

Integrable probability and KPZ universality class

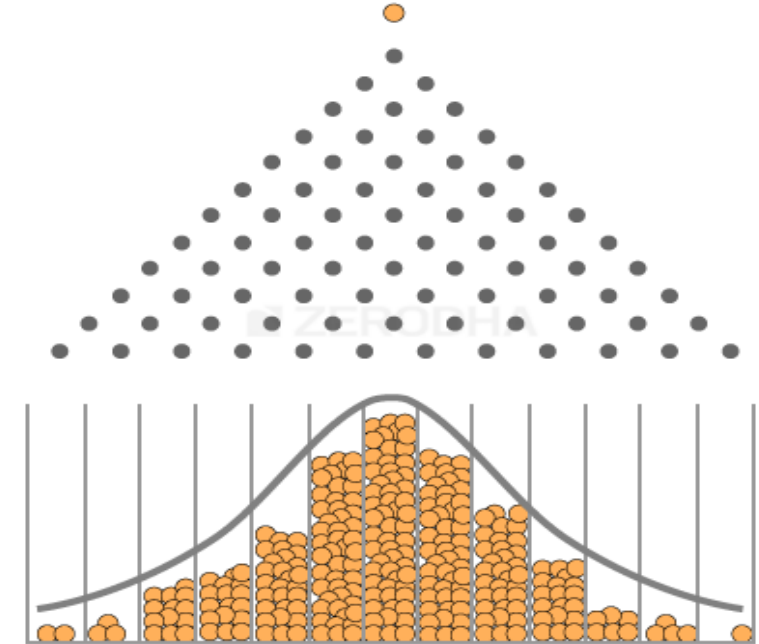
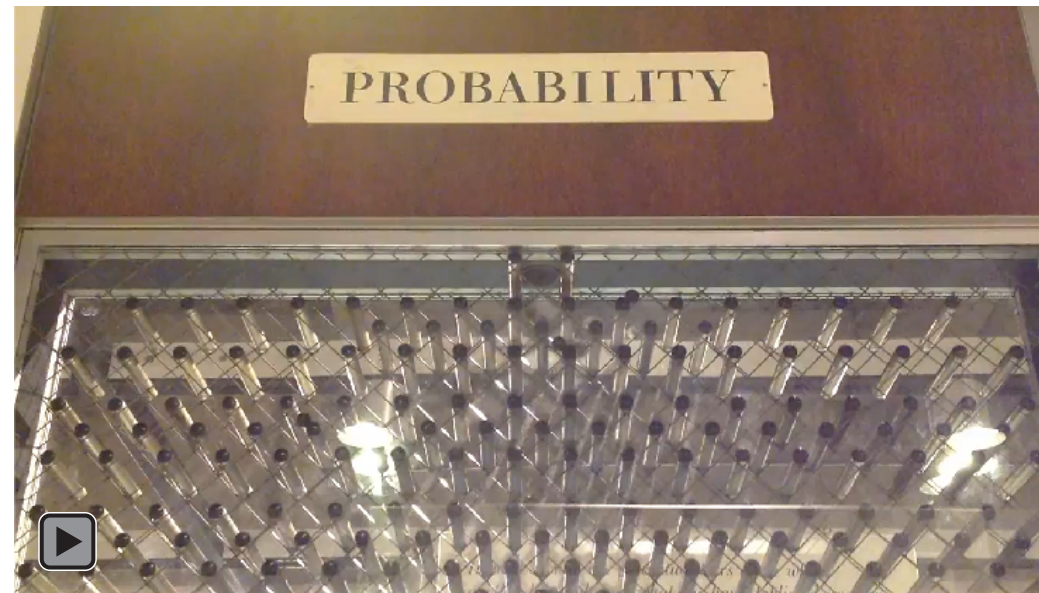
Ivan Corwin (Clay Mathematics Institute, Columbia University)

Classical probability distributions / statistics

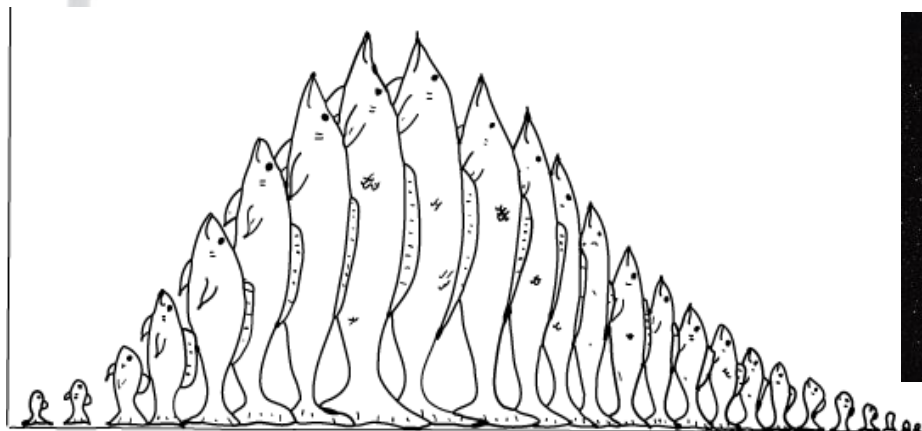
Each distribution has its own universality class



Gaussian



Poisson

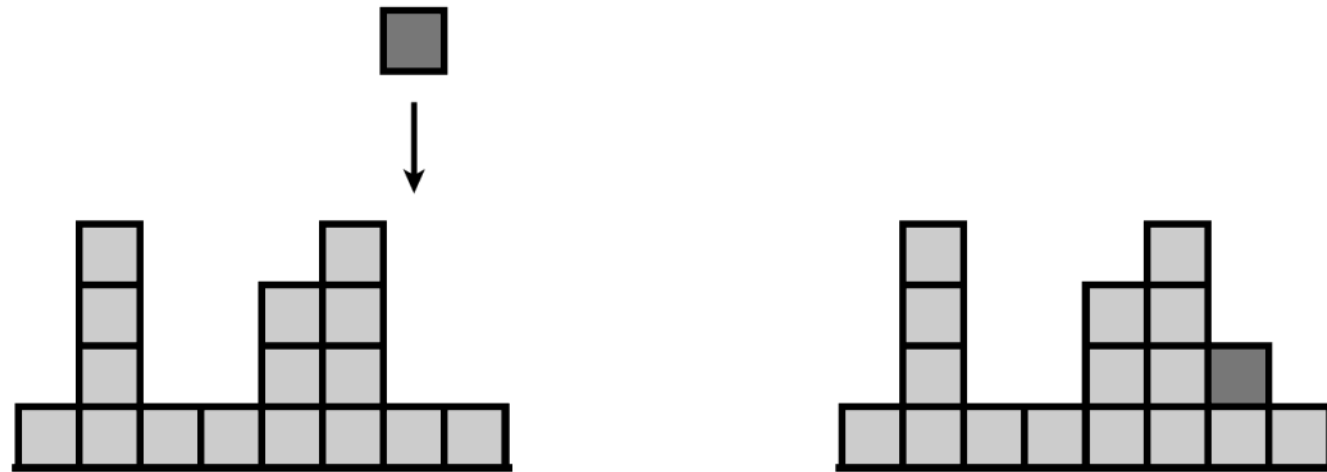


Extreme value



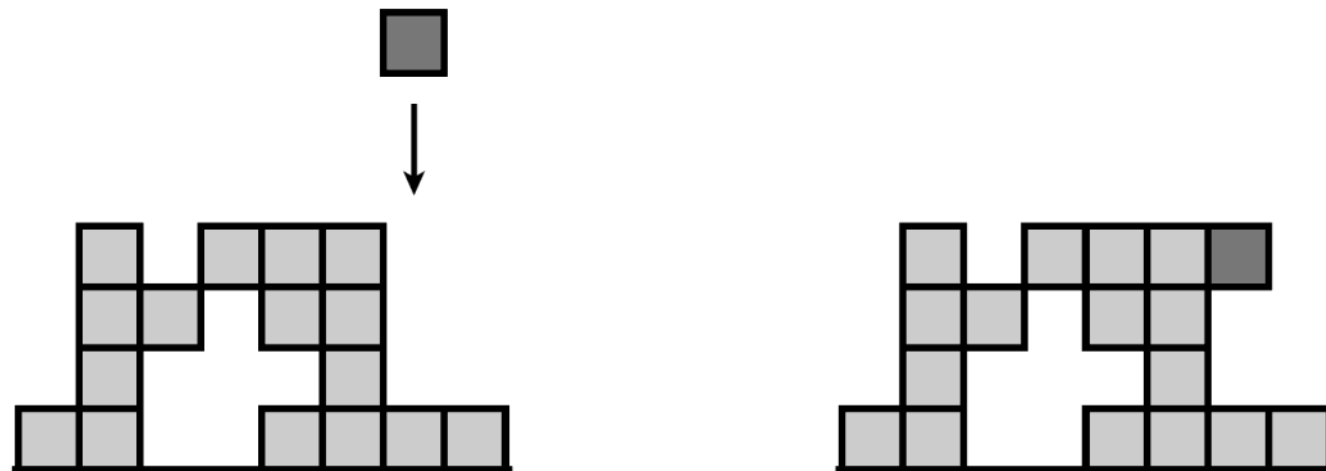
Random growth processes

Blocks fall at random in time / space.

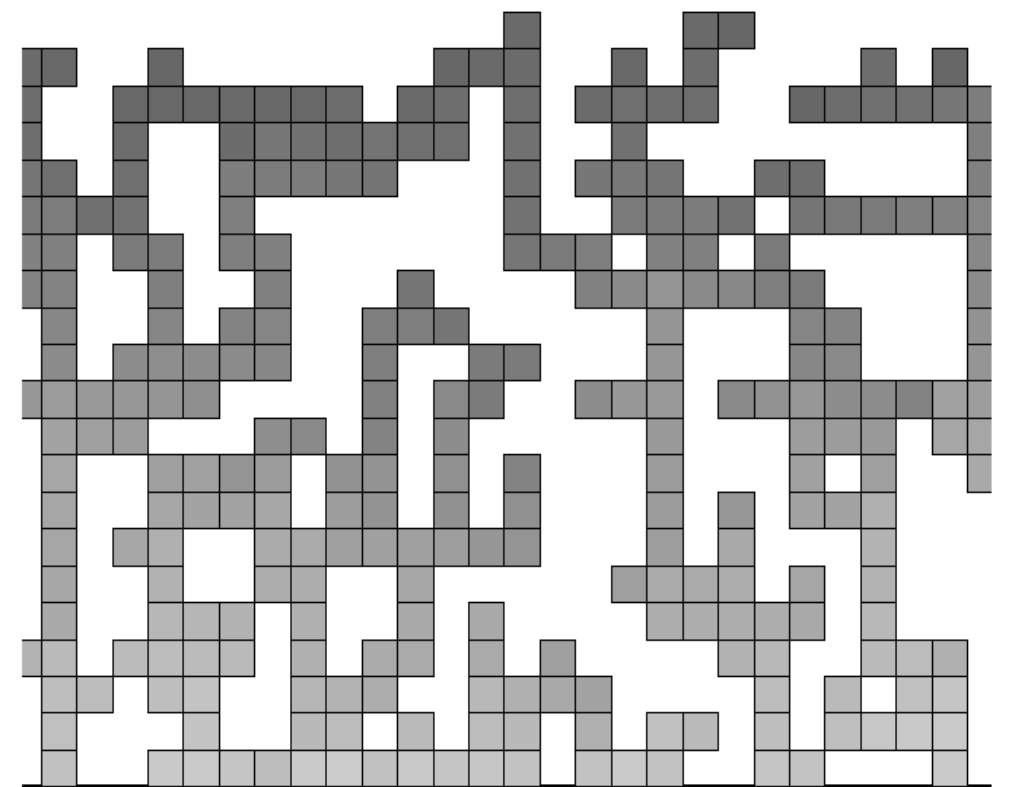
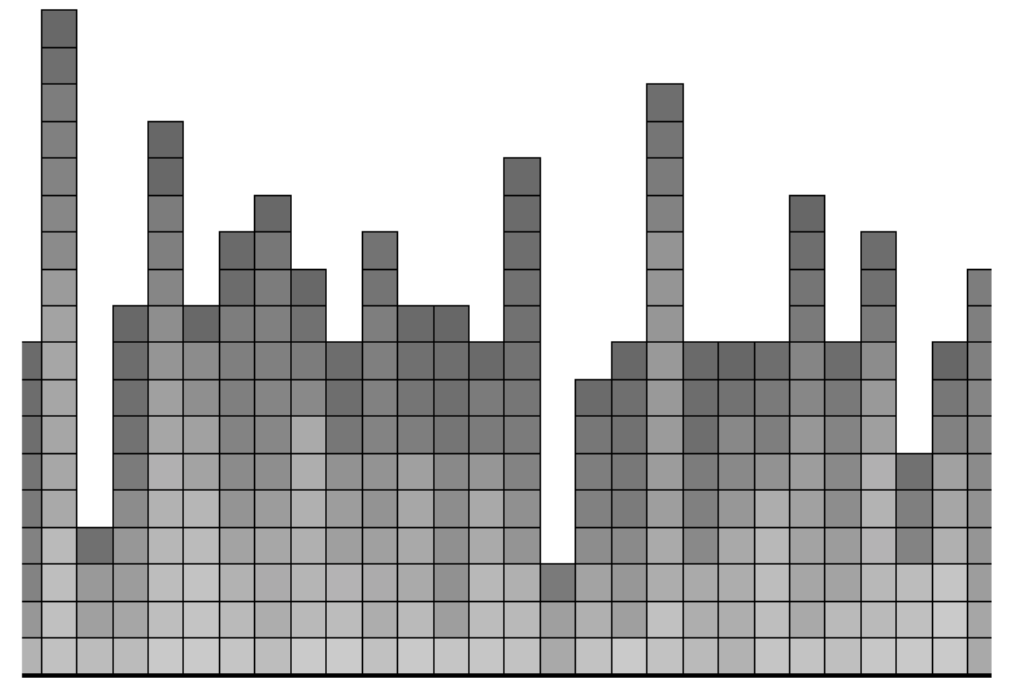


✧ Above: Fall to top of column.

✧ Below: Stick to first incident edge.



After a while we see:



Simulation

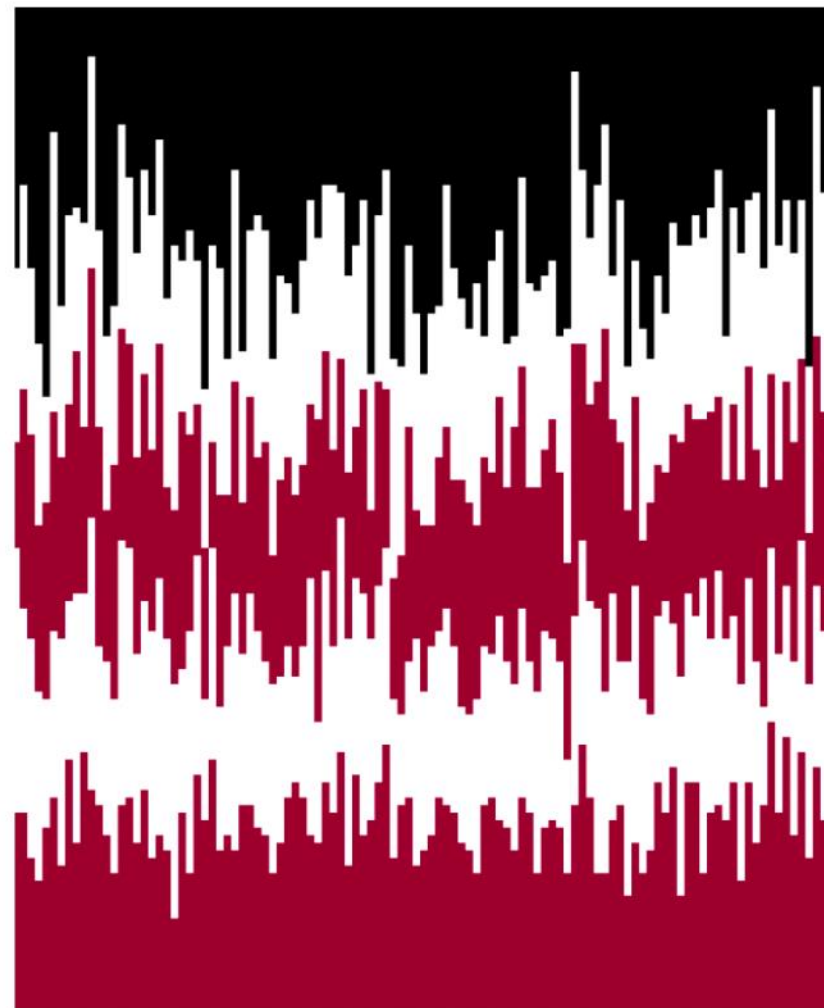


Long-time behavior

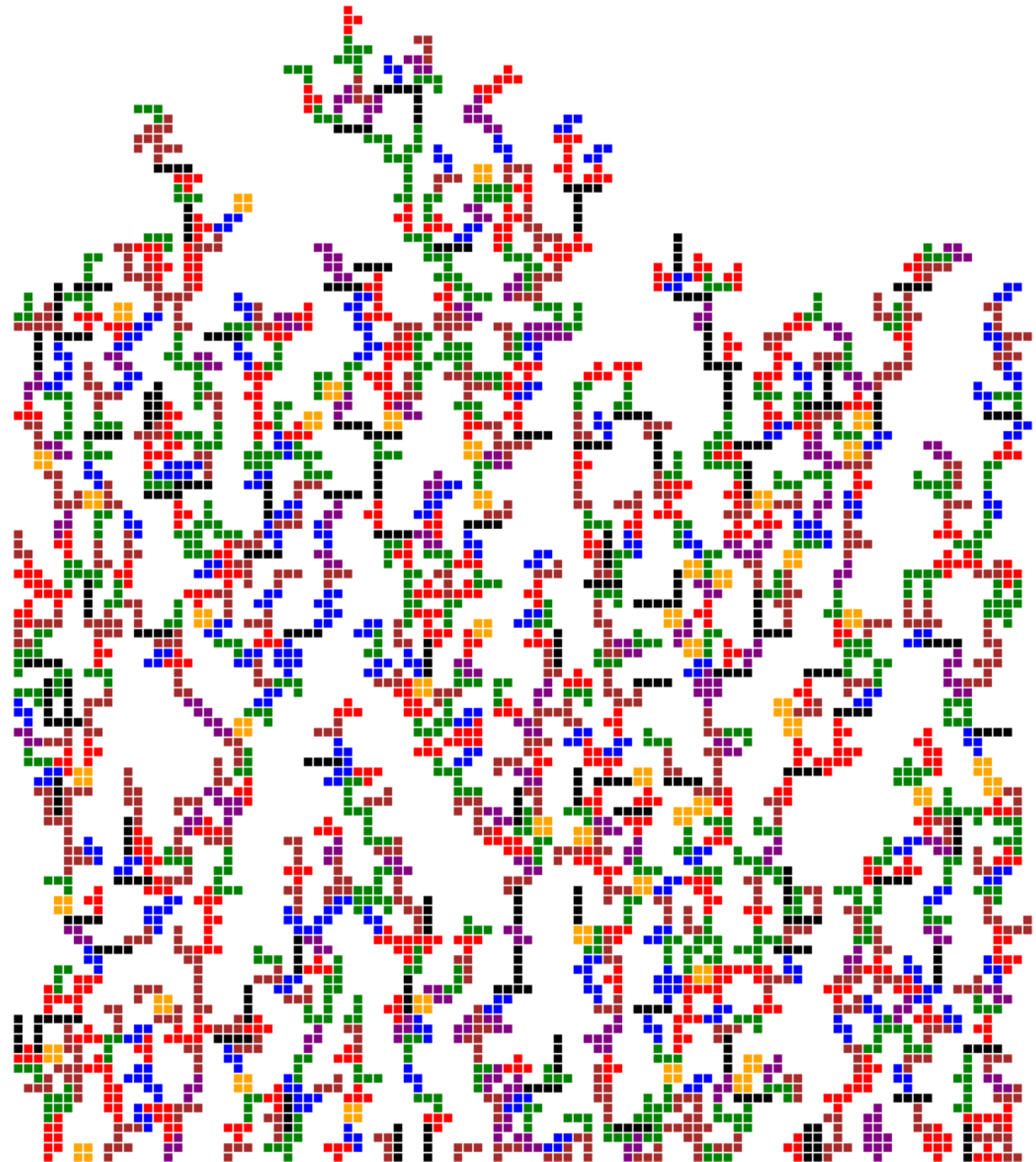
After a long time, they look very different.

- ✧ First example: *No spatial structure* and each column is governed by CLT: $t^{1/2}$ and *Gaussian*.
- ✧ Second example: Fast growth (empty space), smoother interface with *smaller fluctuations and wide spatial correlation*.

What are the scalings and statistics in the second case and do they have their own universality class?



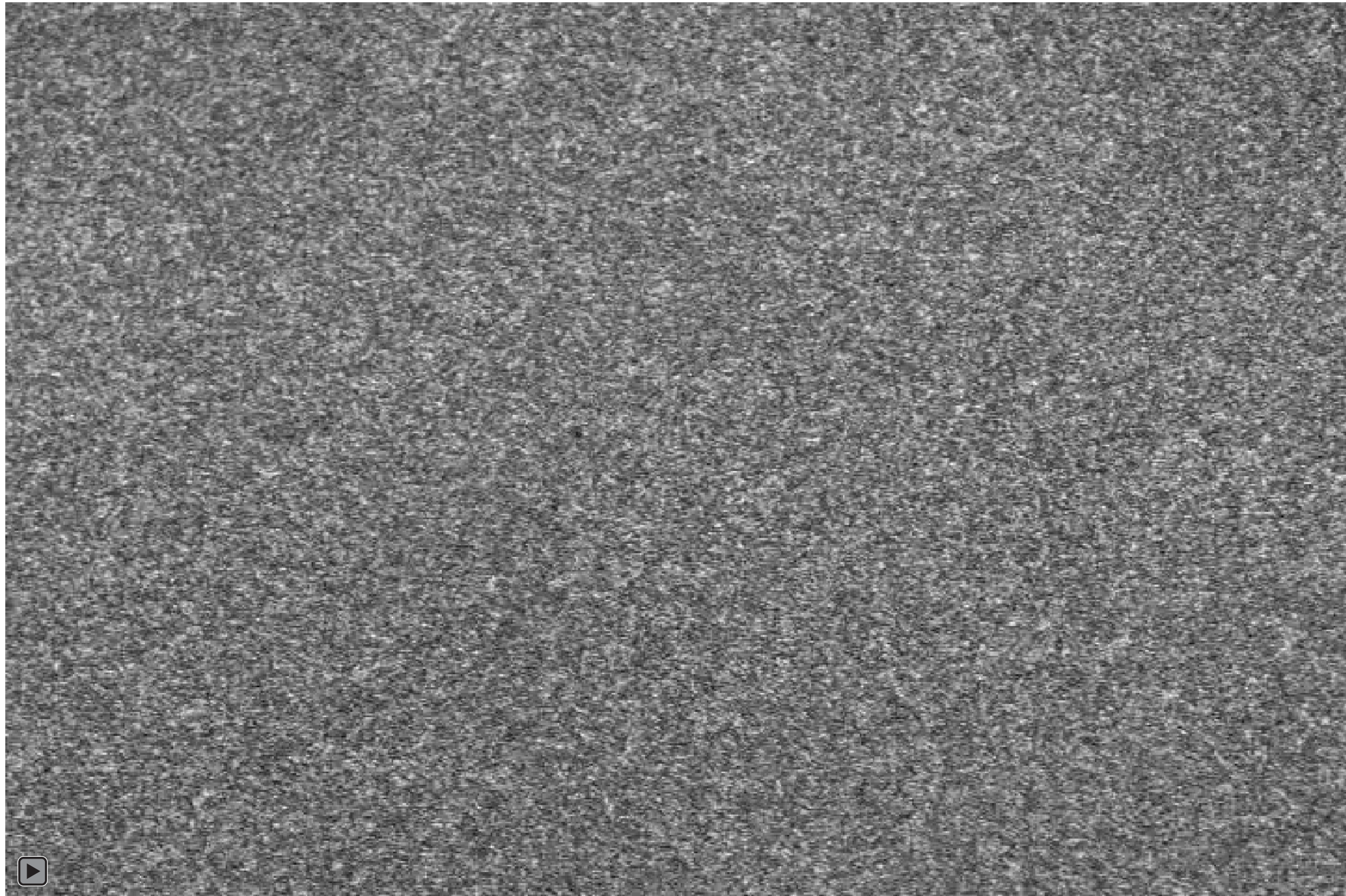
Is random tetris in the same universality class?



What about freezing rain?

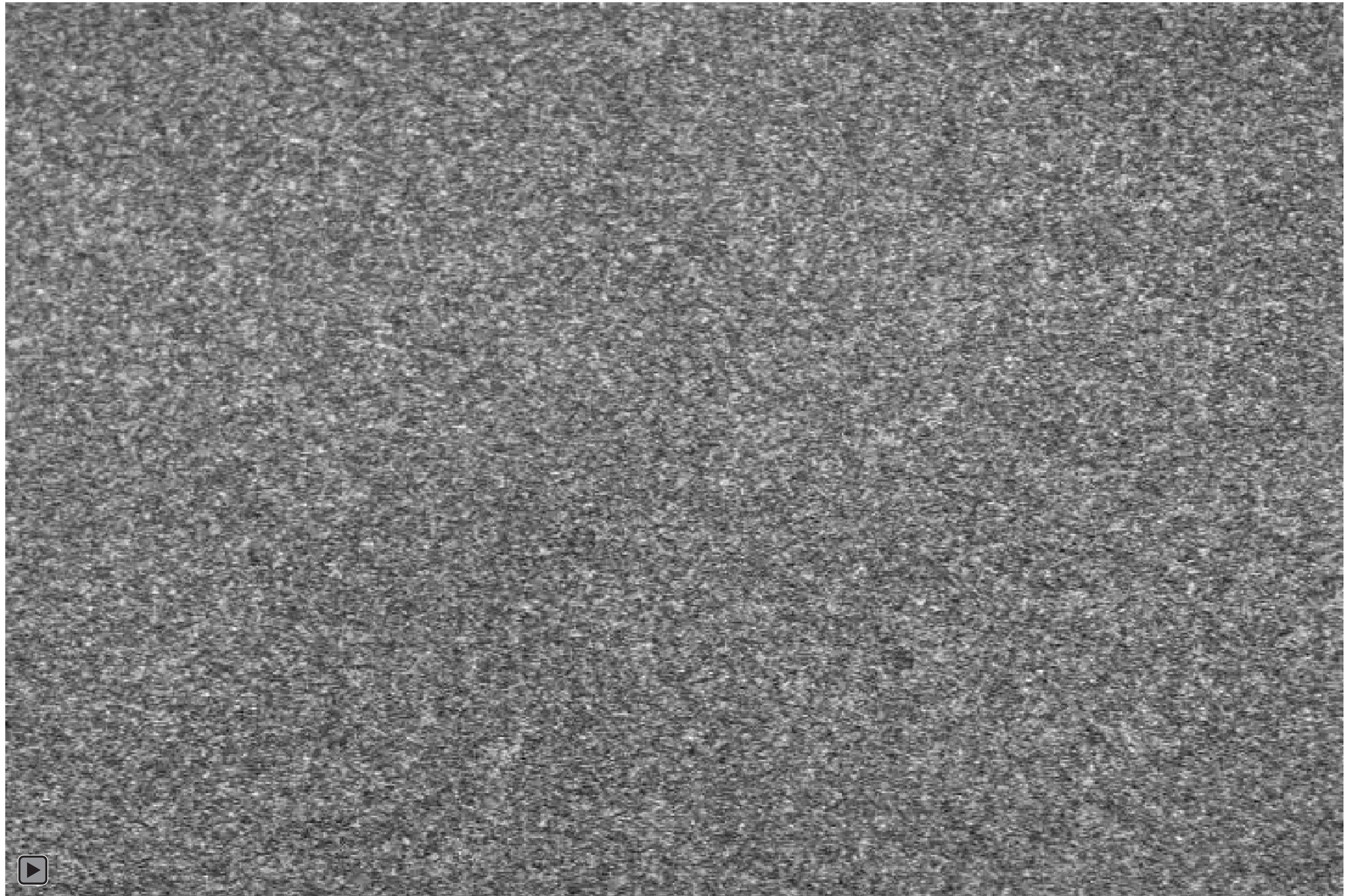


Disordered liquid crystal growth



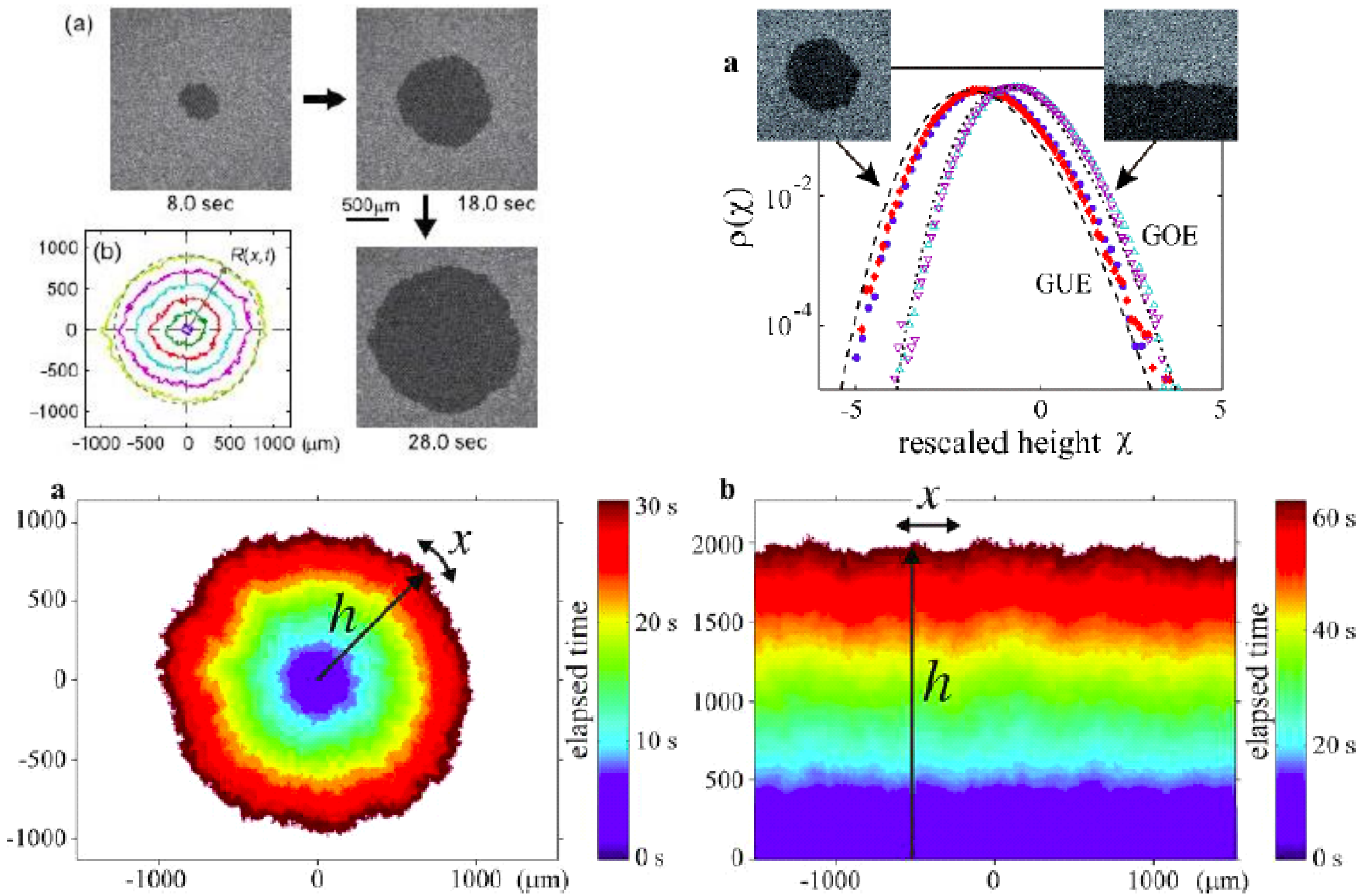
[Takeuchi-Sano-Sasamoto-Spohn 2011]

Disordered liquid crystal growth



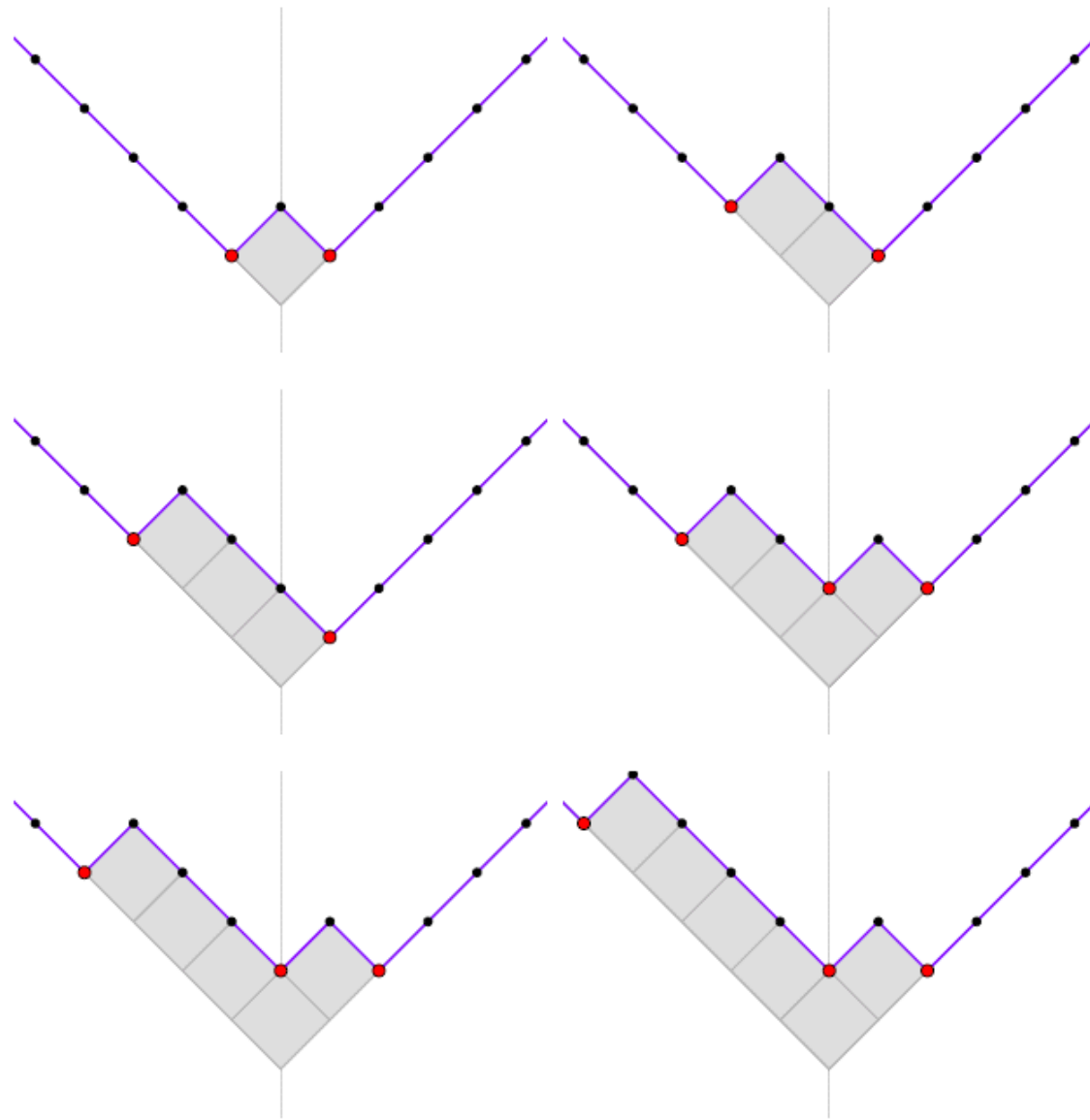
[Takeuchi-Sano-Sasamoto-Spohn 2011]

Disordered liquid crystal growth

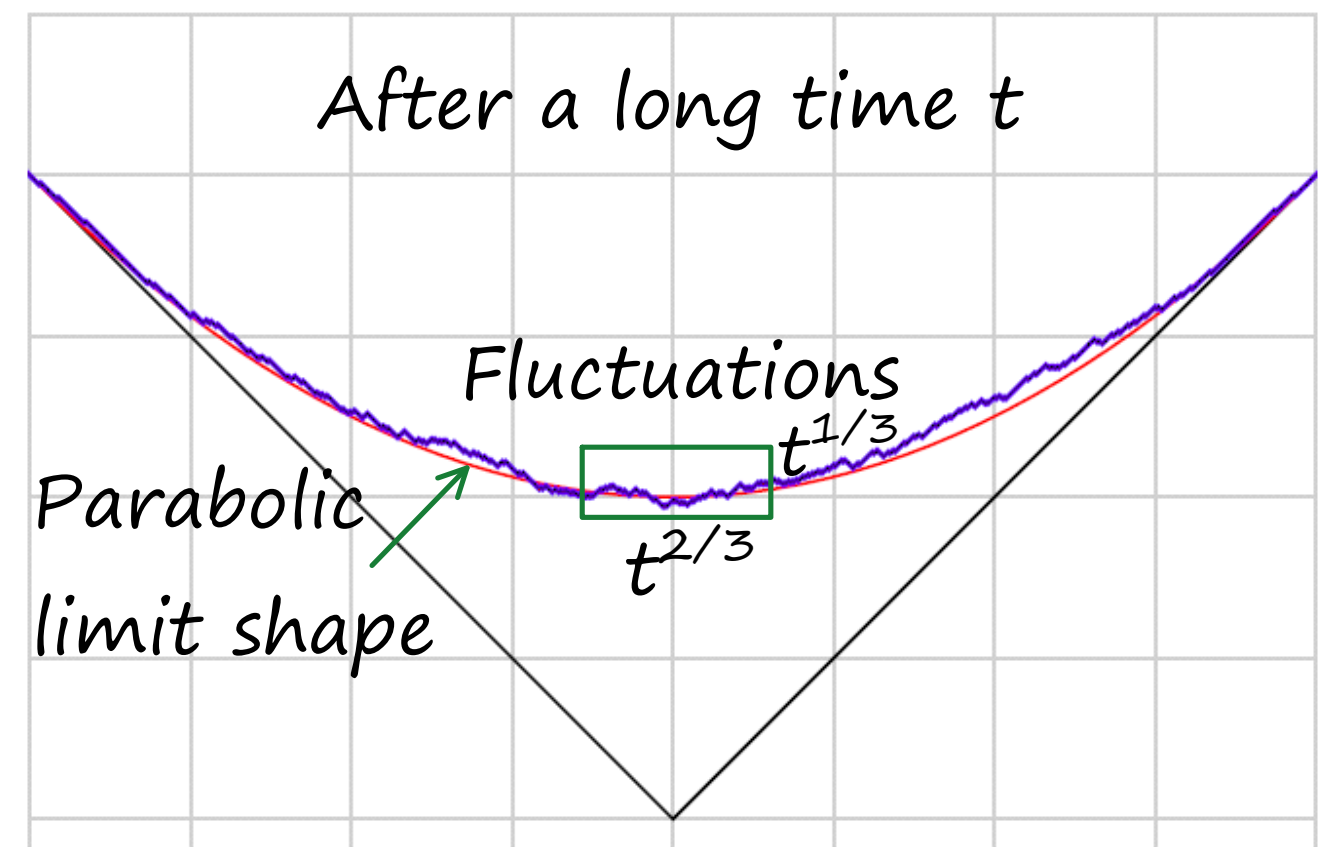
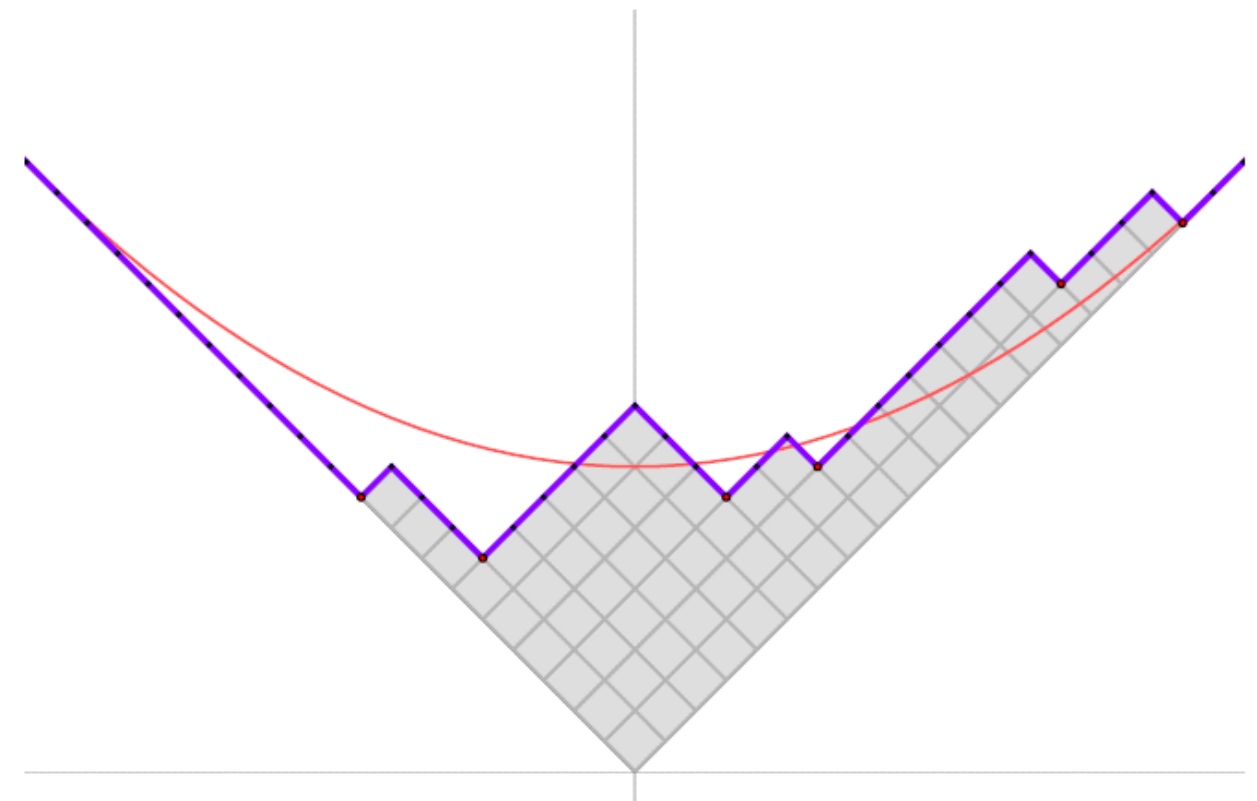


[Takeuchi-Sano-Sasamoto-Spohn 2011]

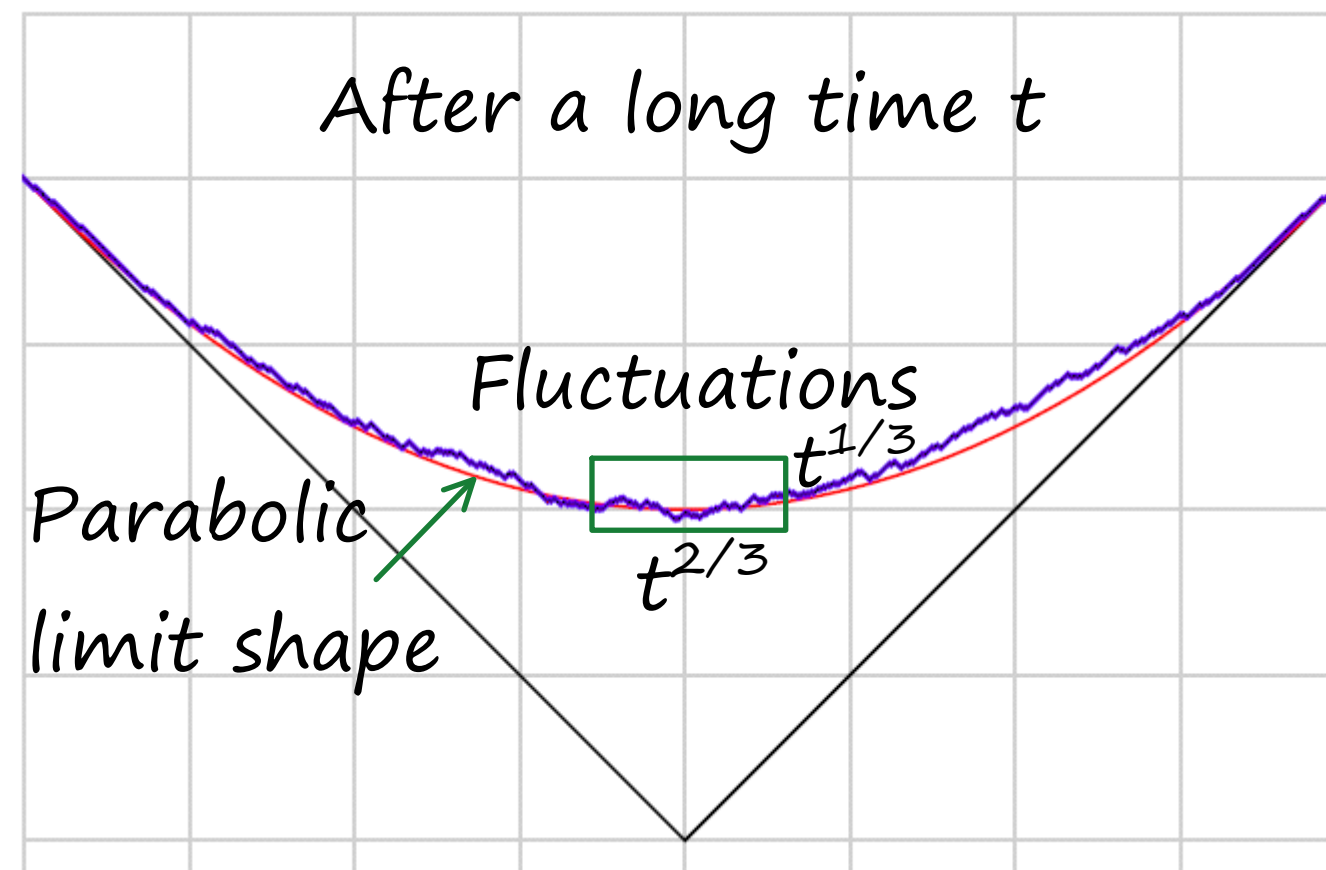
Corner growth model – an integrable example



Minima turn into maxima
after random (exponentially
distributed) waiting times.



Corner growth model – an integrable example



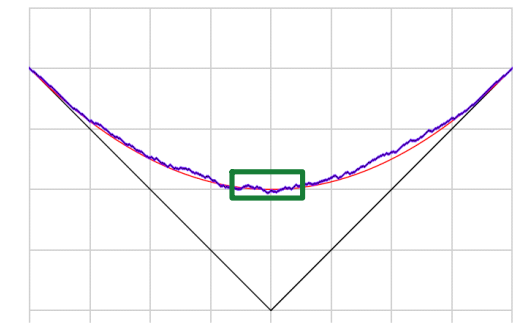
Define the rescaled height function $h_L(t, x) = L^{-1/3} \left[h(Lt, L^{2/3}x) - \frac{Lt}{2} \right]$

Theorem [Johansson 1999]: For wedge initial data as L grows

$$\text{Probability } (h_L(1, 0) > -s) \rightarrow F_{\text{GUE}}(s).$$

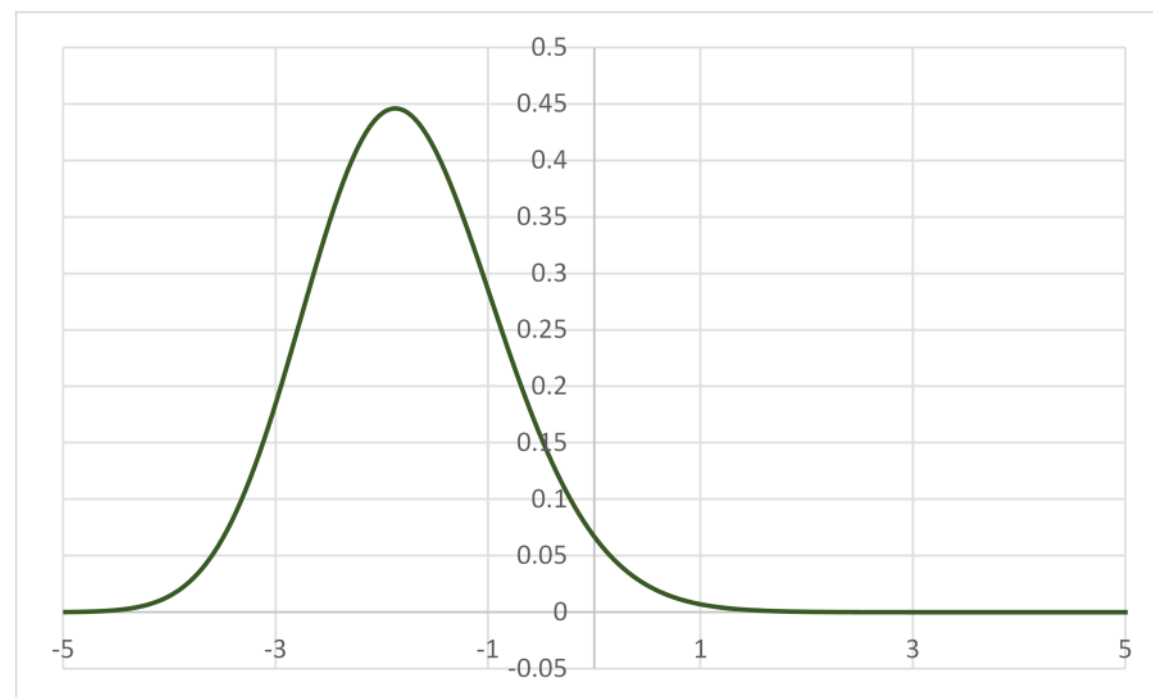
The $t^{1/3}$ scaling and GUE distribution represent KPZ class behavior.

QUE Tracy-Widom distribution

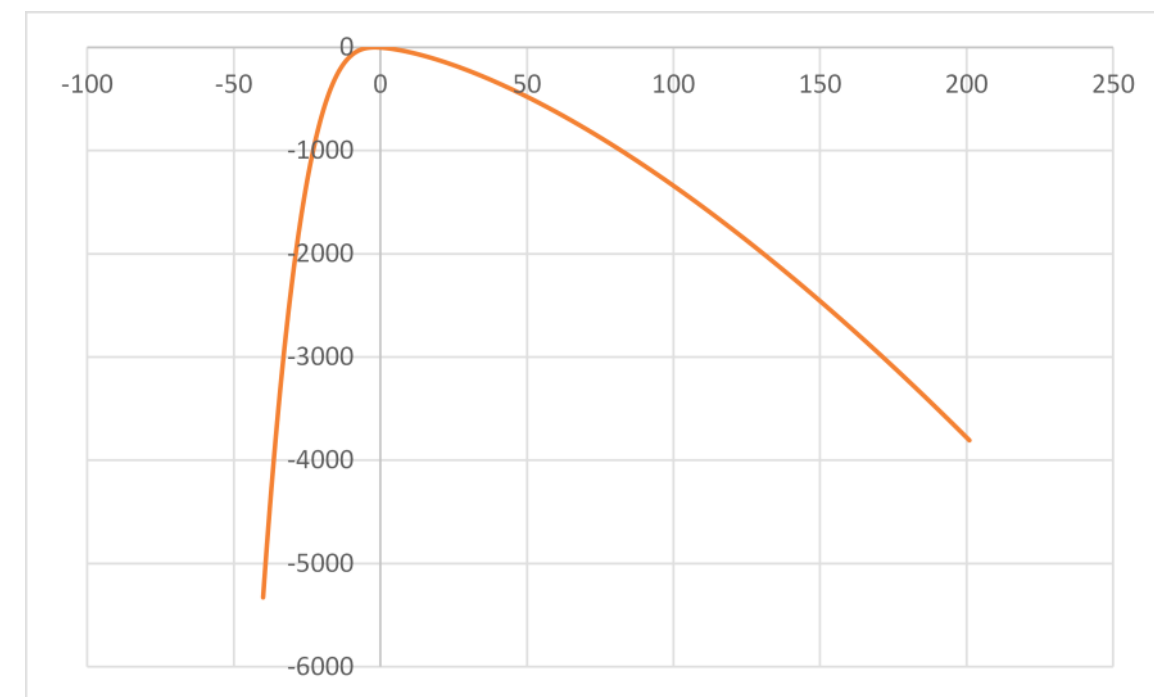


After large time t , the marginal distribution of the height above the origin, when centered and scaled by $t^{1/3}$ convergences to the **QUE Tracy-Widom distribution** [Johansson 1999] which also describes the largest eigenvalues of certain random matrices [Tracy-Widom 1993].

Probability density function $p(x)$



$\text{Log}[p(x)]$

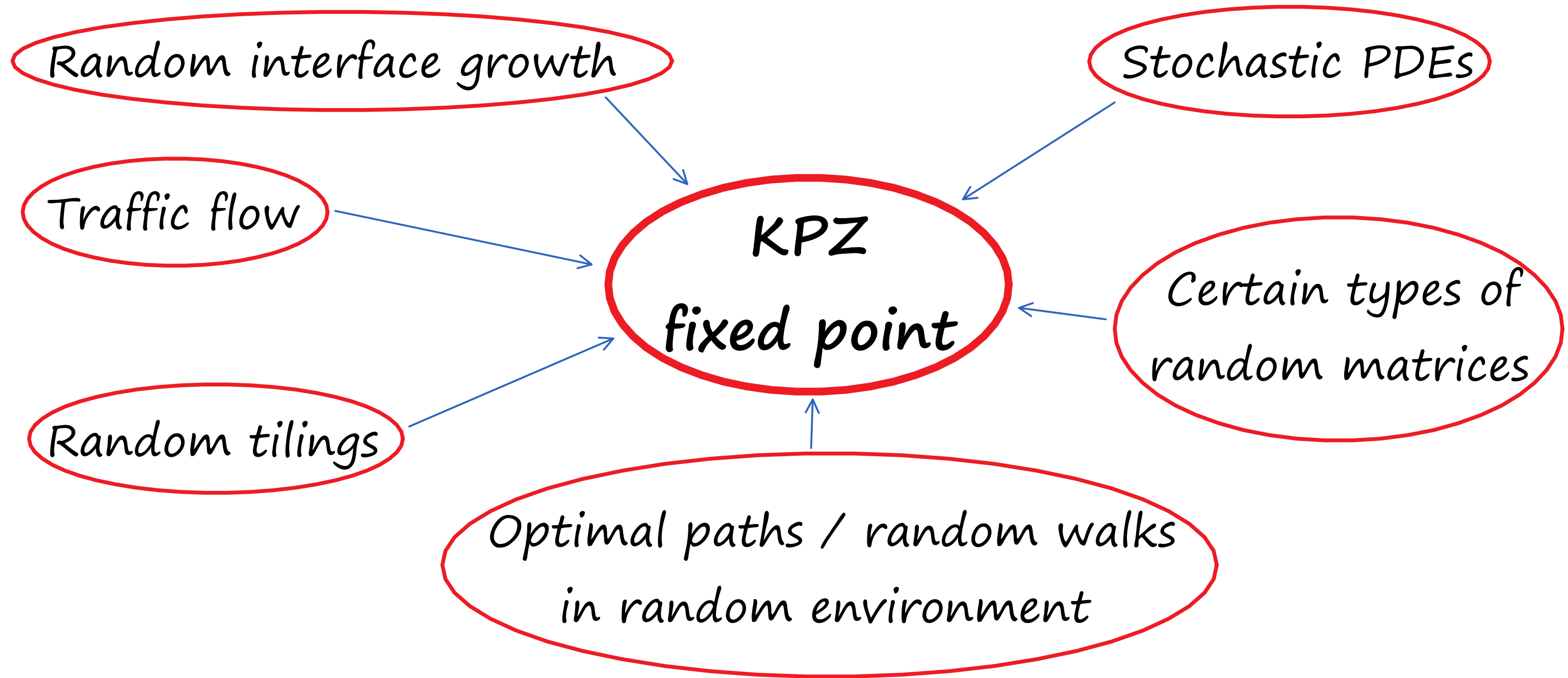


Numerics show this scaling / statistics arises in the 2nd block model.

1+1 dimensional Kardar-Parisi-Zhang universality class

- $3 : 2 : 1$ scaling of *time : space : fluctuation* is called '*KPZ scaling*'.
- Entire growth processes should have a limit – the *KPZ fixed point*.
- Believed to arise in 1+1 dimensional growth processes which enjoy
 - Local dynamics
 - Smoothing
 - Slope dependent (or lateral) growth rate
 - Space-time random driving forces
- There are a number of other types of systems which can (at least in special cases or approximations) be maps into growth processes. Hence these become included into the universality class too.


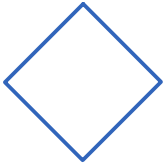
Filling in the KPZ universality class

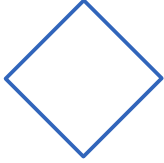



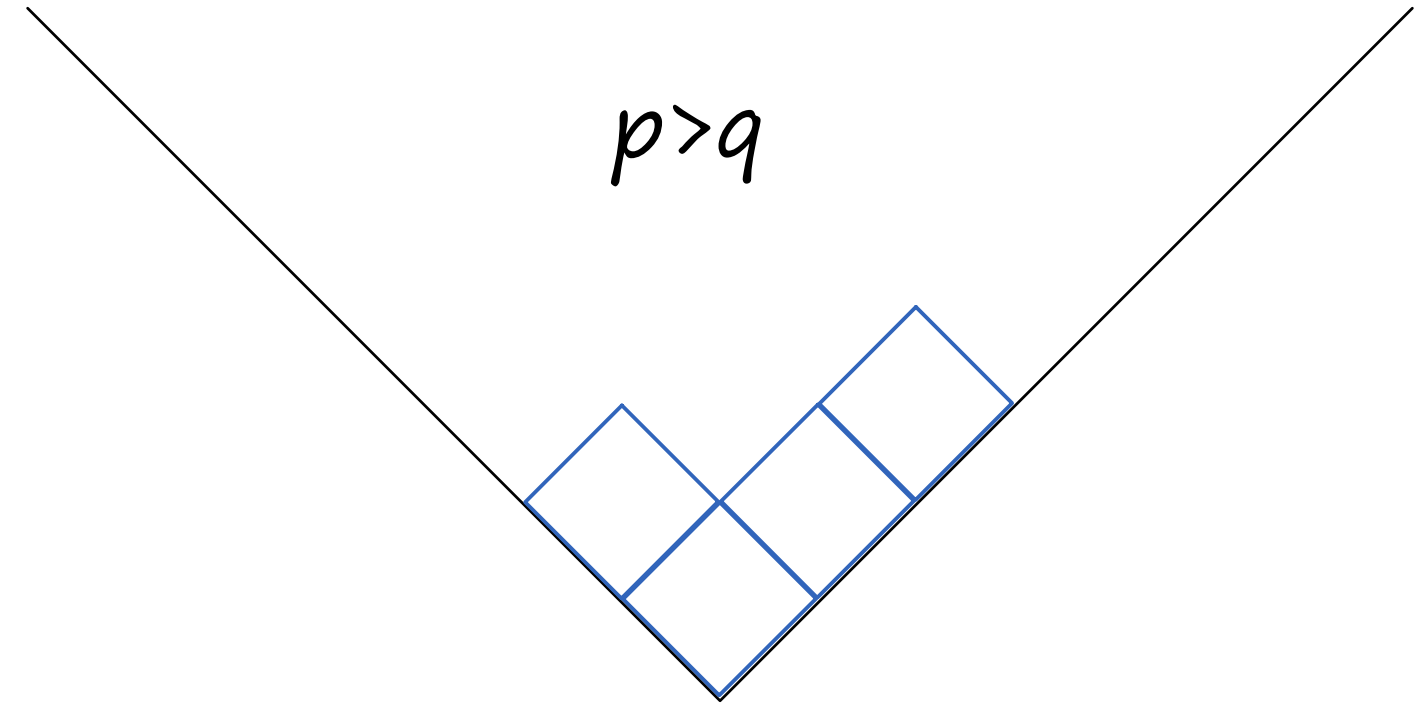
KPZ fixed point should be the universal limit under 3:2:1 scaling.
This is mainly conjectural and only proved for integrable models.

Random interface growth

- Partially asymmetric corner growth model:

✧ Each  turns into  after an exponential rate p waiting time.

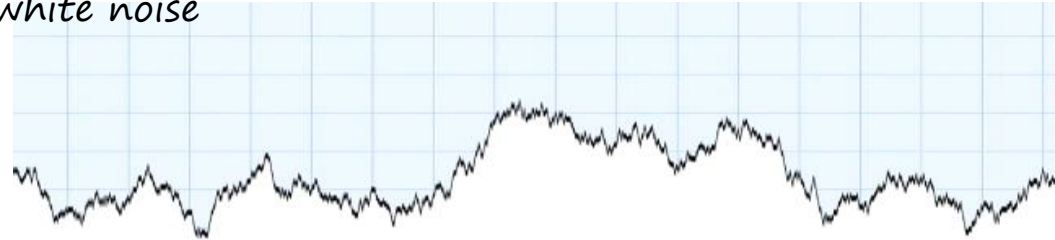
✧ Each  turns into  after an exponential rate q waiting time.



Theorem [Tracy-Widom '09]: Same law of large numbers and fluctuation limit theorems hold with $t \rightarrow t/(p-q)$.

When $p=q$ the law of large numbers and fluctuations change nature. This corresponds with the Edwards-Wilkinson universality class which has 4:2:1 scaling and Gaussian limiting behavior.

Kardar-Parisi-Zhang stochastic partial differential equation

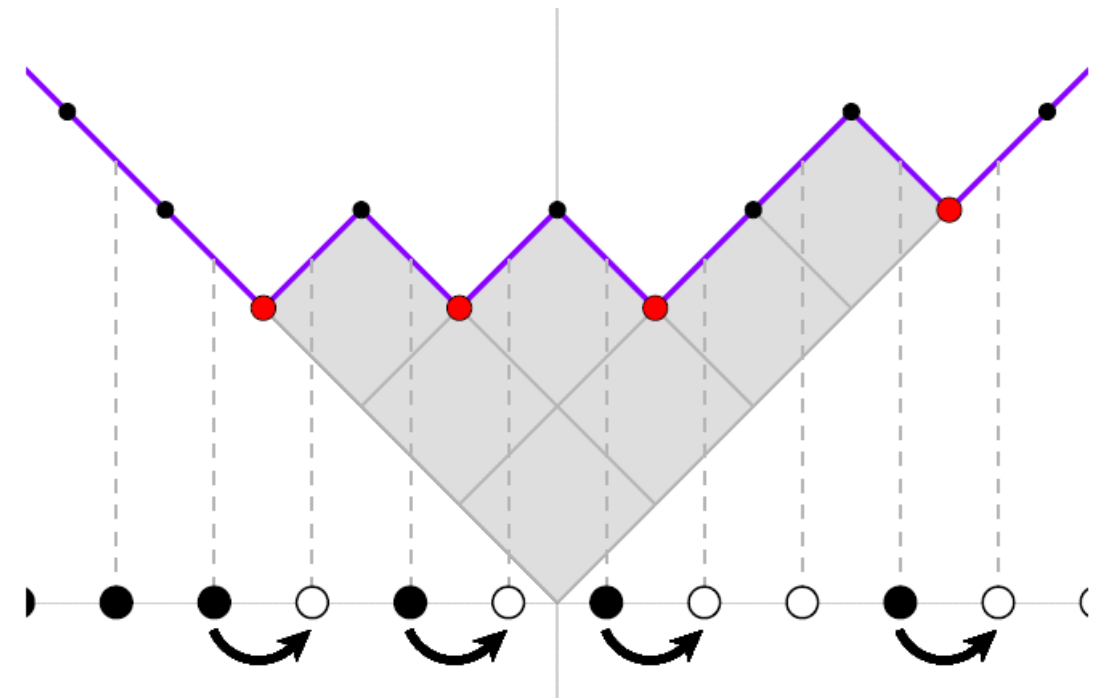
$$\partial_t h(t,x) = \frac{1}{2} \partial_{xx} h(t,x) + \frac{1}{2} (\partial_x h(t,x))^2 + \overset{\text{space-time white noise}}{\xi(t,x)}$$


- ✧ Continuum growth model introduced by [Kardar-Parisi-Zhang 1986] and studied (mathematically non-rigorously) using dynamical renormalization group methods of [Forster-Nelson-Stephen '77].
- ✧ Predicted scaling of **time : space : fluctuations**. Models (like corner growth) which have this behavior are in the KPZ universality class.
- ✧ Only recently did we prove that the KPZ equation is in the KPZ universality class [Amir-C-Quastel '10] and show the GUE statistics.

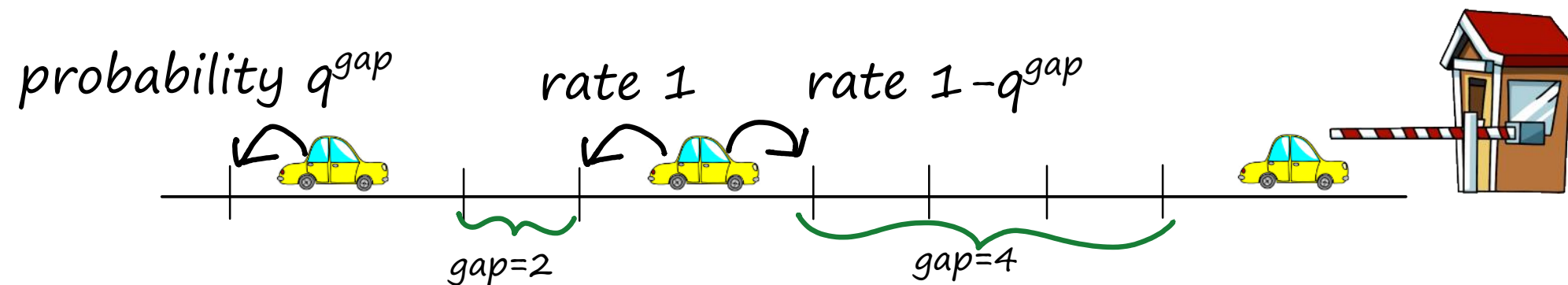
First non-linear SPDE which has been solved (i.e. computed exact formula for distributions and studied asymptotics).

Exactly solvable traffic models

Some growth processes can be coupled with **traffic models for one-lane roads**. The simplest comes from the corner growth model.

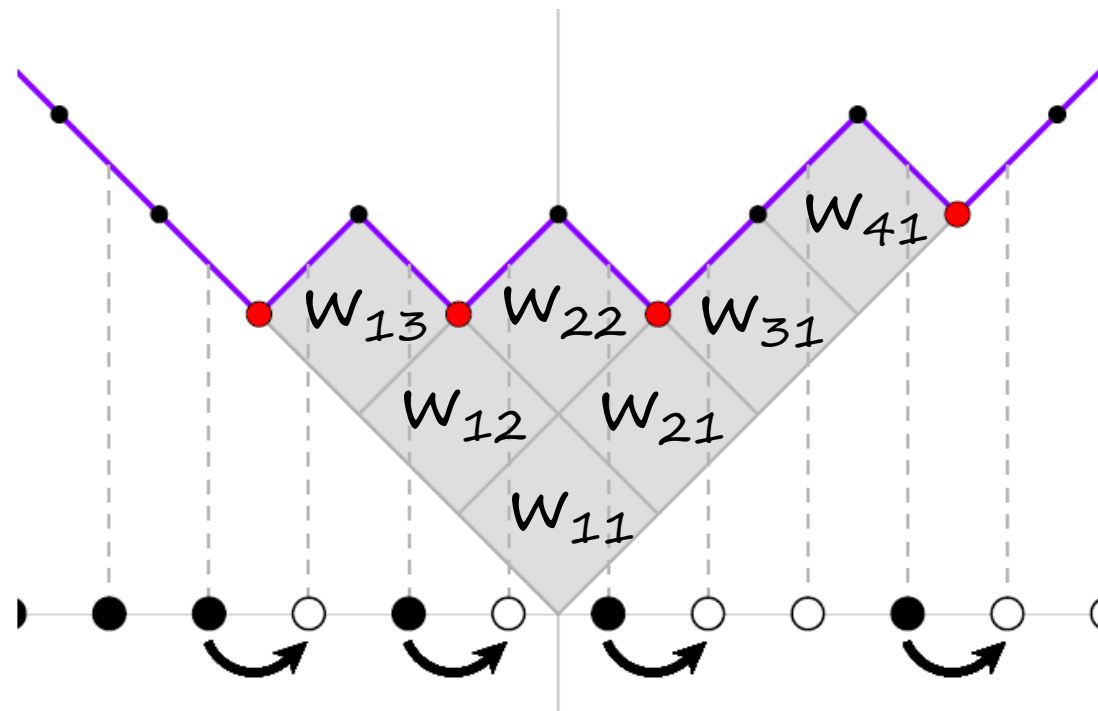


Here is a more realistic traffic models (**slowing down and breaking**) which is integrable and in the KPZ class [Borodin-C '11, C-Petrov '14].



KPZ class behavior: For step initial data, the number of particles to cross origin behaves like $ct + c't^{1/3}\chi$ where χ is F_{QUE} distributed.

Optimal paths in random environment



Last passage percolation [Rost '81]

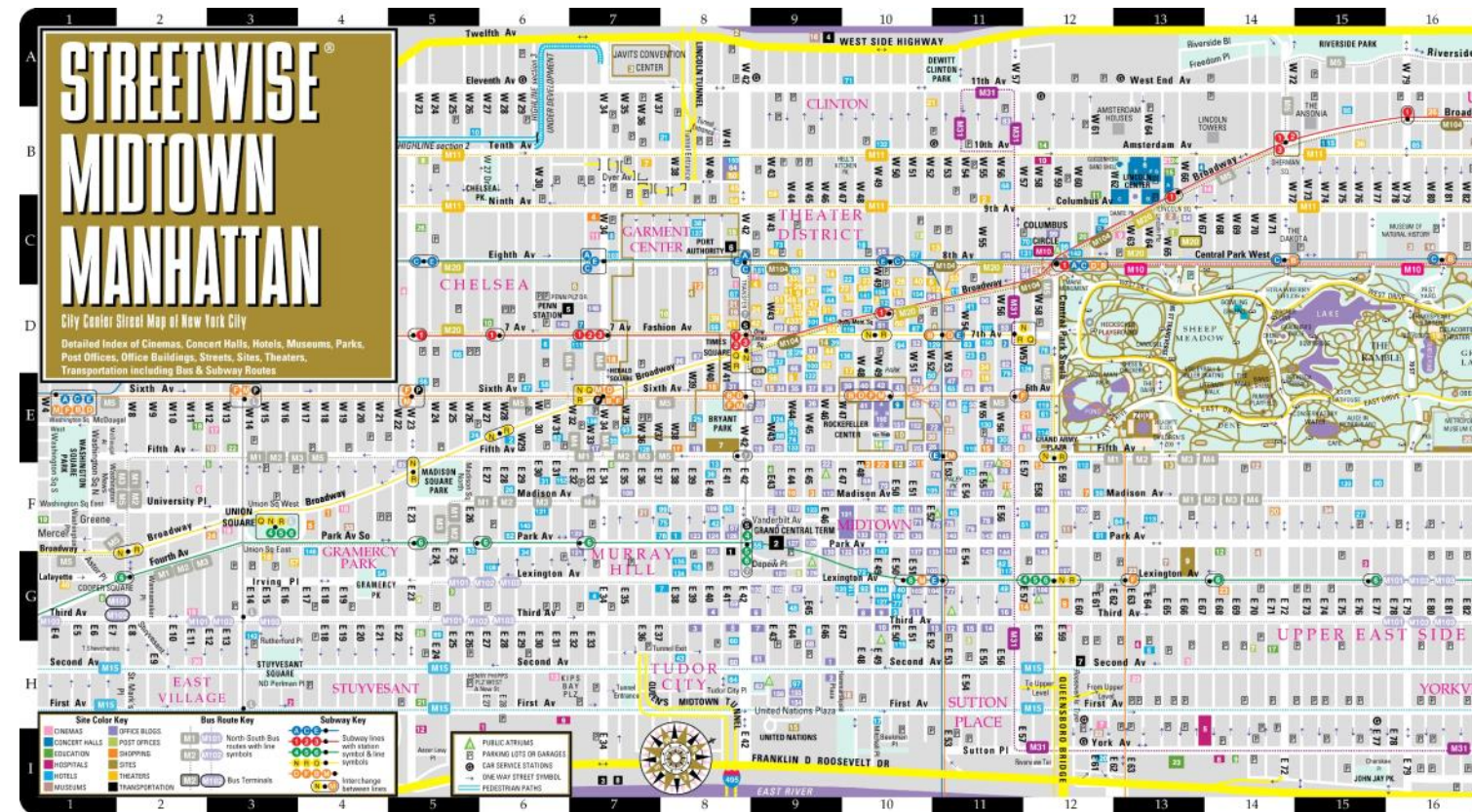
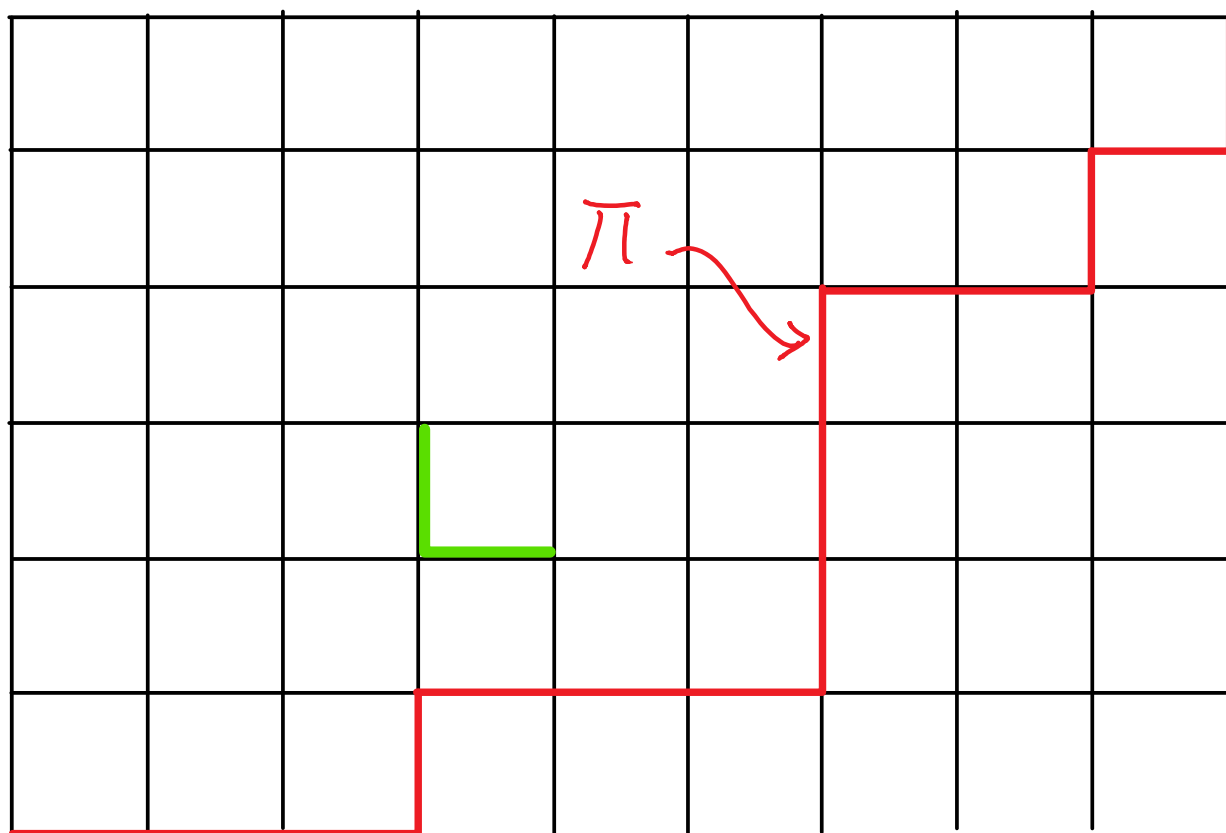
- w_{ij} : time for box (i,j) to grow, once it can (exponential rate 1).
- $L(x,y)$: time when box x,y is grown.

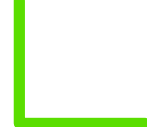
$$\text{Recursion: } L(x,y) = \max(L(x-1,y), L(x,y-1)) + w_{xy}$$

$$\text{Iterating: } L(x,y) = \max_{\pi: (1,1) \rightarrow (x,y)} \sum_{(i,j) \in \pi} w_{ij}$$

KPZ class behavior: $L(xt, yt)$ behaves like $ct + c't^{1/3} \chi$ where χ is F_{GUE} distributed and the constants depend on x, y .

Optimal paths in random environment

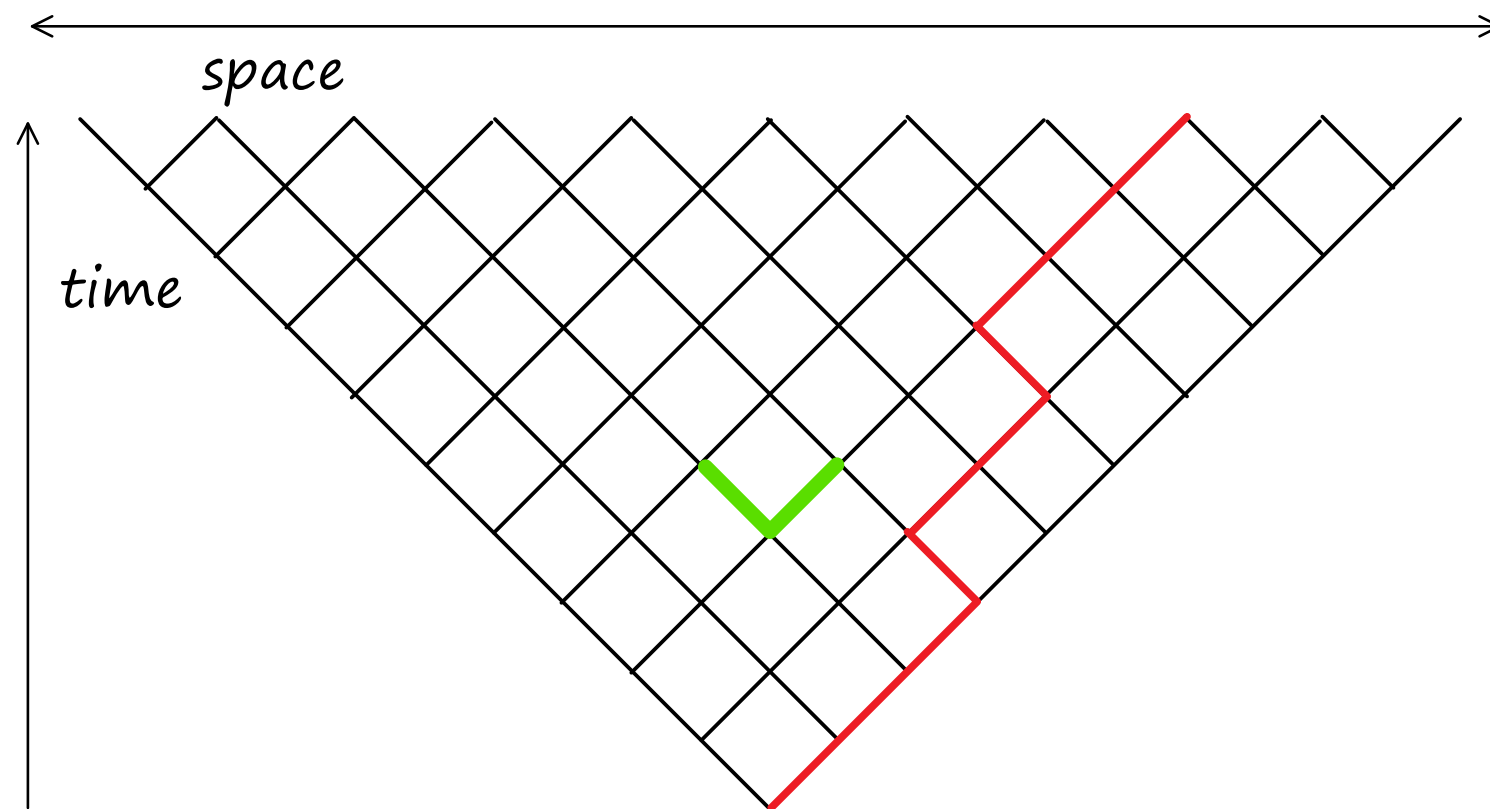


[Barraquand-C '15]: Assign edge weights to each  so with probability $1/2$, horizontal weight is 0 and vertical is $\exp(1)$; otherwise reversed.

Minimal passage time $P(x,y) = \min_{\pi: (0,0) \rightarrow (x,y)} \sum_{e \in \pi} w_e.$

KPZ class behavior: For $x \neq y$, $P(xt,yt)$ behaves like $ct + c't^{1/3} \chi$ where χ is F_{QUE} distributed and the constants depend on x,y .

Random walk in random environment



- For each (space,time)-vertex choose u_{ys} uniform on $[0,1]$.
- Take independent random walks $X^{(1)}, X^{(2)}, \dots$ where at time s and position y , move left with probability u_{ys} , right with $1-u_{ys}$.
- Let $M(t,N) = \max (X^{(1)}, \dots , X^{(N)})$.

KPZ class behavior: For $0 < r < 1$, $M(t, e^{rt})$ behaves like $ct + c' t^{1/3} \chi$ where χ is F_{QUE} and the constants depend on r [Barraquand-C '15].

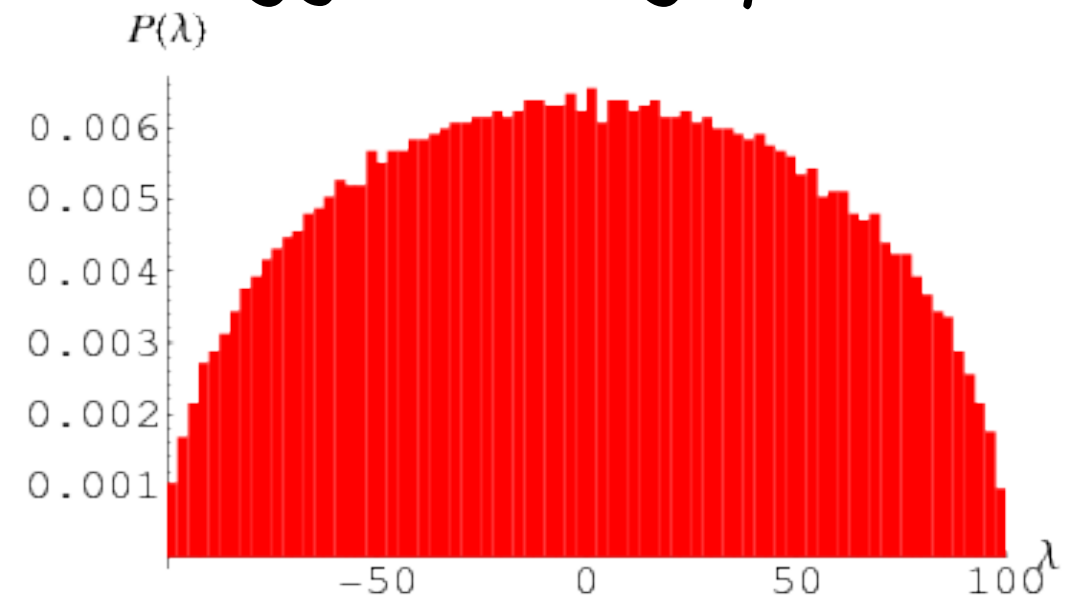
Certain types of random matrices

Gaussian Unitary Ensemble (GUE) on $N \times N$ complex matrices:

$$H^{(N)} = [H_{ij}^{(N)}]_{i,j=1}^N \text{ where } H_{ij} = \overline{H_{ji}} = \begin{cases} \mathcal{N}(0, N) & i=j \\ \mathcal{N}(0, N/2) + i\sqrt{N/2}\mathcal{N}(0, N/2) & i \neq j \end{cases}$$

Introduced by [Wigner '55] to model the energy levels/gaps of atoms too complicated to solve analytically.

Let $\lambda_1^{(N)} \geq \dots \geq \lambda_N^{(N)}$ denote the (random) real eigenvalues of $H^{(N)}$.



KPZ class behavior: $\lambda_1^{(N)}$ behaves like $2N + N^{1/3}\chi$ where χ is F_{GUE} .

Relationship to growth processes is much less apparent here.

Certain types of random matrices

Complex Wishart Ensemble (or sample covariance) on $N \times M$ matrices:

$$H^{(N,M)} = [H_{ij}^{(N,M)}]_{\substack{1 \leq i \leq N \\ 1 \leq j \leq M}} \text{ where } H_{ij} = \mathcal{N}(0, 1/2) + \sqrt{-1} \mathcal{N}(0, 1/2)$$

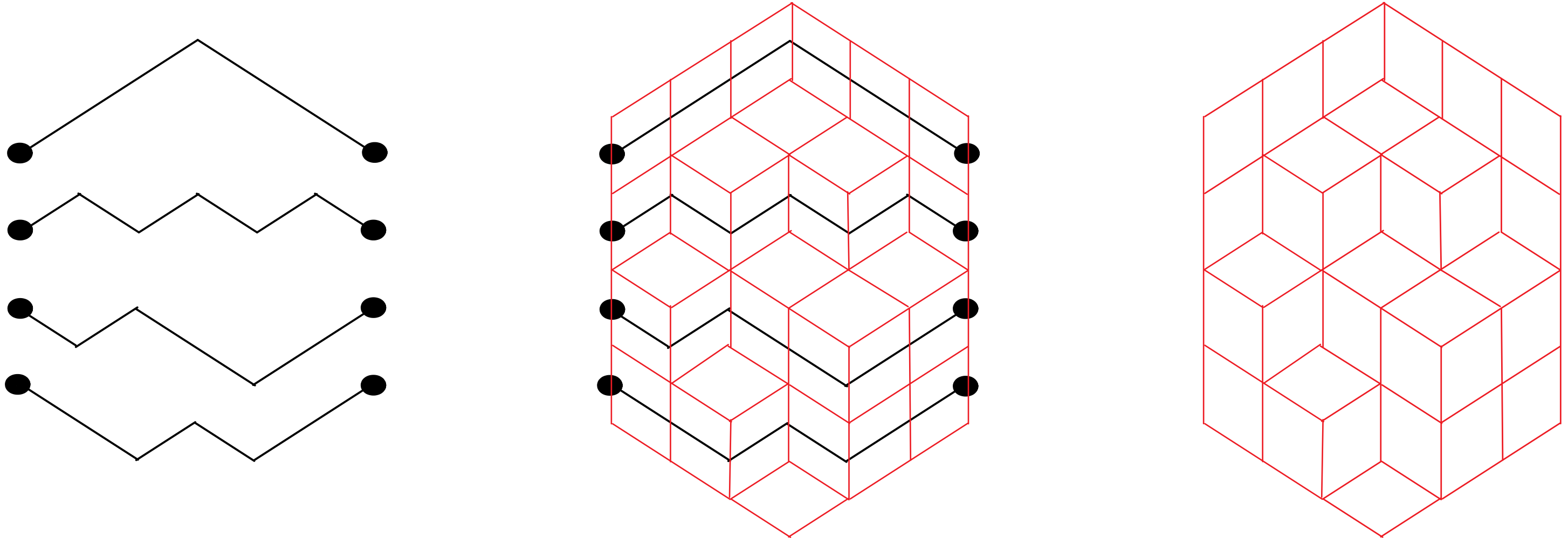
Introduced by [Wishart '28] within statistics. Provides a base-line for noisy data against which to compare **Principal Component Analysis**

Let $\sigma_1^{(N,M)} \geq \dots \geq \sigma_N^{(N,M)}$ denote the (random) real positive singular values of $H^{(N,M)}$ (i.e., the square-roots of eigenvalues of $H^{(N,M)} (H^{(N,M)})^*$).

Surprise [Johansson '00]: The distribution of $\sigma_1^{(N,M)}$ equals that of $L(N,M)$.

$$\text{E.G. } N=M=1, \text{ Probability}(\sigma_1^{(1,1)} \leq s) = \frac{1}{\pi} \int_{x^2+y^2 \leq s^2} e^{-x^2-y^2} dx dy = \int_0^{s^2} e^{-r^2} 2r dr = \int_0^s e^{-t} dt$$

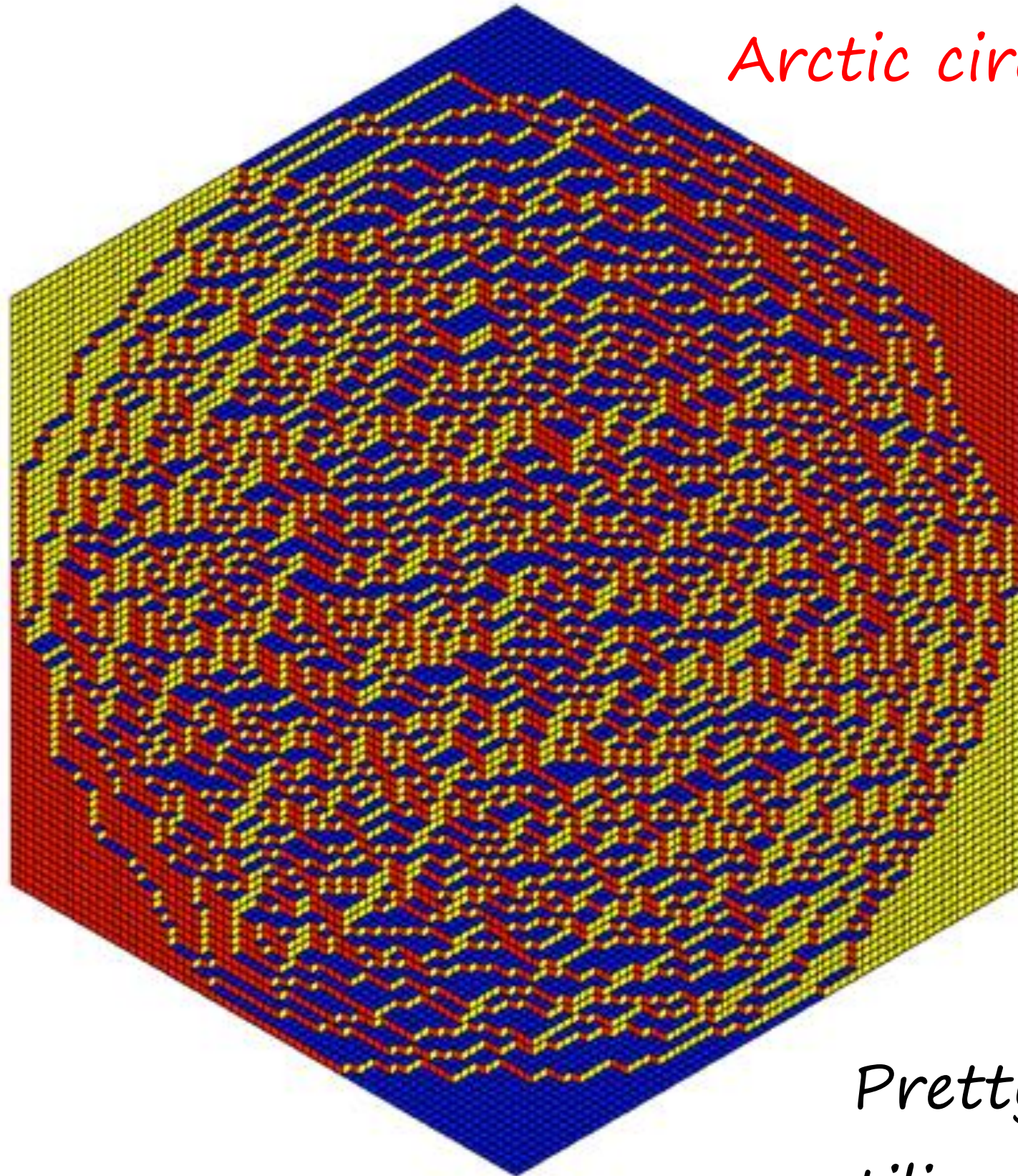
Vicious walkers and random tilings



Consider N random walks with fixed starting and ending points, conditioned not to touch. This gives rise to a uniform measure on fillings of a box, or tilings of a hexagon by three types of rhombi.

Vicious walkers and random tilings

Arctic circle theorem [Cohn-Larsen-Propp '98]



KPZ class behavior: The top walker (or edge of the arctic circle) has fluctuations of order $N^{1/3}$ and limiting F_{GUE} distribution. [Baik-Kriecherbauer-McLaughlin-Miller '07], [Petrov '12]

Pretty pictures (and math) when tiling various types of domains

Open problems

- **Higher dimension** (e.g. random surface growth)
- **KPZ universality** (scale, distribution, entire space-time limit)
 - Growth processes (e.g. ballistic deposition, Eden model)
 - Interacting particle systems (e.g. non-nearest neighbor exclusion)
 - Last/first passage percolation, RWRE with general weights
- Full description of **KPZ fixed point**
 - Complete space-time multipoint distribution
 - Unique characterization of fixed point
- **Weak universality** of the KPZ equation
 - Under critical weak tuning of the strength of model parameters
- Discover **new integrable examples and tools in their analyses**