Computational Information Games

Houman Owhadi

TexAMP 2015

Main Question

Can we turn the process of discovery of a scalable numerical method into a UQ problem and, to some degree, solve it as such in an automated fashion?

Can we use a computer, not only to implement a numerical method but also to find the method itself?

Problem: Find a method for solving (1) as fast as possible to a given accuracy

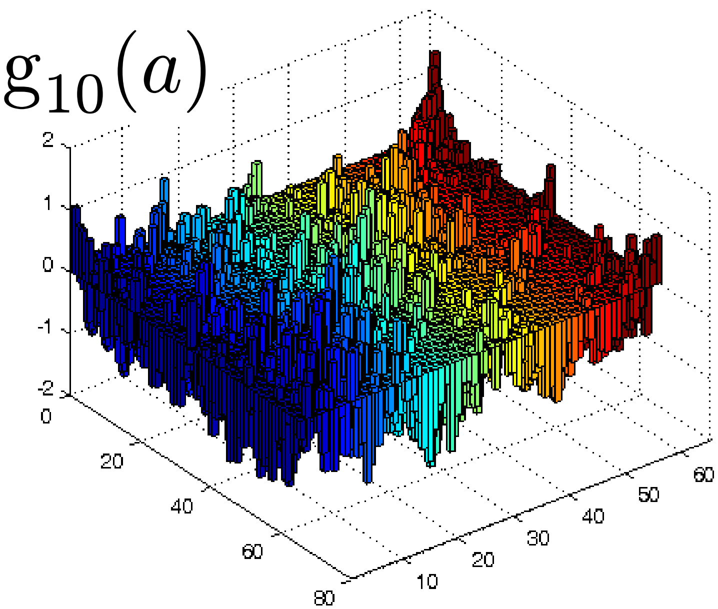
$$
\text{(1)} \quad \boxed{\begin{cases} -\operatorname{div}(a\nabla u) = g, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}}
$$

$\Omega \subset \mathbb{R}^d$ $\qquad$ $\partial\Omega$ is piec. Lip.

a unif. ell.

$a_{i,j} \in L^\infty(\Omega)$

$\log_{10}(a)$



Multigrid Methods

Multigrid: [Fedorenko, 1961, Brandt, 1973, Hackbusch, 1978]

Multiresolution/Wavelet based methods

[Brewster and Beylkin, 1995, Beylkin and Coult, 1998, Averbuch et al., 1998]

$$\mathbb{R}^m$$

- **Linear complexity with smooth coefficients**

Problem **Severely affected by lack of smoothness**

Robust/Algebraic multigrid

[Mandel et al., 1999,Wan-Chan-Smith, 1999,
Xu and Zikatanov, 2004, Xu and Zhu, 2008], [Ruge-Stüben, 1987]
[Panayot - 2010]

Stabilized Hierarchical bases, Multilevel preconditioners

[Vassilevski - Wang, 1997, 1998]

[Panayot - Vassilevski, 1997]

[Chow - Vassilevski, 2003]

[Aksoylu- Holst, 2010]

- Some degree of robustness but problem remains open with rough coefficients

Why? **Interpolation operators are unknown**

Don't know how to bridge scales with rough coefficients!

Low Rank Matrix Decomposition methods

Fast Multipole Method: [Greengard and Rokhlin, 1987]

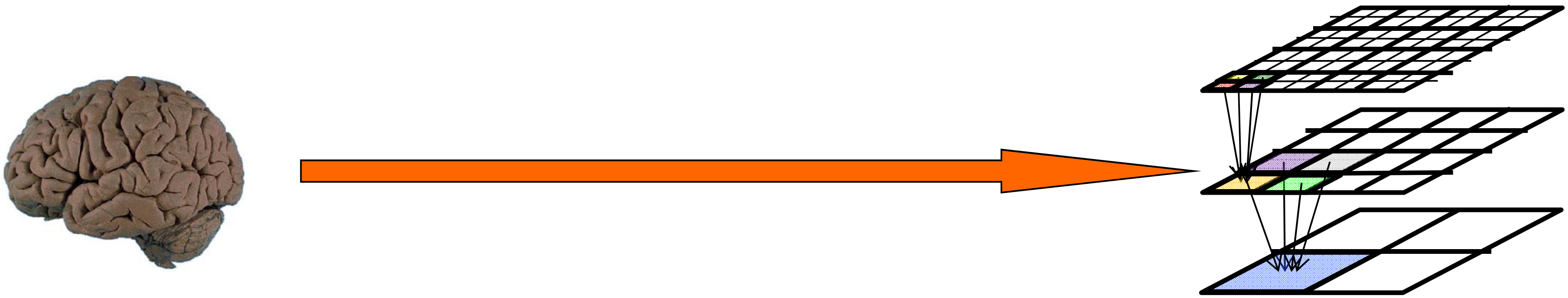
Hierarchical Matrix Method: [Hackbusch et al., 2002]

[Bebendorf, 2008]:

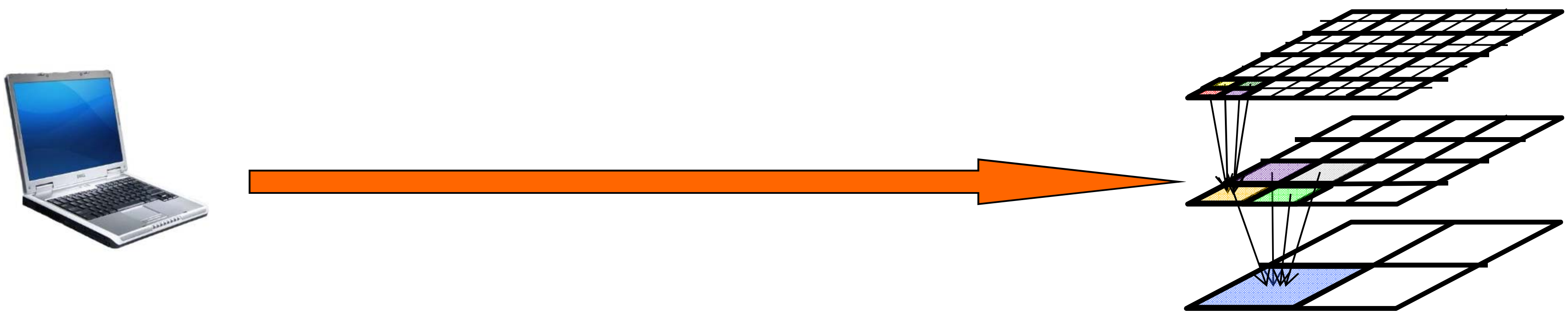
$$\boxed{N \ln^{d+3} N \text{ complexity}}$$

Common theme between these methods

Their process of discovery is based on intuition, brilliant insight, and guesswork



Can we turn this process of discovery into an algorithm?



Answer: YES

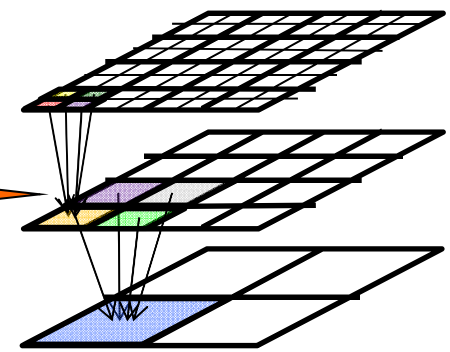
Compute fast

Play adversarial Information game

Compute with partial information

Identify game

Find optimal strategy



[Owhadi 2015, Multi-grid with rough coefficients and Multiresolution PDE decomposition from Hierarchical Information Games, arXiv:1503.03467]

Resulting method: $\boxed{N \ln^2 N \text{ complexity}}$

This is a theorem

$$\begin{cases} -\operatorname{div}(a\nabla u) = g \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

Resulting method:

$$\boxed{H_0^1(\Omega) = \mathfrak{W}^{(1)} \oplus_a \mathfrak{W}^{(2)} \oplus_a \cdots \oplus_a \mathfrak{W}^{(k)} \oplus_a \cdots}$$

$$< \psi, \chi >_a := \int_\Omega (\nabla\psi)^T a \nabla\chi = 0 \text{ for } (\psi,\chi) \in \mathfrak{W}^{(i)} \times \mathfrak{W}^{(j)},\ i \neq j$$

Theorem For $v \in \mathfrak{W}^{(k)}$

$$\boxed{\frac{C_1}{2^k} \leq \frac{\|v\|_a}{\|\operatorname{div}(a\nabla v)\|_{L^2(\Omega)}} \leq \frac{C_2}{2^k}}$$

$$\|v\|_a^2 :=< v, v >_a = \int_\Omega (\nabla v)^T a \nabla v$$

Looks like an eigenspace decomposition

$$\boxed{u = w^{(1)} + w^{(2)} + \cdots + w^{(k)} + \cdots}$$

$w^{(k)} =$ F.E. sol. of PDE in $\mathfrak{W}^{(k)}$

Can be computed independently

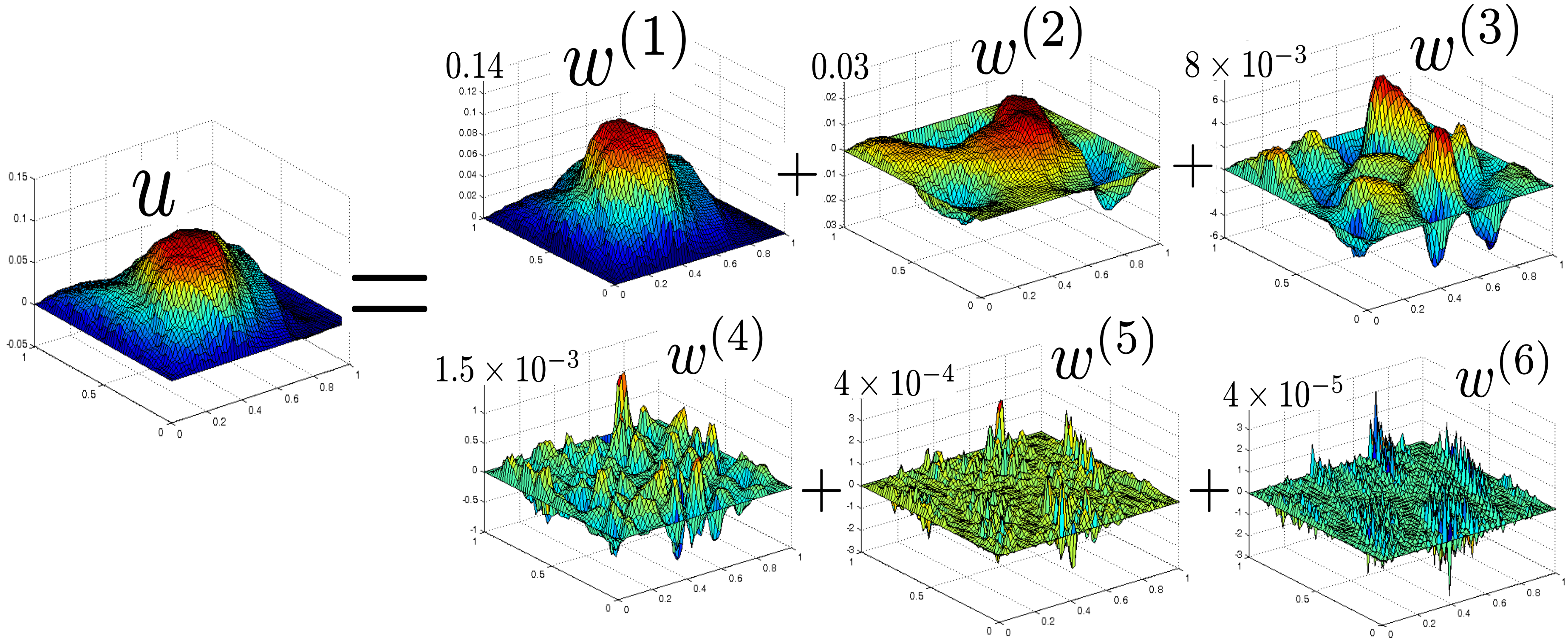
$B^{(k)}$: Stiffness matrix of PDE in $\mathfrak{W}^{(k)}$

Theorem $$\boxed{\frac{\lambda_{\max}(B^{(k)})}{\lambda_{\min}(B^{(k)})} \leq C}$$

Just relax in $\mathfrak{W}^{(k)}$ to find $w^{(k)}$

Quacks like an eigenspace decomposition

Multiresolution decomposition of solution space



Solve time-discretized wave equation (implicit time steps) with rough coefficients in $\mathcal{O}(N \ln^2 N)$-complexity

Swims like an eigenspace decomposition

$\boxed{\mathfrak{V}\text{: F.E. space of } H_0^1(\Omega) \text{ of dim. } N}$

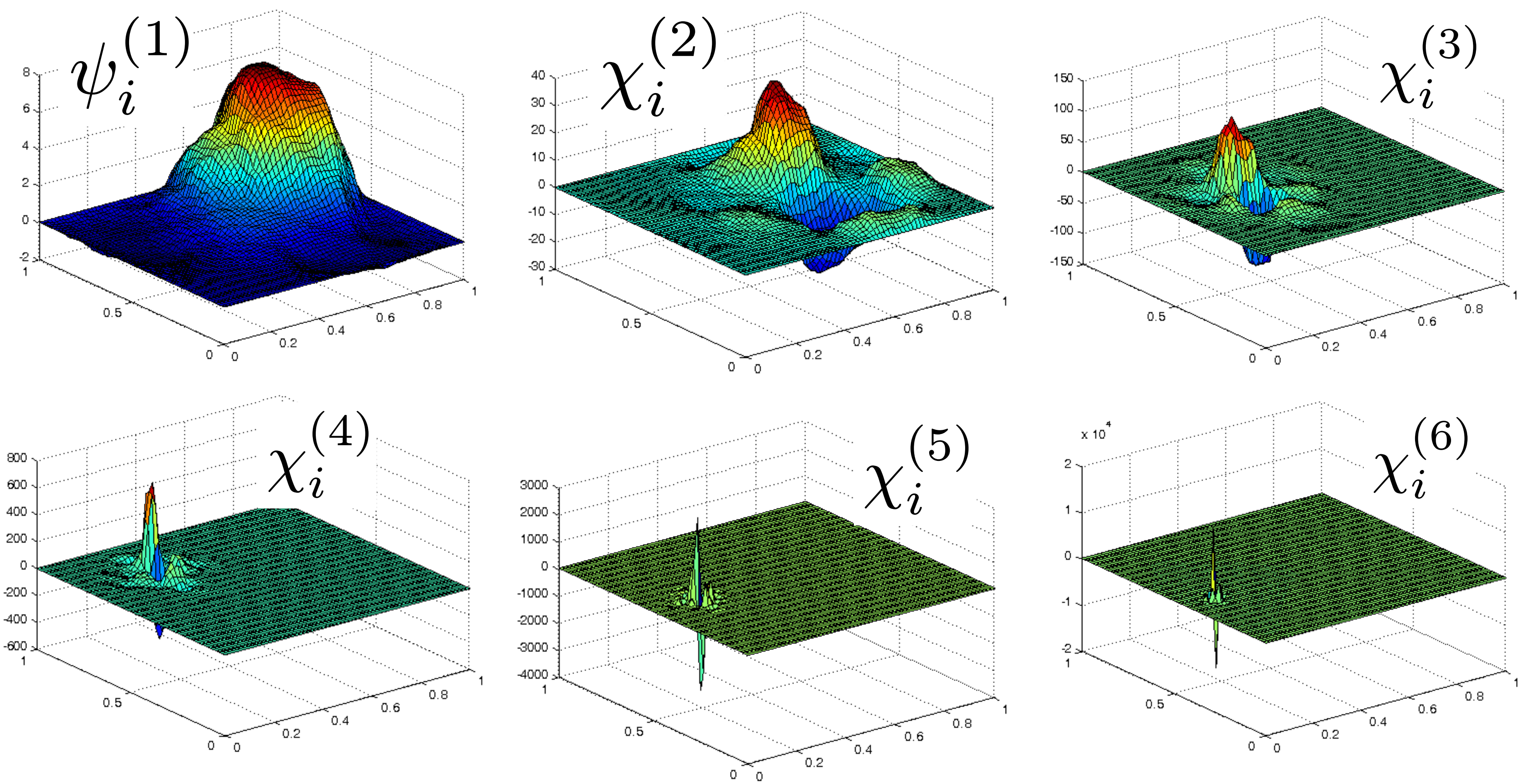
Theorem The decomposition

$$\boxed{\mathfrak{V} = \mathfrak{W}^{(1)} \oplus_a \mathfrak{W}^{(2)} \oplus_a \cdots \oplus_a \mathfrak{W}^{(k)}}$$

Can be performed and stored in

$$\boxed{\mathcal{O}(N \ln^2 N) \text{ operations}}$$

Doesn’t have the complexity of an eigenspace decomposition



Basis functions look like and behave like wavelets: Localized and can be used to compress the operator and locally analyze the solution space

$$\begin{tikzcd} H_0^1(\Omega) \arrow[rr, "\mathrm{div}(a\nabla\cdot)"] & & H^{-1}(\Omega) \\ u \arrow[rr, bend left, "\text{Reduced operator}"] & & g \arrow[d] \\ \mathbb{R}^m \ni u_m \arrow[u] & & g_m \in \mathbb{R}^m \arrow[ll, "\textbf{Inverse Problem}"'] \end{tikzcd}$$

Numerical implementation requires computation with partial information.

$$\phi_1, \ldots, \phi_m \in L^2(\Omega)$$

$$u_m = \left(\int_\Omega \phi_1 u, \ldots, \int_\Omega \phi_m u\right)$$

$$u_m \in \mathbb{R}^m \xrightarrow{\boxed{\text{Missing information}}} u \in H_0^1(\Omega)$$

Discovery process

Identify underlying information game

$$\begin{cases} -\operatorname{div}(a\nabla u) = g \text{ in } \Omega, \\ \qquad\qquad\qquad u = 0 \text{ on } \partial\Omega, \end{cases}$$

Measurement functions: $\phi_1, \ldots, \phi_m \in L^2(\Omega)$

Player A

Chooses

$g \in L^2(\Omega)$

$\|g\|_{L^2(\Omega)} \leq 1$

Player B

Sees $\int_\Omega u\phi_1, \ldots, \int_\Omega u\phi_m$

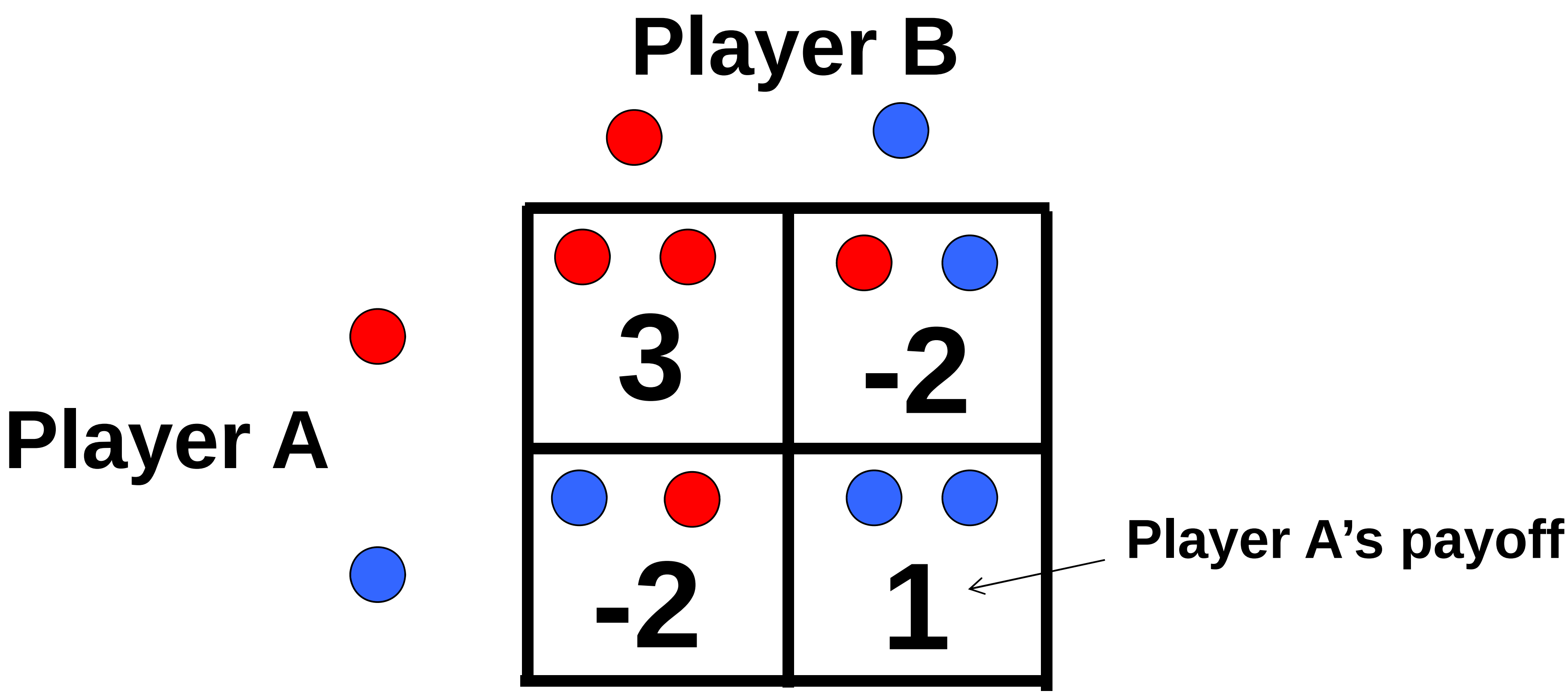
Chooses $u^* \in L^2(\Omega)$

Max *Min*

$$\left\| u - u^* \right\|_a$$

$\|f\|_a^2 := \int_\Omega (\nabla f)^T a \nabla f$

Deterministic zero sum game



Player A & B both have a blue and a red marble

At the same time, they show each other a marble

How should A & B play the (repeated) game?

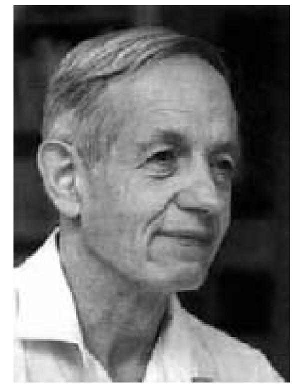
Game theory

Optimal strategies are mixed strategies

Optimal way to play is at random

		Player B	
		q	$1-q$
Player A	p	**3**	**-2**
	$1-p$	**-2**	**1**

John Von Neumann



John Nash

A's expected payoff

$$= 3pq + (1-p)(1-q) - 2p(1-q) - 2q(1-p)$$

$$= 1 - 3q + p(8q-3) \;= -\frac{1}{8} \qquad \text{for } q = \frac{3}{8}$$

Player A

Chooses
$g \in L^2(\Omega)$
$\|g\|_{L^2(\Omega)} \leq 1$

Player B

Sees $\int_\Omega u\phi_1, \ldots, \int_\Omega u\phi_m$
Chooses $u^* \in L^2(\Omega)$

$$\left\| u - u^* \right\|_a$$

Continuous game but as in decision theory under compactness it can be approximated by a finite game

Abraham Wald

The best strategy for A is to play at random

Player B's best strategy live in the Bayesian class of estimators

Player B's class of mixed strategies

Pretend that player A is choosing g at random

$$g \in L^2(\Omega) \iff \xi\text{: Random field}$$

$$\begin{cases} -\operatorname{div}(a\nabla u) = g \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \iff \begin{cases} -\operatorname{div}(a\nabla v) = \xi \text{ in } \Omega, \\ v = 0 \text{ on } \partial\Omega, \end{cases}$$

Player B's bet

$$u^*(x) := \mathbb{E}\Big[v(x) \Big| \int_\Omega v(y)\phi_i(y)\,dy = \int_\Omega u(y)\phi_i(y)\,dy, \forall i\Big]$$

Player's B optimal strategy?

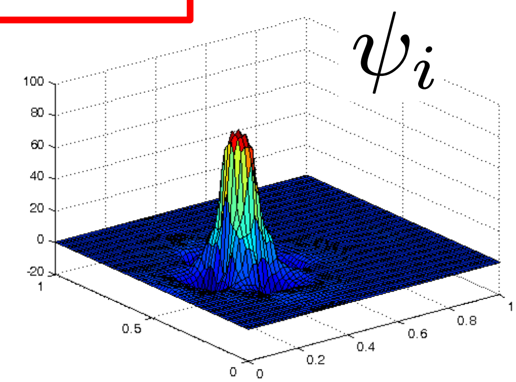
Player B's best bet? $\iff$ min max problem over distribution of ξ

Computational efficiency → $\boxed{\xi \sim \mathcal{N}(0,\Gamma)}$

Elementary gambles form deterministic basis functions for player B's bet

Theorem

$$\boxed{u^*(x) = \sum_{i=1}^{m} \psi_i(x) \int_\Omega u(y)\phi_i(y)\, dy}$$



Gamblets

ψ_i: Elementary gambles/bets

Player B's bet if $\int_\Omega u\phi_j = \delta_{i,j}$, $j = 1,\ldots,m$

$$\boxed{\psi_i(x) := \mathbb{E}_{\xi\sim\mathcal{N}(0,\Gamma)}\Big[v(x)\Big|\int_\Omega v(y)\phi_j(y)\,dy = \delta_{i,j},\ j\in\{1,\ldots,m\}\Big]}$$

What are these gamblets?

Depend on

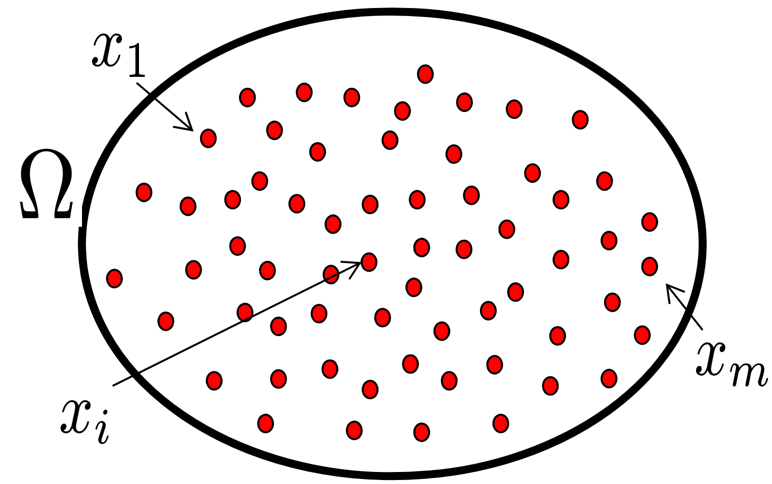
- Γ: Covariance function of ξ (Player B's decision)
- $(\phi_i)_{i=1}^m$: Measurements functions (rules of the game)

Example

[Owhadi, 2014] arXiv:1406.6668

$$\Gamma(x,y) = \delta(x-y)$$

$$\phi_i(x) = \delta(x-x_i)$$



$a = I_d \Longleftrightarrow \psi_i$: Polyharmonic splines

[Harder-Desmarais, 1972] [Duchon 1976, 1977,1978]

$a_{i,j} \in L^\infty(\Omega) \Longleftrightarrow \psi_i$: Rough Polyharmonic splines

[Owhadi-Zhang-Berlyand 2013]

What is Player B's best strategy?

What is Player B's best choice for

$$\Gamma(x,y)=\mathbb{E}\big[\xi(x)\xi(y)\big] \quad \textbf{?}$$

$$\boxed{\Gamma=\mathcal{L}} \quad \Longleftrightarrow \quad \int_\Omega \xi(x)f(x)\,dx \sim \mathcal{N}(0,\|f\|_a^2)$$

$$\|f\|_a^2 := \int_\Omega (\nabla f)^T a \nabla f$$

$$\Updownarrow$$

$$\boxed{\mathcal{L}=-\operatorname{div}(a\nabla\cdot)}$$

Why? $\Longleftrightarrow$ **See algebraic generalization**

The recovery is optimal (Galerkin projection)

Theorem If $\Gamma = \mathcal{L}$ then

$u^*(x)$ is the F.E. solution of (1) in $\mathrm{span}\{\mathcal{L}^{-1}\phi_i | i = 1, \ldots, m\}$

$$\boxed{\|u - u^*\|_a = \inf_{\psi \in \mathrm{span}\{\mathcal{L}^{-1}\phi_i : i \in \{1,\ldots,m\}\}} \|u - \psi\|_a}$$

$$\boxed{\mathcal{L} = -\,\mathrm{div}(a\nabla \cdot)}$$

(1)
$$\boxed{\begin{cases} -\,\mathrm{div}(a\nabla u) = g, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}}$$

Optimal variational properties

Theorem

$\sum_{i=1}^{m} w_i \psi_i$ minimizes $\|\psi\|_a$
over all ψ such that $\int_\Omega \phi_j \psi = w_j$ for $j = 1, \ldots, m$

Variational characterization

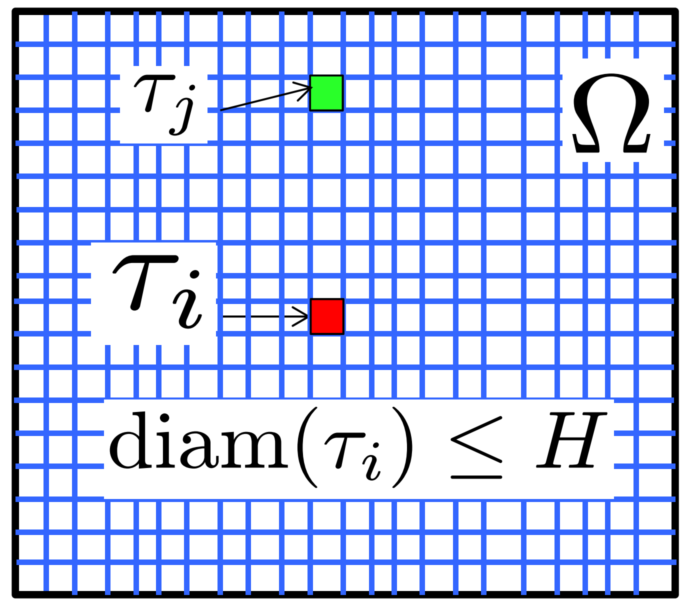
Theorem ψ_i: Unique minimizer of

$$\begin{cases} \text{Minimize} & \|\psi\|_a \\ \text{Subject to} & \psi \in H_0^1(\Omega) \text{ and } \int_\Omega \phi_j \psi = \delta_{i,j}, \quad j = 1, \ldots, m \end{cases}$$

Selection of measurement functions

Example Indicator functions of a Partition of Ω of resolution H

$$\boxed{\phi_i = 1_{\tau_i}}$$

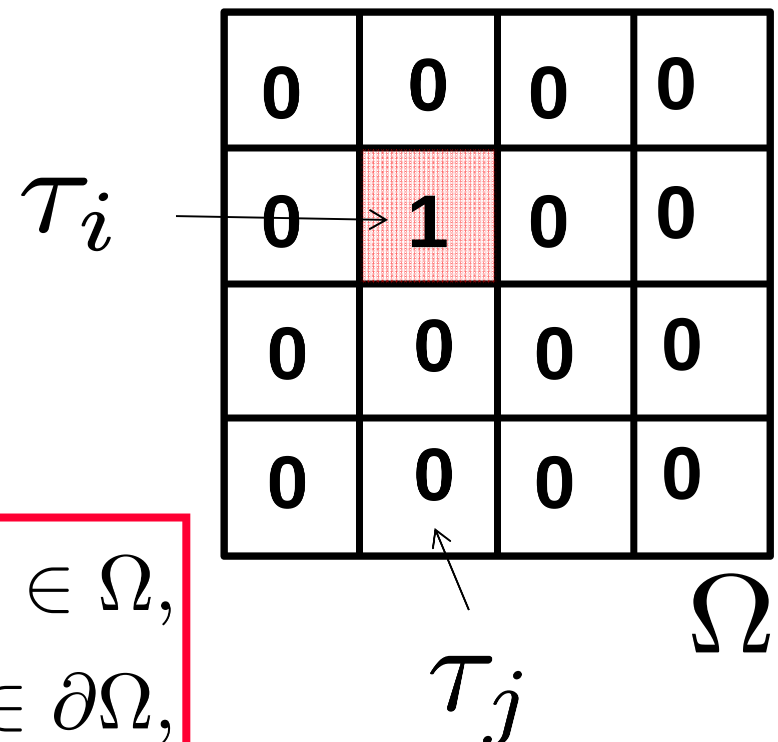


Theorem

$$\boxed{\|u - u^*\|_a \leq \frac{H}{\lambda_{\min}(a)} \|g\|_{L^2(\Omega)}}$$

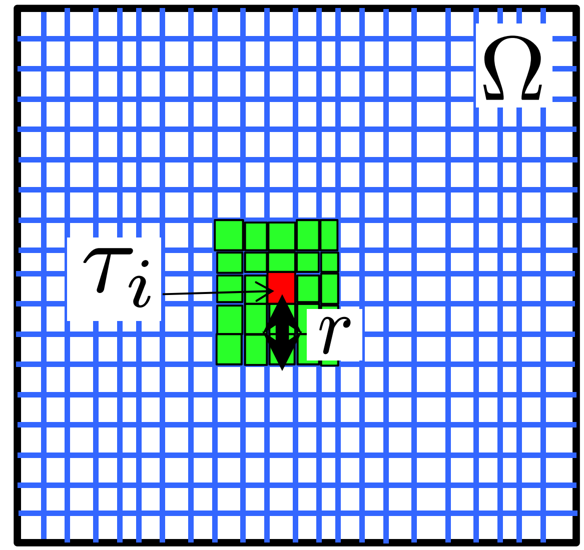
Elementary gamble

$\boxed{\psi_i}$ Your best bet on the value of u given the information that $\int_{\tau_i} u = 1$ and $\int_{\tau_j} u = 0$ for $j \neq i$



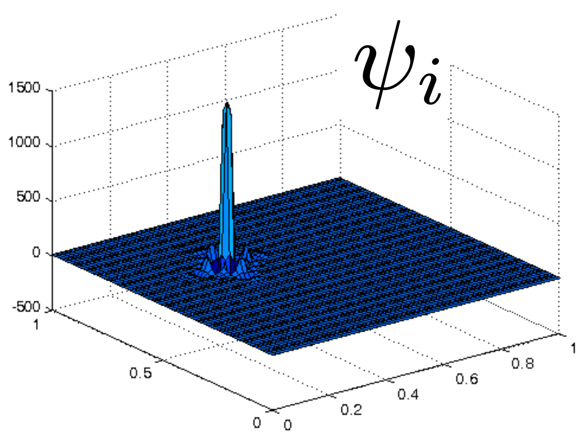
(1)
$$\boxed{\begin{cases} -\operatorname{div}(a\nabla u) = g, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}}$$

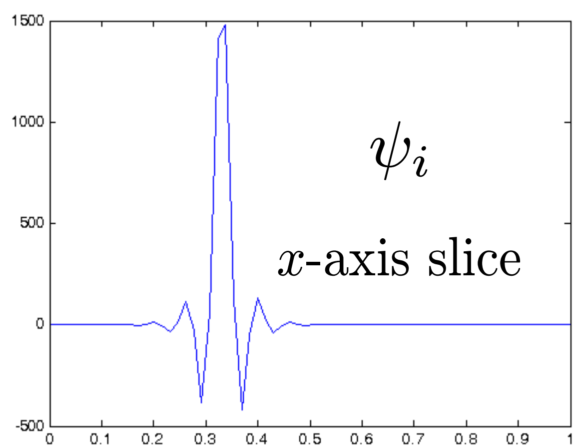
Exponential decay of gamblets

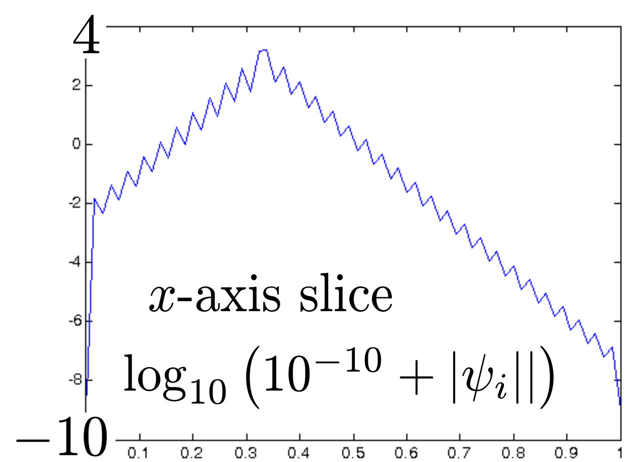


Theorem

$$\boxed{\int_{\Omega\cap(B(\tau_i,r))^c}(\nabla\psi_i)^T a\nabla\psi_i \le e^{-\frac{r}{lH}}\,\|\psi_i\|_a^2}$$



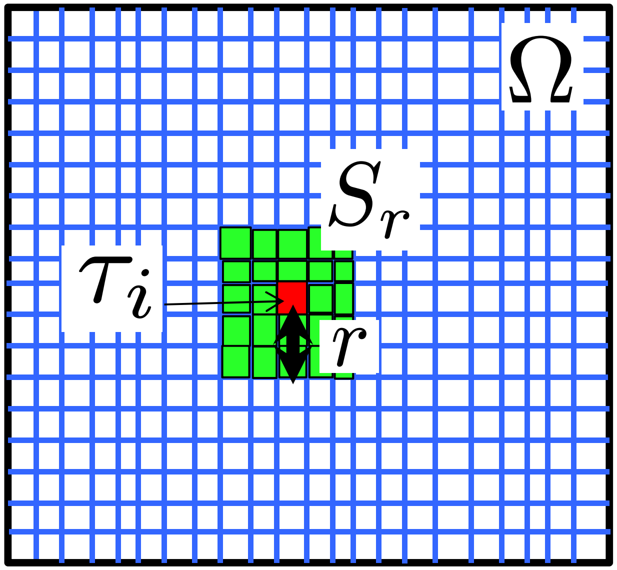




Localization of the computation of gamblets

$\psi_i^{\text{loc},r}$: Minimizer of

$$\begin{cases} \text{Minimize} & \|\psi\|_a \\ \text{Subject to} & \psi \in H^1_0(S_r) \text{ and } \int_{S_r} \phi_j \psi = \delta_{i,j} \\ & \qquad\qquad\qquad \text{for } \tau_j \in S_r \end{cases}$$



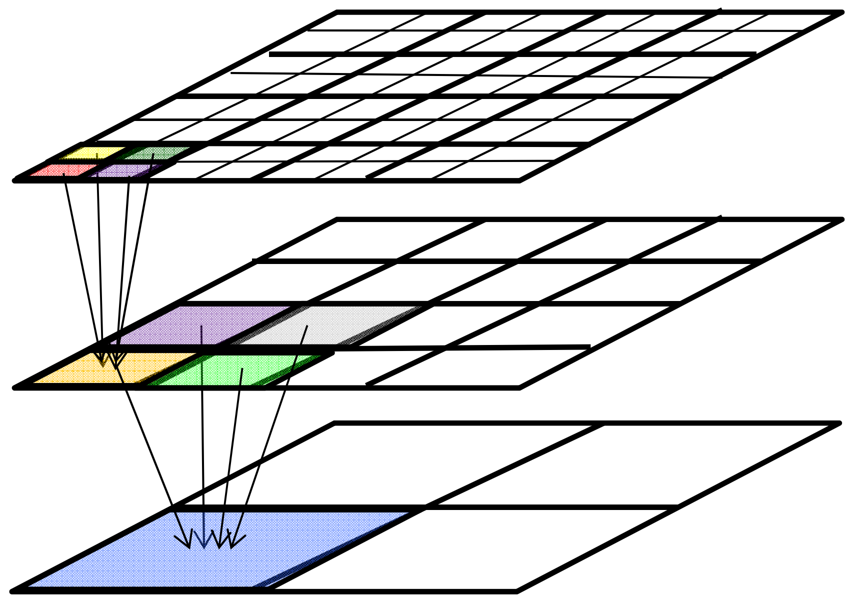
No loss of accuracy if localization $\sim H \ln \frac{1}{H}$

$$u^{*,\text{loc}}(x) = \sum_{i=1}^m \psi_i^{\text{loc,r}}(x) \int_\Omega u(y)\phi_i(y)\,dy$$

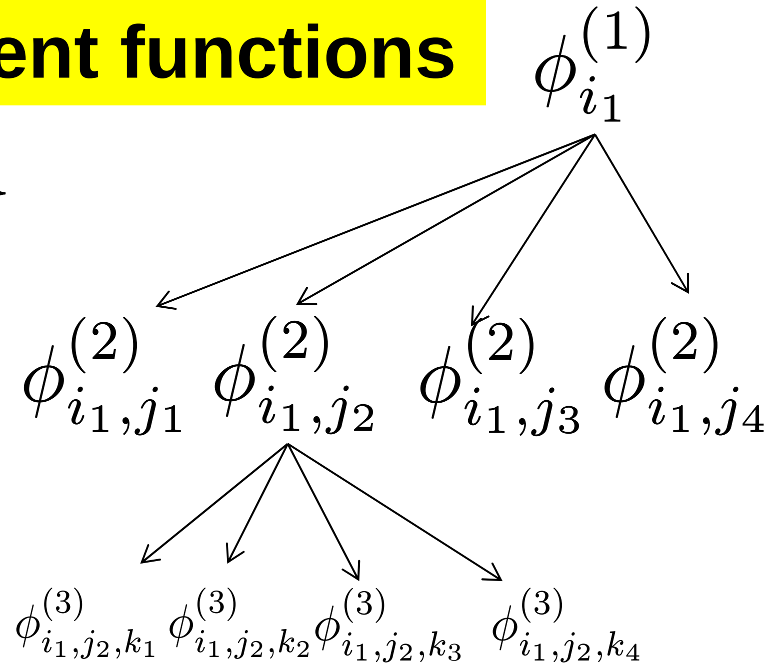
Theorem If $r \geq C H \ln \frac{1}{H}$

$$\|u - u^{*,\text{loc}}\|_a \leq \frac{1}{\sqrt{\lambda_{\min}(a)}} H \|g\|_{L^2(\Omega)}$$

Formulation of the hierarchical game



Hierarchy of nested Measurement functions

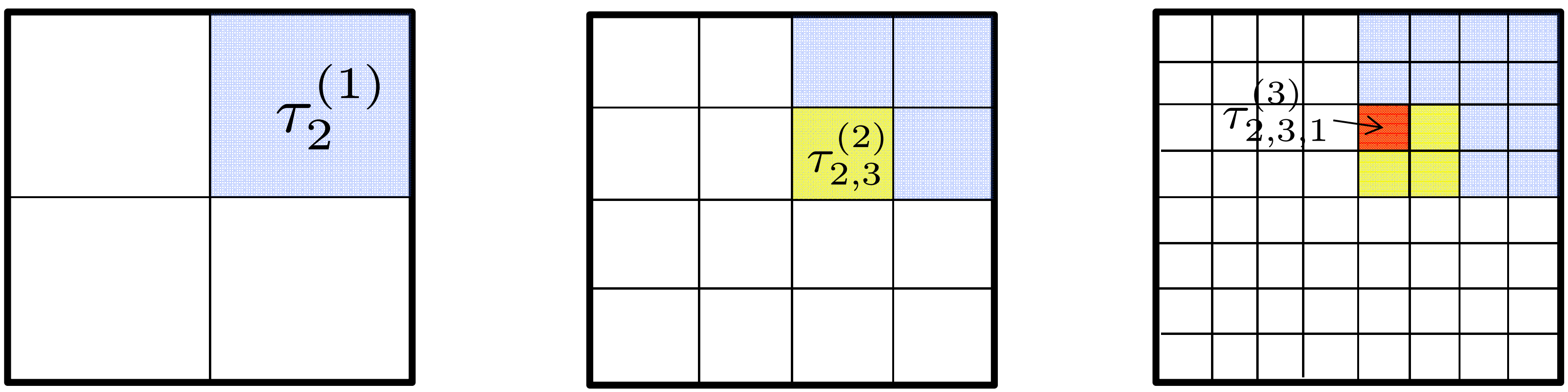


$$\phi^{(k)}_{i_1,\ldots,i_k} \quad \text{with } k \in \{1,\ldots,q\}$$

$$\boxed{\phi^{(k)}_i = \sum_j c_{i,j}\phi^{(k+1)}_{i,j}}$$

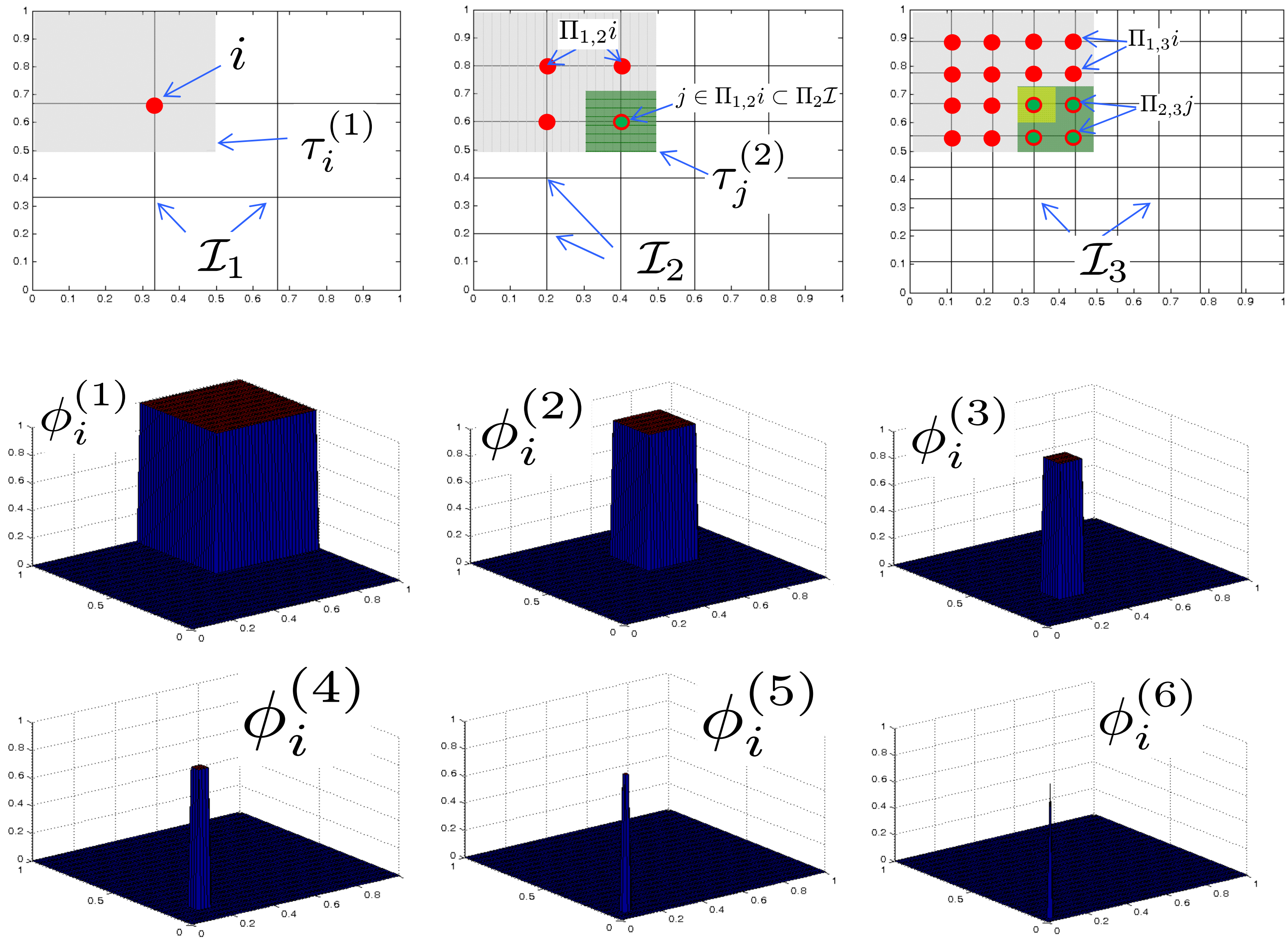
Example

$\phi^{(k)}_i$: Indicator functions of a hierarchical nested partition of Ω of resolution $H_k = 2^{-k}$



$$\phi^{(1)}_2 = 1_{\tau^{(1)}_2} \qquad \phi^{(2)}_{2,3} = 1_{\tau^{(2)}_{2,3}} \qquad \phi^{(3)}_{2,3,1} = 1_{\tau^{(3)}_{2,3,1}}$$

In the discrete setting simply aggregate elements (as in algebraic multigrid)



Formulation of the hierarchy of games

$$\begin{cases} -\operatorname{div}(a\nabla u) = g \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

Player A

Chooses

$g \in L^2(\Omega)$

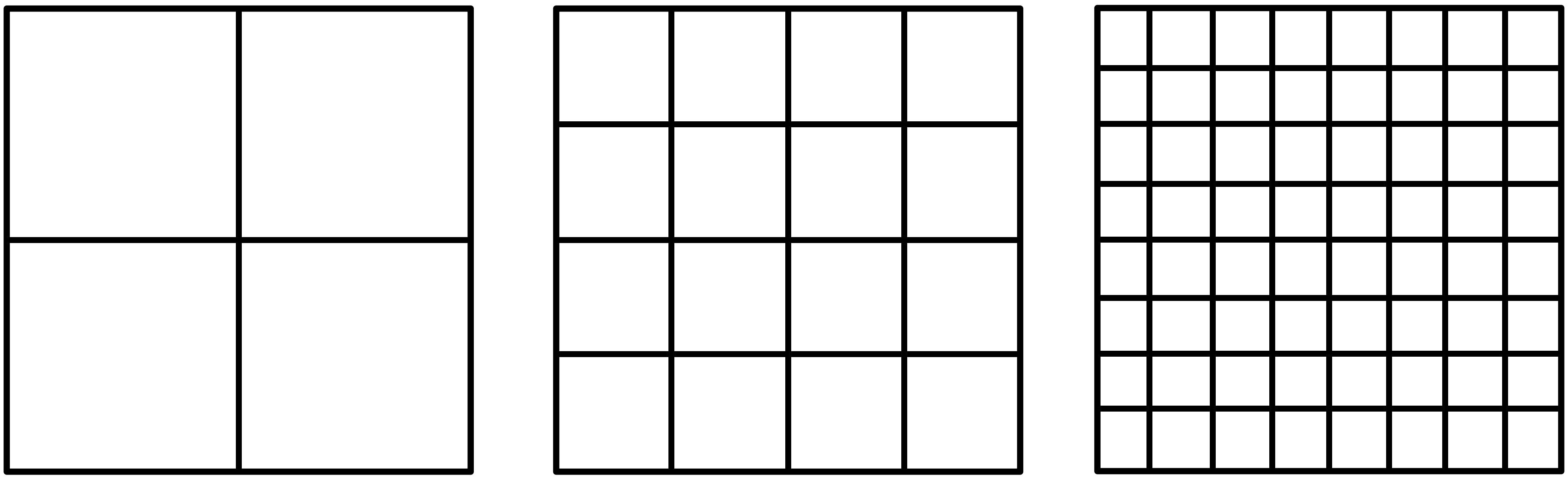
$\|g\|_{L^2(\Omega)} \leq 1$

Player B

Sees $\{\int_\Omega u\phi_i^{(k)},\ i \in \mathcal{I}_k\}$

Must predict

u and $\{\int_\Omega u\phi_j^{(k+1)},\ j \in \mathcal{I}_{k+1}\}$



Player B's best strategy

$$\xi \sim \mathcal{N}(0, \mathcal{L})$$

$$\boxed{\begin{cases} -\operatorname{div}(a\nabla u) = g \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}} \Longleftrightarrow \boxed{\begin{cases} -\operatorname{div}(a\nabla v) = \xi \text{ in } \Omega, \\ v = 0 \text{ on } \partial\Omega, \end{cases}}$$

Player B's bets

$$\boxed{u^{(k)}(x) := \mathbb{E}\Big[v(x)\Big| \int_\Omega v(y)\phi_i^{(k)}(y)\,dy = \int_\Omega u(y)\phi_i^{(k)}(y)\,dy,\ i \in \mathcal{I}_k\Big]}$$

The sequence of approximations forms a martingale under the mixed strategy emerging from the game

$$\mathcal{F}_k = \sigma\Big(\int_\Omega v\phi_i^{(k)},\ i \in \mathcal{I}_k\Big) \qquad \boxed{v^{(k)}(x) := \mathbb{E}\big[v(x)\big|\mathcal{F}_k\big]}$$

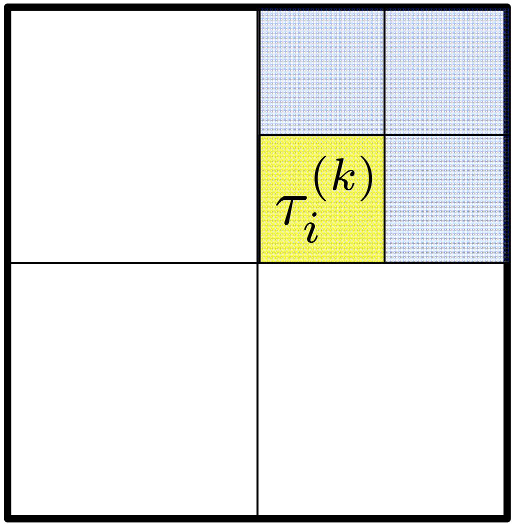
Theorem

$$\boxed{\mathcal{F}_k \subset \mathcal{F}_{k+1}}$$

$$\boxed{v^{(k)}(x) := \mathbb{E}\big[v^{(k+1)}(x)\big|\mathcal{F}_k\big]}$$

Accuracy of the recovery

Theorem

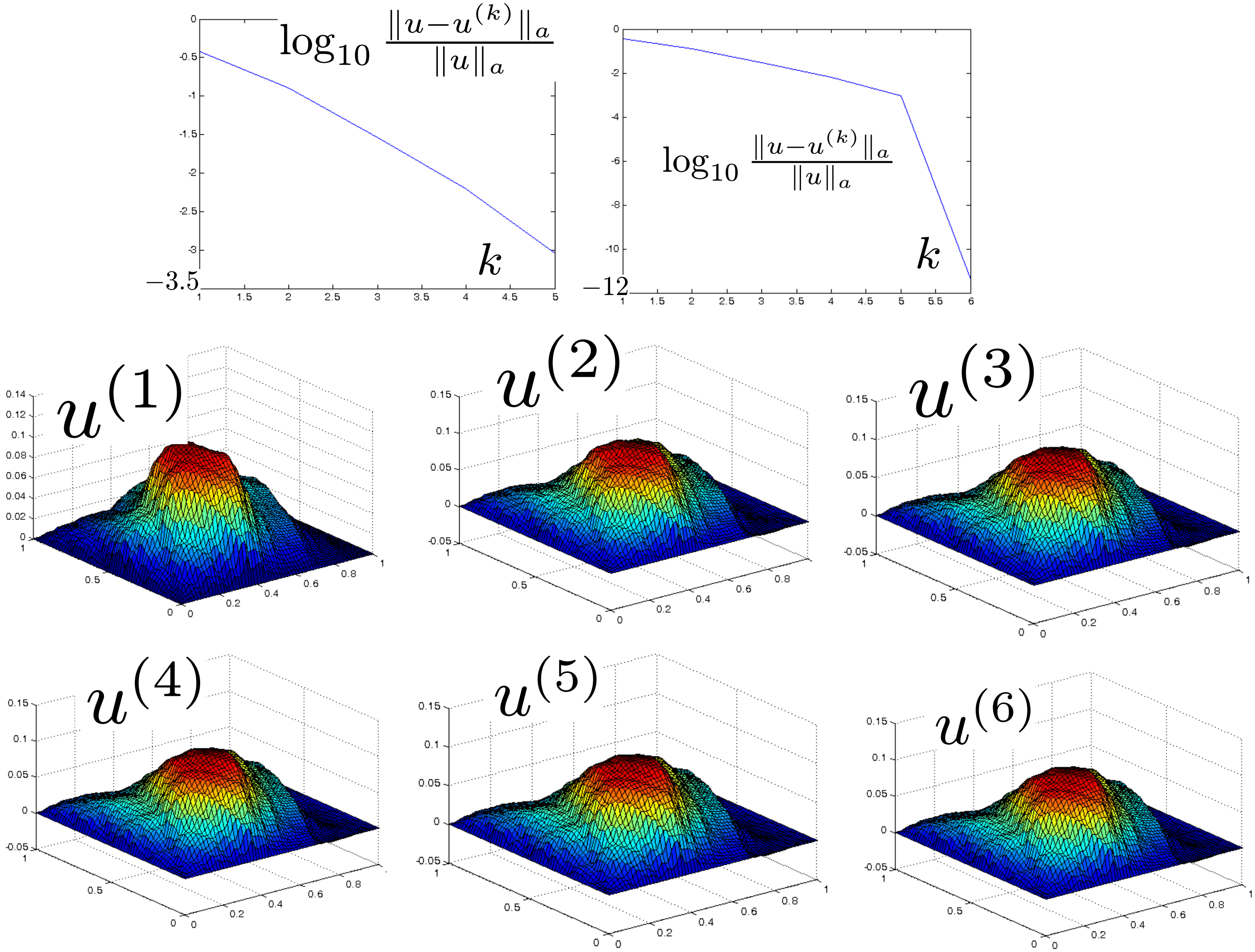


$$\boxed{\|u - u^{(k)}\|_a \leq \frac{H_k}{\lambda_{\min}(a)} \|g\|_{L^2(\Omega)}}$$

$$\boxed{H_k := \max_i \operatorname{diam}(\tau_i^{(k)})}$$

$$\phi_i^{(k)} = 1_{\tau_i^{(k)}} \qquad \operatorname{diam}(\tau_i^{(k)}) \leq H_k$$

In a discrete setting the last step of the game recovers the solution to numerical precision



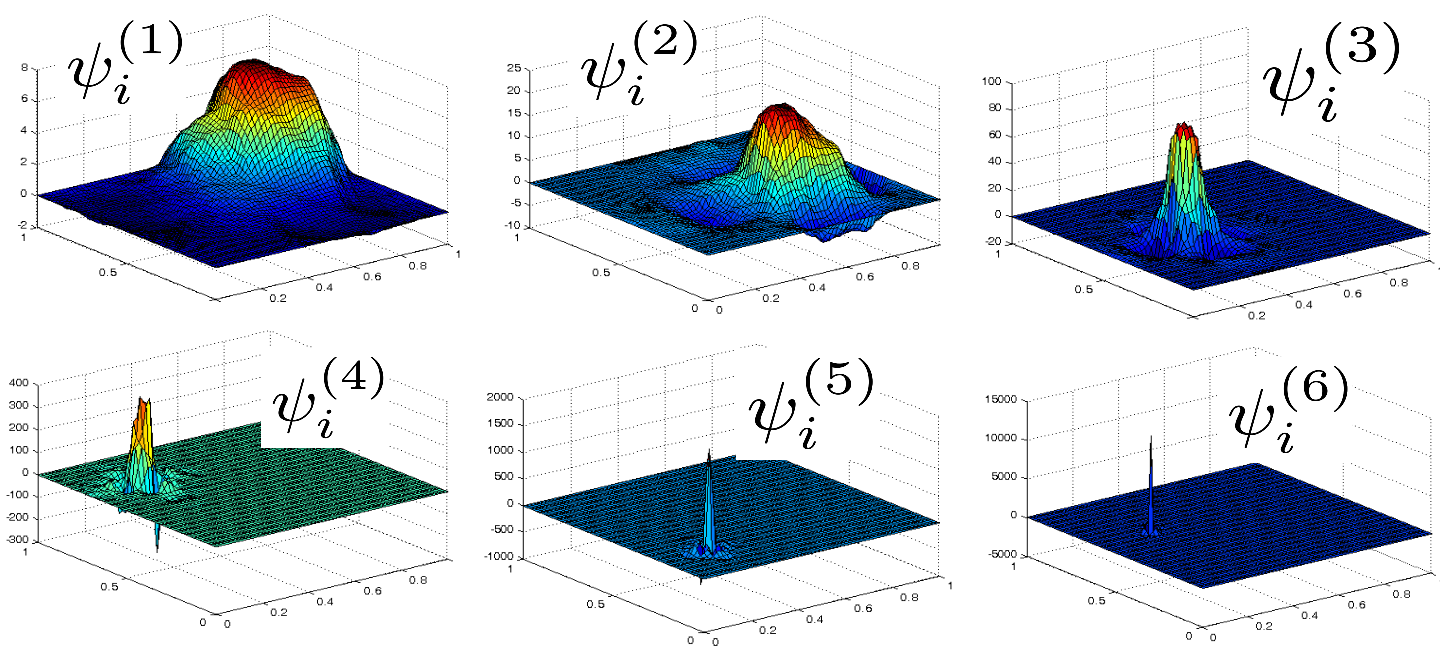
Gamblets **Elementary gambles form a hierarchy of deterministic basis functions for player B's hierarchy of bets**

Theorem

$$u^{(k)}(x)=\sum_i \psi_i^{(k)}(x)\int_\Omega u(y)\phi_i^{(k)}(y)\,dy$$

$\psi_i^{(k)}$: Elementary gambles/bets at resolution $H_k=2^{-k}$

$$\psi_i^{(k)}(x):=\mathbb{E}\Big[v(x)\Big|\int_\Omega v(y)\phi_j^{(k)}(y)\,dy=\delta_{i,j},\, j\in\mathcal{I}_k\Big]$$



Gamblets are nested

$$\boxed{\mathfrak{V}^{(k)} := \operatorname{span}\{\psi_i^{(k)}, i \in \mathcal{I}_k\}}$$

$$\psi_{i_1}^{(1)}$$

$$\psi_{i_1,j_1}^{(2)} \quad \psi_{i_1,j_2}^{(2)} \quad \psi_{i_1,j_3}^{(2)} \quad \psi_{i_1,j_4}^{(2)}$$

$$\psi_{i_1,j_2,k_1}^{(3)} \quad \psi_{i_1,j_2,k_2}^{(3)} \quad \psi_{i_1,j_2,k_3}^{(3)} \quad \psi_{i_1,j_2,k_4}^{(3)}$$

Theorem

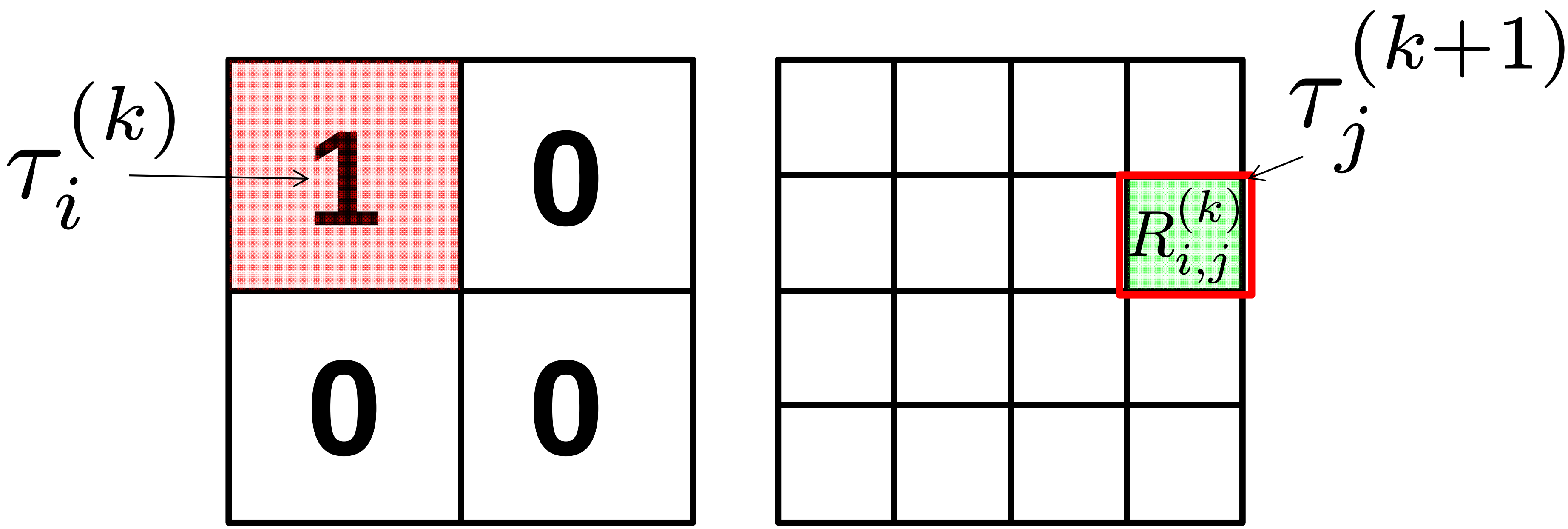
$$\boxed{\mathfrak{V}^{(k)} \subset \mathfrak{V}^{(k+1)}}$$

$$\boxed{\psi_i^{(k)}(x) = \sum_{j \in \mathcal{I}_{k+1}} R_{i,j}^{(k)} \psi_j^{(k+1)}(x)}$$

Interpolation/Prolongation operator

$$R_{i,j}^{(k)} = \mathbb{E}\left[\int_\Omega v(y)\phi_j^{(k+1)}(y)\,dy \,\middle|\, \int_\Omega v(y)\phi_l^{(k)}(y)\,dy = \delta_{i,l},\ l \in \mathcal{I}_k\right]$$

$\boxed{R_{i,j}^{(k)}}$ Your best bet on the value of $\int_{\tau_j^{(k+1)}} u$ given the information that $\int_{\tau_i^{(k)}} u = 1$ and $\int_{\tau_l} u = 0$ for $l \neq i$



At this stage you can finish with classical multigrid

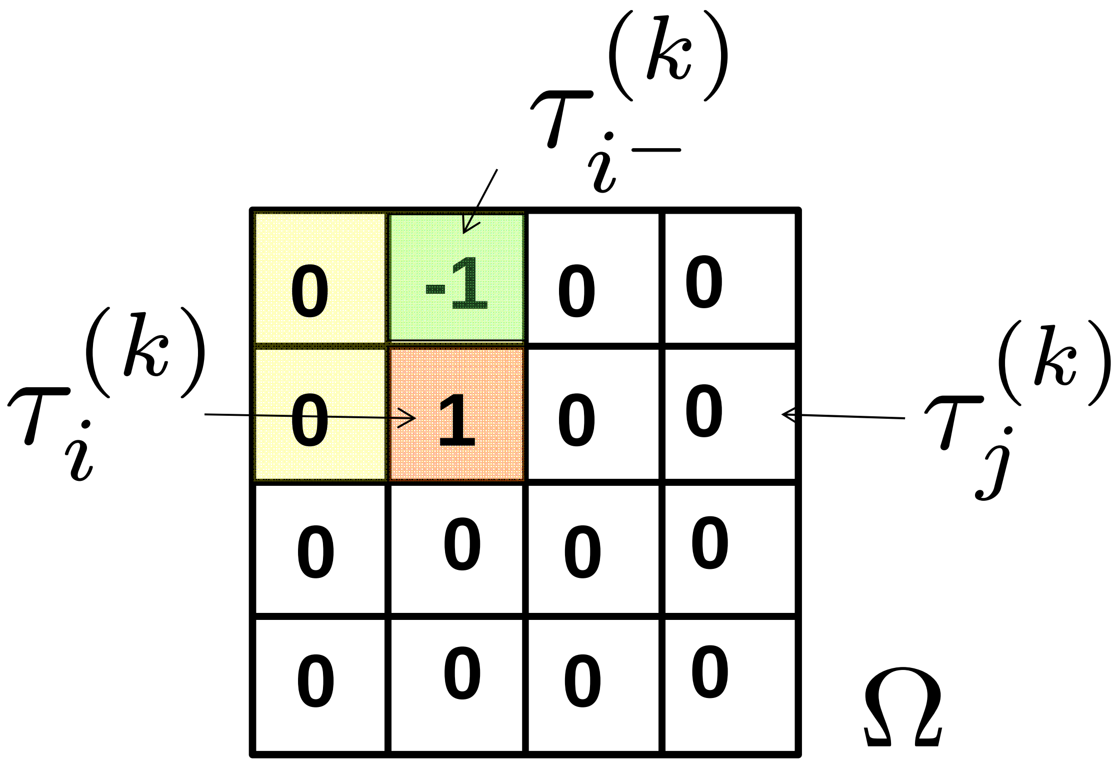
But we want multiresolution decomposition

Elementary gamble

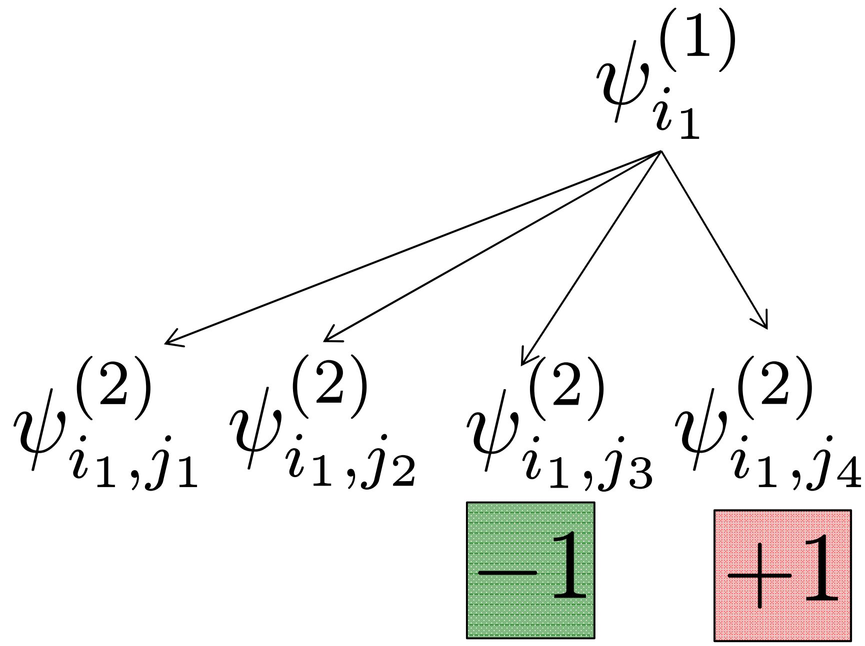
$$\boxed{\chi_i^{(k)}}$$

Your best bet on the value of u given the information that

$$\int_{\tau_i^{(k)}} u = 1,\ \int_{\tau_{i-}^{(k)}} u = -1 \text{ and } \int_{\tau_j^{(k)}} u = 0 \text{ for } j \neq i$$

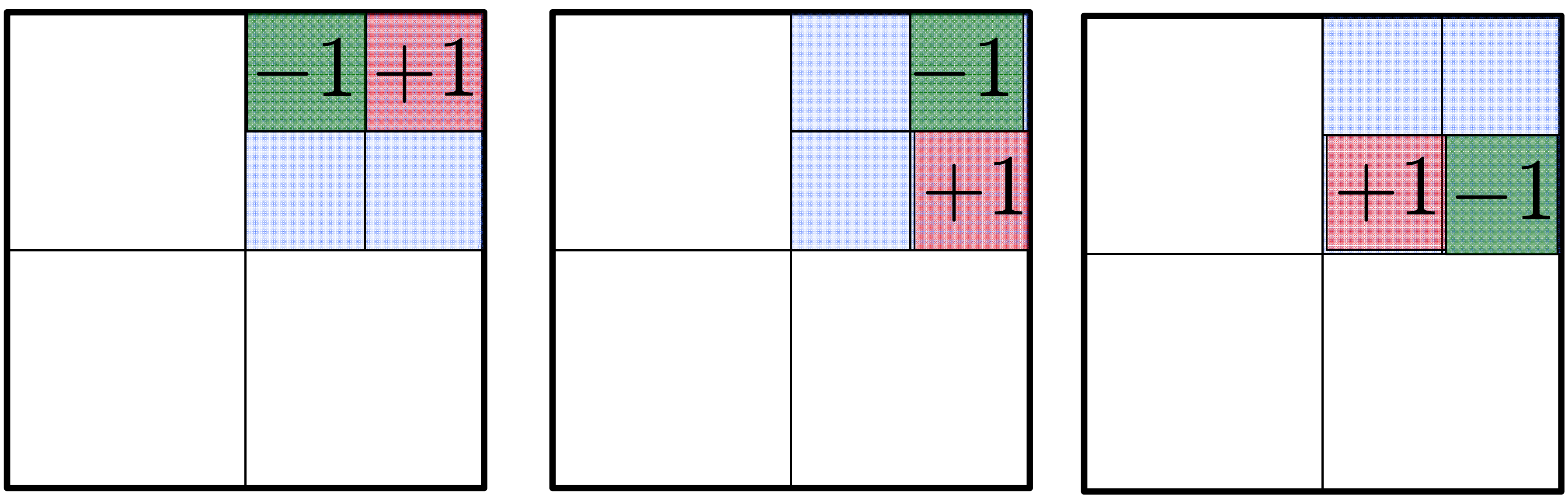


$$\boxed{\chi_i^{(k)} = \psi_i^{(k)} - \psi_{i^-}^{(k)}}$$

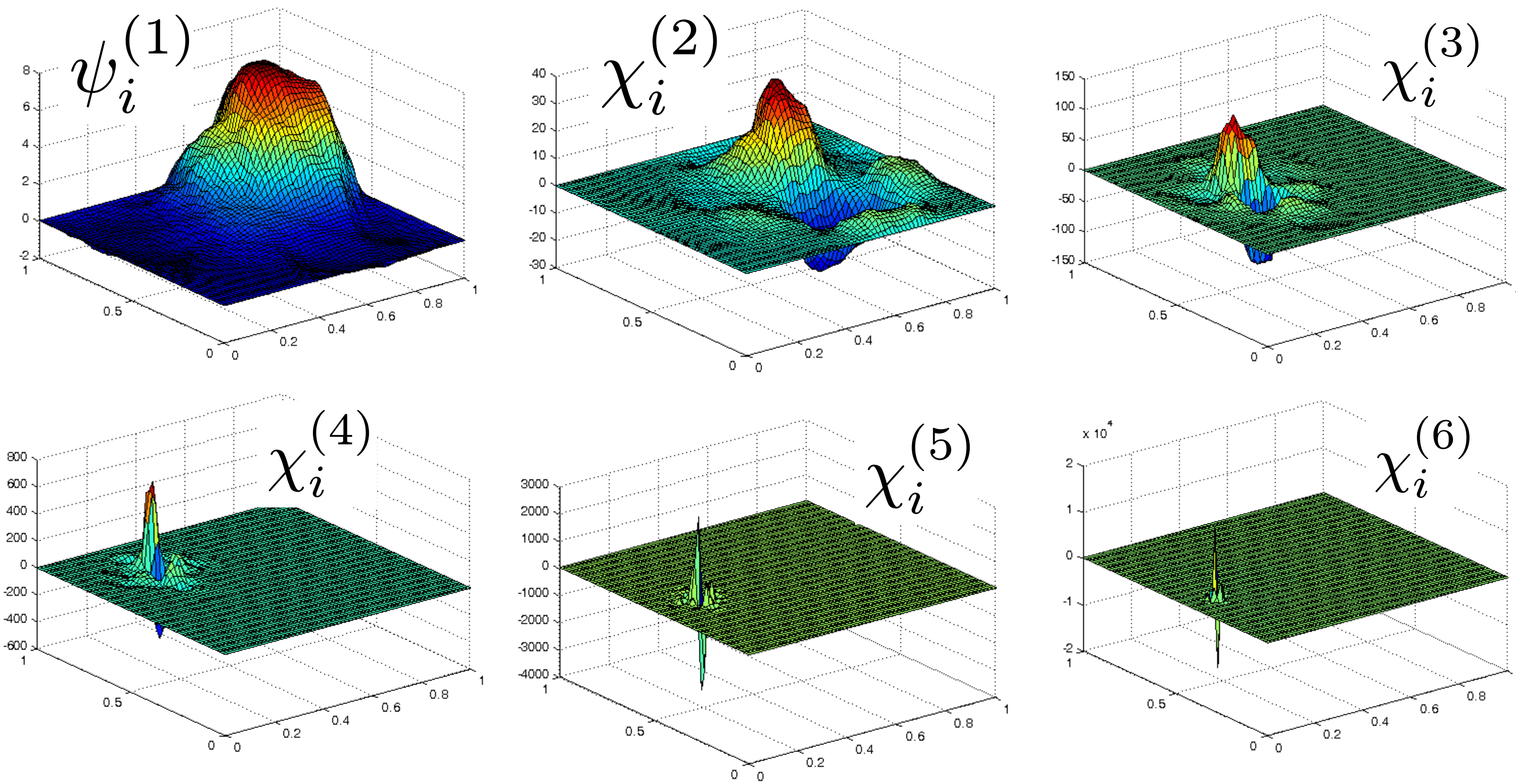


$$i = (i_1, \ldots, i_{k-1}, i_k)$$

$$i^- = (i_1, \ldots, i_{k-1}, i_k - 1)$$



$$\boxed{\chi_i^{(k)} = \psi_i^{(k)} - \psi_{i^-}^{(k)}}$$



Multiresolution decomposition of the solution space

$$\boxed{\mathfrak{V}^{(k)} := \operatorname{span}\{\psi_i^{(k)}, i \in \mathcal{I}_k\}}$$

$$\boxed{\mathfrak{W}^{(k)} := \operatorname{span}\{\chi_i^{(k)}, i\}}$$

$\mathfrak{W}^{(k+1)}$: Orthogonal complement of $\mathfrak{V}^{(k)}$ in $\mathfrak{V}^{(k+1)}$ with respect to $< \psi, \chi >_a := \int_\Omega (\nabla \psi)^T a \nabla \chi$

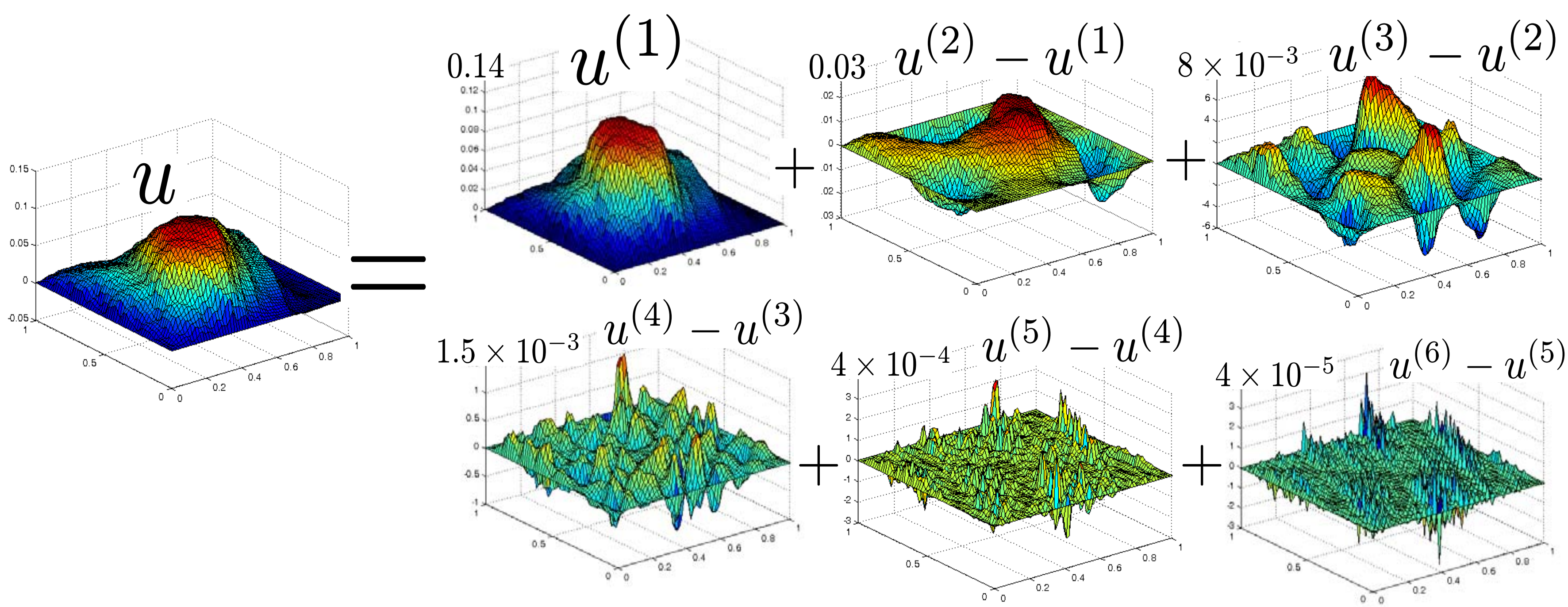
Theorem

$$\boxed{H_0^1(\Omega) = \mathfrak{V}^{(1)} \oplus_a \mathfrak{W}^{(2)} \oplus_a \cdots \oplus_a \mathfrak{W}^{(k)} \oplus_a \cdots}$$

Multiresolution decomposition of the solution

Theorem

$$u^{(k+1)} - u^{(k)} = \text{F.E. sol. of PDE in } \mathfrak{W}^{(k+1)}$$



Subband solutions $u^{(k+1)} - u^{(k)}$ can be computed independently

Uniformly bounded condition numbers

$$A_{i,j}^{(k)} := \langle \psi_i^{(k)}, \psi_j^{(k)} \rangle_a \qquad B_{i,j}^{(k)} := \langle \chi_i^{(k)}, \chi_j^{(k)} \rangle_a$$

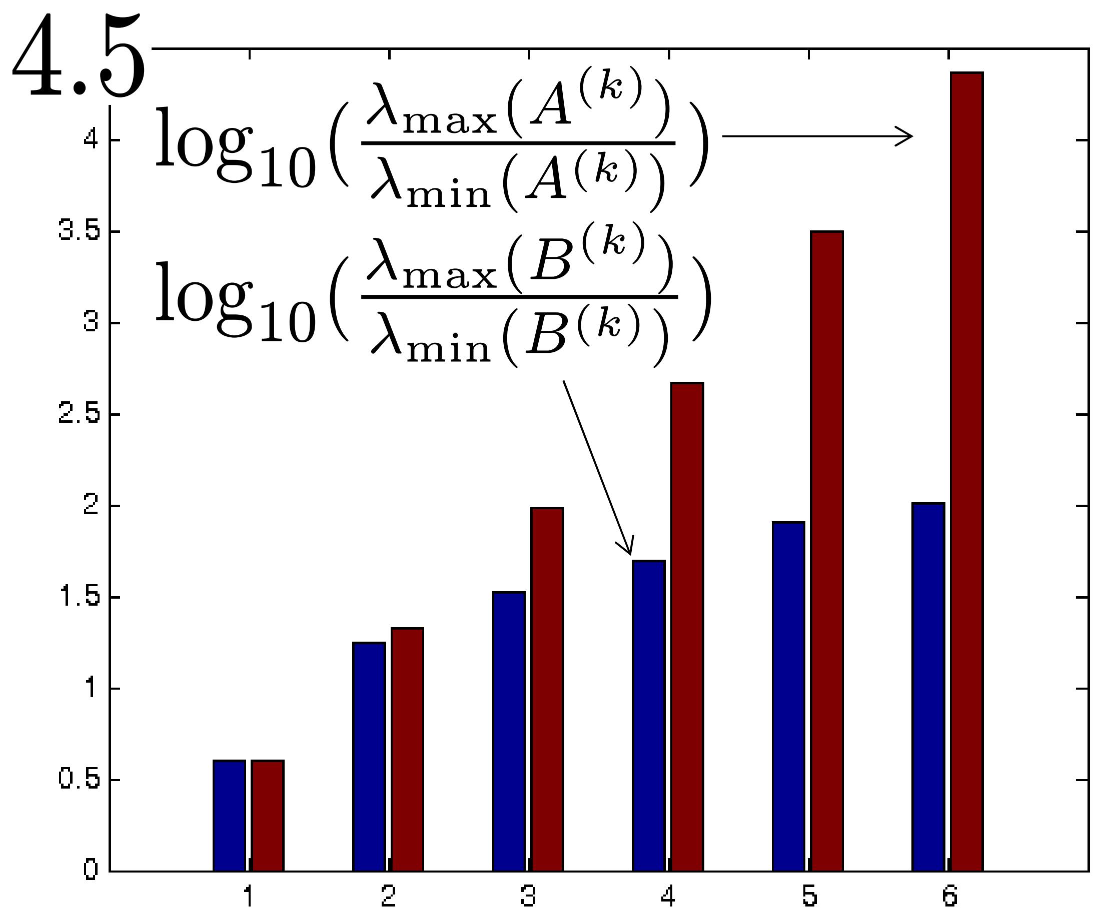
Theorem

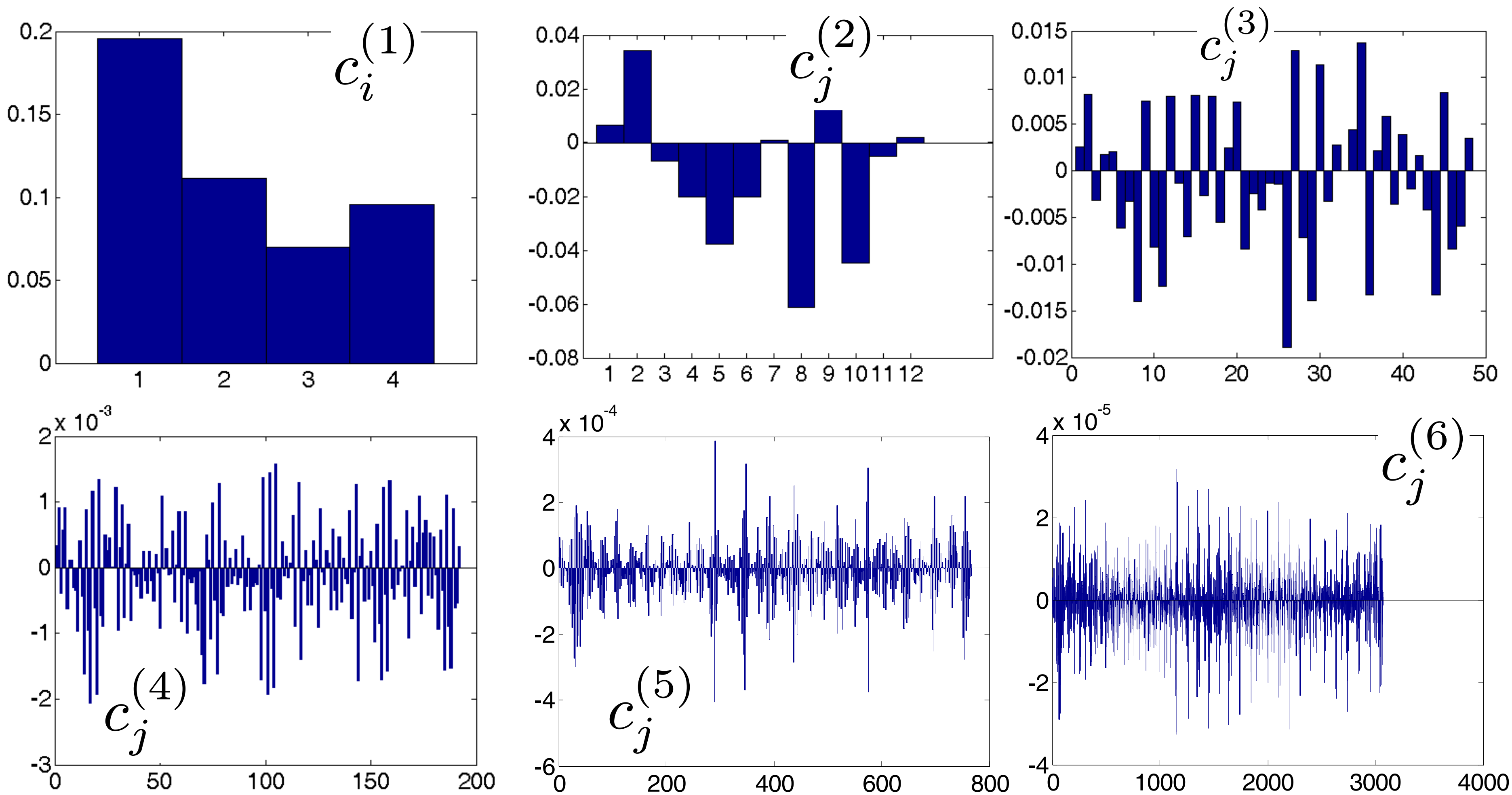
$$\boxed{\frac{\lambda_{\max}(B^{(k)})}{\lambda_{\min}(B^{(k)})} \leq C}$$

⇓

Just relax!

In $v \in \mathfrak{W}^{(k)}$
to get
$u^{(k)} - u^{(k-1)}$



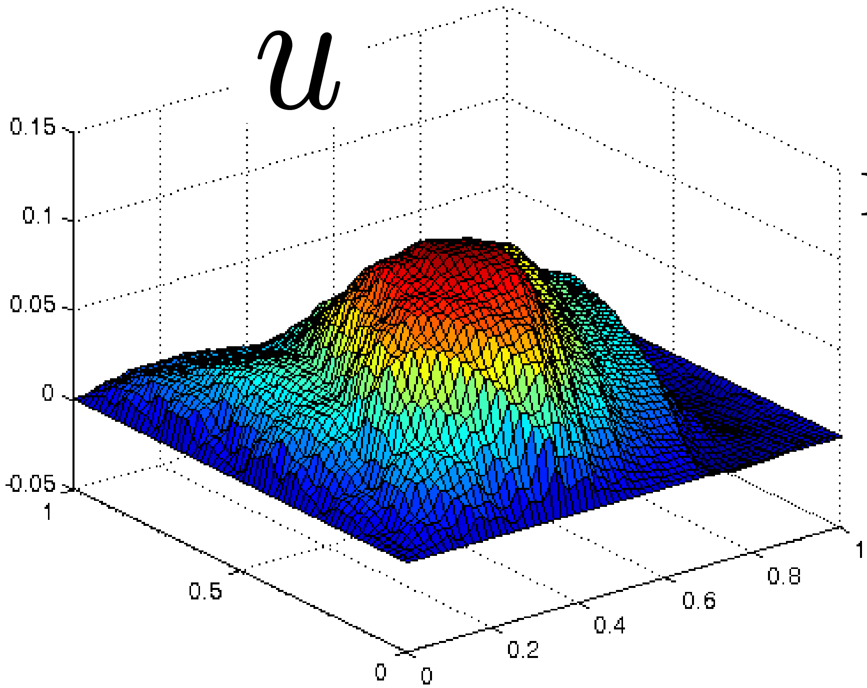


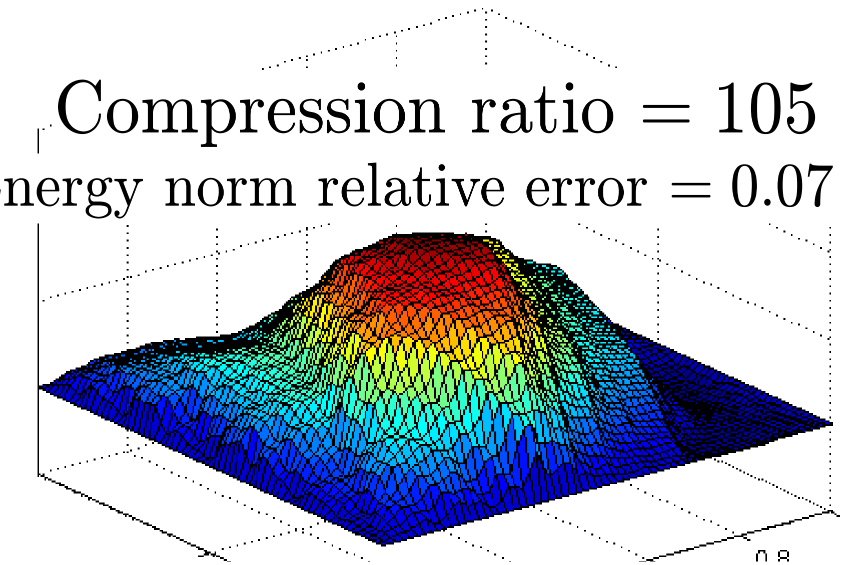
$$u = \sum_i c_i^{(1)} \frac{\psi_i^{(1)}}{\|\psi_i^{(1)}\|_a} + \sum_{k=2}^{q} \sum_j c_j^{(k)} \frac{\chi_j^{(k)}}{\|\chi_j^{(k)}\|_a}$$

Coefficients of the solution in the gamblet basis

Operator Compression

Gamblets behave like wavelets but they are adapted to the PDE and can compress its solution space





Throw 99% of the coefficients

Fast gamblet transform

$$\boxed{\mathcal{O}(N\ln^2 N)\ \text{complexity}}$$

Nesting

$$\boxed{A^{(k)}=(R^{(k,k+1)})^T A^{(k+1)} R^{(k,k+1)}}$$

Level(k) gamblets and stiffness matrices can be computed from level(k+1) gamblets and stiffness matrices

Well conditioned linear systems

Underlying linear systems have uniformly bounded condition numbers

$$\boxed{\psi_i^{(k)}=\psi_{(i,1)}^{(k+1)}+\sum_j C_{i,j}^{(k+1),\chi}\chi_j^{(k+1)}}$$

$$\boxed{C^{(k+1),\chi}=(B^{(k+1)})^{-1}Z^{(k+1)}}$$

$$Z_{j,i}^{(k+1)}:=-(e_j^{(k+1)}-e_{j^-}^{(k+1)})^T A^{(k+1)} e_{(i,1)}^{(k+1)}$$

Localization

The nested computation can be localized without compromising accuracy or condition numbers

$$
\begin{tikzcd}
\varphi_i, A^h, M^h \arrow[r] & \psi_i^{(q)},\ A^{(q)} \arrow[r] \arrow[d] & \chi_i^{(q)}, B^{(q)} \arrow[r] \arrow[dl] & u^{(q)} - u^{(q-1)} \\
& \psi_i^{(q-1)},\ A^{(q-1)} \arrow[r] & \chi_i^{(q-1)}, B^{(q-1)} \arrow[r] & u^{(q-1)} - u^{(q-2)} \\
& \vdots & \vdots & \vdots \\
& \psi_i^{(3)},\ A^{(3)} \arrow[r] \arrow[d] & \chi_i^{(3)}, B^{(3)} \arrow[r] \arrow[dl] & u^{(3)} - u^{(2)} \\
& \psi_i^{(2)},\ A^{(2)} \arrow[r] \arrow[d] & \chi_i^{(2)}, B^{(2)} \arrow[r] \arrow[dl] & u^{(2)} - u^{(1)} \\
& \psi_i^{(1)},\ A^{(1)} \arrow[rr] & & u^{(1)}
\end{tikzcd}
$$

Parallel operating diagram both in space and in frequency