; rough statement)

Suppose that $p < 1 + 4/d$, and that E is a real number bigger than the ground state energy at a given mass m. If we attempt to choose a function uniformly at random from all functions satisfying $M(u) = m$ and $H(u) = E$, by first discretizing the problem and then passing to the infinite volume continuum limit, then the resulting sequence of discrete random functions (equivalence classes) converges in the L^∞ norm to the ground state soliton of mass m.

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- The situation is similar to the dynamical version of the soliton resolution conjecture: If initial data has mass m and energy E, then it has the same mass and energy at all times, but looks more and more like the ground state soliton as $t \to \infty$.

How to discretize?

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$$H(u) := \frac{h^d}{2} \sum_{\substack{x,y \in V_n \\ |x-y|=1}} \left| \frac{u(x) - u(y)}{h} \right|^2 - \frac{h^d}{p+1} \sum_{x \in V_n} |u(x)|^{p+1}.$$

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$$S_{\epsilon,h,n}(E,m) := \{u \in \mathbb{C}^{V_n} : |M(u) - m| \leq \epsilon, |H(u) - E| \leq \epsilon\}.$$

- Let f be a random function chosen uniformly from the finite volume set $S_{\epsilon,h,n}(E,m)$.
- Extend f to a step function $\tilde{f}$ on $\mathbb{R}^d$ in the natural way.
- There are three discretization parameters involved here:
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- The main theorem says that the equivalence class corresponding to this random function $\tilde{f}$ converges to the ground state soliton of mass m if (ϵ, h, nh) is taken to $(0, 0, \infty)$ in an appropriate manner.

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  - In this statistical sense, the theorem resolves SRC.

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- Note that $f \approx f^{\text{large}}$ in the L^∞ norm.

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- Given any $m' \in [0, m]$ and $E' \in [0, E]$, let $A(m', E')$ be the set of all f with mass m and energy E, such that mass of f^{large} is m' and energy of f^{large} is E'.

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- Suppose that the size of the largest of these sets overwhelmingly dominates the rest. Let (m^*, E^*) be the pair where this maximum is attained. Then $|A| \approx |A(m^*, E^*)|$, which means that if a function f is chosen uniformly from A, then with high probability f^{large} has mass m^* and energy E^*.

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- Estimating the sizes of $A(m', E')$ via large deviation calculations takes a 50-page chunk of the paper. At the end, it turns out that the above picture is indeed correct, and the pair (m^*, E^*) satisfies the condition that $E^* =$ the ground state energy at mass m^*.

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- These smoothness estimates are used, in conjunction with the stability of the continuum ground state soliton, to complete the argument.