Periodic-orbit evaluation of a spectral statistic of quantum graphs without the semiclassical limit

Jon Harrison[1] and Tori Hudgins[2]

[1]Baylor University, [2]University Dallas

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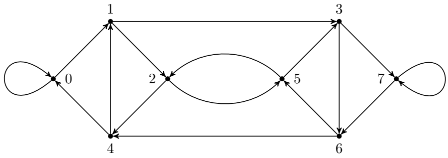
Dynamical approach to spectral statistics

- '71 Gutzwiller's trace formula for the density of states in the semiclassical limit.
- '85 Berry - Diagonal approximation to the form factor using Hannay-Ozorio de Almeida sum rule.
- '99 Kottos and Smilansky - trace formula for the density of states of quantum graphs.
- '01 Sieber and Richter - 2nd order contribution to the small parameter asymptotics of the form factor from figure 8 orbits with one self-intersection.
- '03 Berkolaiko, Schanz and Whitney - 2nd and 3rd order contributions on quantum graphs.
- '04 Müller, Heusler, Braun, Haake and Altland - all higher order contributions.

4-regular quantum graph model



- 4-regular directed graph: 2 incoming and 2 outgoing bonds at each vertex. (Always possible as admits Euler tour.)
- Assign length $L_b > 0$ to each bond, set of bond lengths incommensurate.
- To quantize assign 2×2 unitary vertex scattering matrix at each vertex,

$$\sigma^{(v)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} .$$

Characteristic polynomial

Combine vertex scattering matrices into an $B \times B$ matrix Σ,

$$\Sigma_{b,b'} = \begin{cases} \sigma^{(v)}_{b,b'} & v = t(b') = o(b) \\ 0 & \text{otherwise} \end{cases},$$

Quantum evolution op. $U(k) = \mathrm{e}^{\mathrm{i}kL}\Sigma$, with $L = \mathrm{diag}\{L_1, \dots, L_B\}$.

Characteristic polynomial

$$F_\xi(k) = \det(\xi \mathrm{I} - U(k)) = \sum_{n=0}^{B} a_n \, \xi^{B-n}$$

- Spectrum corresponds to roots of $F_1(k) = 0$.
- Riemann-Siegel lookalike formula $a_B = a_B a^*_{B-n}$.

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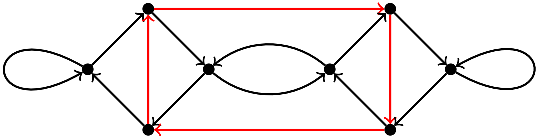
Periodic orbits

- A *periodic orbit* $\gamma = (b_1, \dots, b_m)$ is an equivalence class of closed paths under cyclic shifts.
- A *primitive periodic orbit* is a periodic orbit that is not a repetition of a shorter orbit.
- *Topological length* of γ is m.
- *Metric length* of γ is $L_\gamma = \sum_{b_j \in \gamma} L_{b_j}$.
- *Stability amplitude* is $A_\gamma = \Sigma_{b_2 b_1} \Sigma_{b_3 b_2} \dots \Sigma_{b_m b_{m-1}} \Sigma_{b_1 b_m}$.

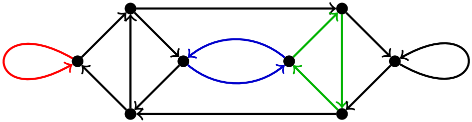
Example: primitive periodic orbit with 4 bonds.



Pseudo orbits

- A *pseudo orbit* $\tilde{\gamma} = \{\gamma_1, \dots, \gamma_M\}$ is a set of periodic orbits.
- A *primitive pseudo orbit* (PPO) $\bar{\gamma}$ is a set of distinct primitive periodic orbits.
- $m_{\bar{\gamma}} = M$ no. of periodic orbits in $\bar{\gamma}$.
- $\mathcal{P}^n$ set of PPO with n bonds.
- *Metric length* $L_{\tilde{\gamma}} = \sum_{j=1}^{M} L_{\gamma_j}$.
- *Stability amplitude* $A_{\tilde{\gamma}} = \prod_{j=1}^{M} A_{\gamma_j}$.

Example: PPO with 6 bonds consisting of $m_{\bar{\gamma}} = 3$ distinct primitive periodic orbits.



Theorem (Band,H.,Joyner)

Coefficients of the characteristic polynomial $F_\xi(k)$ are given by,

$$a_n = \sum_{\bar{\gamma} \in \mathcal{P}^n} (-1)^{m_{\bar{\gamma}}} A_{\bar{\gamma}} e^{\mathrm{i}kL_{\bar{\gamma}}} \ .$$

- Expand $\det(\xi \mathrm{I} - U(k))$ as a sum over permutations.
- A permutation $\rho \in S_B$ can contribute iff $\rho(b)$ is connected to b for all b in ρ, i.e $t(b) = o(\rho(b))$.
- Representing ρ as a product of disjoint cycles each cycle is a primitive periodic orbit.

Variance of coefficients of the characteristic polynomial

$$\langle a_n \rangle = \sum_{\bar{\gamma} |\, E_{\bar{\gamma}}=n} (-1)^{m_{\bar{\gamma}}} A_{\bar{\gamma}} \lim_{K\to\infty} \frac{1}{K} \int_0^K \mathrm{e}^{\mathrm{i}kL_{\bar{\gamma}}} \mathrm{d}k = \begin{cases} 1 & n=0 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \langle |a_n|^2 \rangle &= \sum_{\bar{\gamma}\in\mathcal{P}^n} (-1)^{m_{\bar{\gamma}}+m_{\bar{\gamma}'}} A_{\bar{\gamma}} \bar{A}_{\bar{\gamma}'} \lim_{K\to\infty} \frac{1}{K} \int_0^K \mathrm{e}^{\mathrm{i}k(L_{\bar{\gamma}}-L_{\bar{\gamma}'})} \mathrm{d}k \\ &= \sum_{\bar{\gamma},\bar{\gamma}'\in\mathcal{P}^n} (-1)^{m_{\bar{\gamma}}+m_{\bar{\gamma}'}} A_{\bar{\gamma}} \bar{A}_{\bar{\gamma}'} \, \delta_{L_{\bar{\gamma}},L_{\bar{\gamma}'}} \\ &= \sum_{\bar{\gamma}\in\mathcal{P}^n} C_{\bar{\gamma}} \qquad (1) \end{aligned}$$

$$C_{\bar{\gamma}} = \sum_{\bar{\gamma}'\in\mathcal{P}_{\bar{\gamma}}} (-1)^{m_{\bar{\gamma}}+m_{\bar{\gamma}'}} A_{\bar{\gamma}} \bar{A}_{\bar{\gamma}'} \qquad (2)$$

where $\mathcal{P}_{\bar{\gamma}}$ is the set of PPO length $L_{\bar{\gamma}}$.

- '99 Variance of coeffs of the characteristic polynomial of graphs – Kottos and Smilansky.
- '00 Spectral statistics of binary graphs – Tanner.
- '02 Variance of coeffs of characteristic polynomial of binary graphs via permanent of transition matrix – Tanner.
- '19 Diagonal contribution for q-nary graphs – Band, H., Sepanski.

Theorem (H., Hudgins)

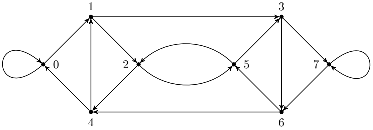
For a 4-regular quantum graph,

$$\langle |a_n|^2 \rangle = \frac{1}{2^n} \left(|\mathcal{P}_0^n| + \sum_{N=1}^{n} 2^N \, |\widehat{\mathcal{P}}_N^n| \right) , \qquad (3)$$

where $\mathcal{P}_0^n$ is the set of PPO length n with no self-intersections and $\widehat{\mathcal{P}}_N^n$ is the set of PPO length n with N self-intersections, all of which are 2-encounters of length zero.

- A PPO with n bonds cannot have $> n$ self-intersections.
- If $\bar{\gamma}$ has no self-intersections $\mathcal{P}_{\bar{\gamma}} = \{\bar{\gamma}\}$ producing the 1st term.
- For most PPO with self-intersections $C_{\bar{\gamma}} = 0$ using parity arguments.
- Exception, PPO where all self-intersections are 2-encounters length zero.

Example 1: Binary de Bruijn graph with 2^3 vertices.



n	$\lvert\mathcal{P}_0^n\rvert$	$\lvert\widehat{\mathcal{P}}_1^n\rvert$	$\lvert\widehat{\mathcal{P}}_2^n\rvert$	$\langle\lvert a_n\rvert^2\rangle$	Numerics	Error
0	1	0	0	1	1.000000	0.000000
1	2	0	0	1	0.999991	0.000009
2	2	0	0	1/2	0.499999	0.000001
3	4	0	0	1/2	0.499999	0.000001
4	8	0	0	1/2	0.499999	0.000001
5	8	8	0	3/4	0.749998	0.000002
6	8	20	0	3/4	0.749986	0.000014
7	16	16	8	5/8	0.624989	0.000011
8	16	16	24	9/16	0.562501	-0.000001

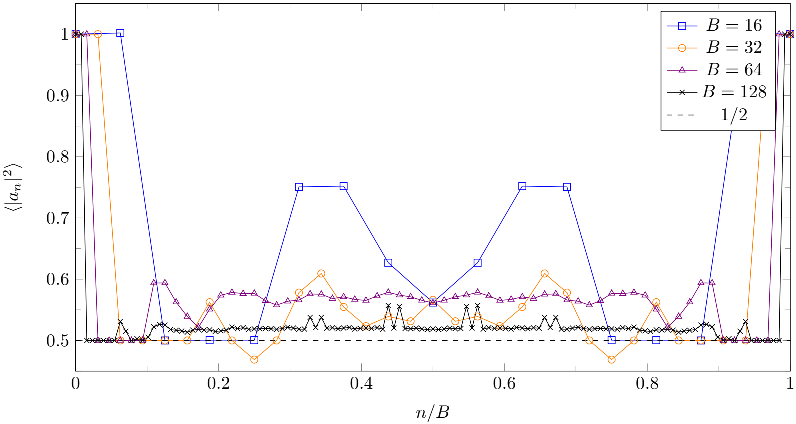
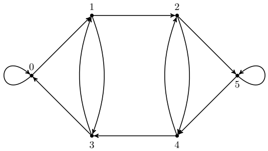


Figure 1: Variance of coefficients of the characteristic polynomial for the family of 4-regular binary de Bruijn graphs with 2^r vertices.

Example 2: Binary graph with $3 \cdot 2$ vertices.



n	$\lvert\mathcal{P}_0^n\rvert$	$\lvert\widehat{\mathcal{P}}_1^n\rvert$	$\langle\lvert a_n\rvert^2\rangle$	Numerics	Error
0	1	0	1	1.000000	0.000000
1	2	0	1	1.000000	0.000000
2	3	0	3/4	0.750001	-0.000001
3	6	0	3/4	0.750003	-0.000003
4	10	4	7/8	0.874999	0.000001
5	8	4	1/2	0.499998	0.000002
6	8	8	3/8	0.374999	0.000001

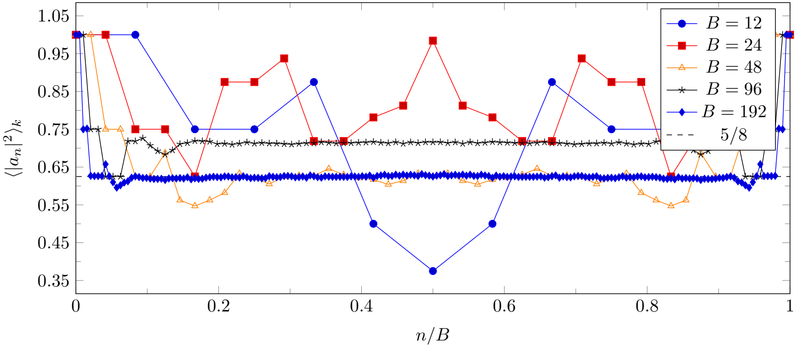
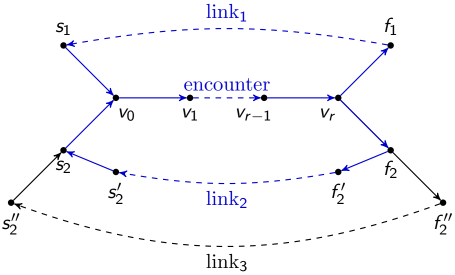


Figure 2: Variance of coefficients of the characteristic polynomial for the family of 4-regular binary graphs with $3 \cdot 2^r$ vertices.

Self-intersections

- A *self-intersection* is a section of a pseudo orbit that is repeated one or more times in the pseudo orbit.
- The maximally repeated section is the *encounter* $\mathrm{enc} = (v_0, \ldots, v_r)$.
- The *length of the encounter* is r and an encounter has length zero when the encounter contains no bonds.
- If the encounter is repeated l times we refer to an *l-encounter*.
- The encounter can be repeated in a single periodic orbit or across multiple orbits in the pseudo orbit.
- An l-encounter with $l \geq 3$ has preceding and subsequent sections repeated $< l$ times as there are only 2 incoming/outgoing bonds at each v.

Examples of pseudo orbits with self-intersections

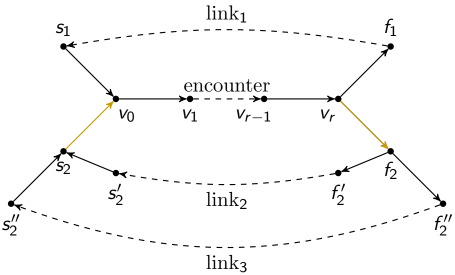


2-encounter: $\bar{\gamma} = (\gamma_1, \ldots, \gamma_m)$ with no self-intersections in $\gamma_2, \ldots, \gamma_m$ and

$$\gamma_1 = (f_1 \ldots, s_1, \mathrm{enc}, f_2, f_2' \ldots, s_2', s_2, \mathrm{enc}, f_1)$$

abbreviated $\gamma_1 = (1, 2)$ for link 1 followed by link 2.

Examples of pseudo orbits with self-intersections



3-encounter: Define $\bar{\gamma}$ similarly but with $\gamma_1 = (1, 2, 3)$.

(Bonds (s_2, v_0) and (v_r, f_2) preceding and following the encounter are repeated twice.)

Semiclassical limit

For quantum graphs the semiclassical limit is the limit of a sequence of graphs with $B \to \infty$. To take the semiclassical limit of the variance we fix n/B and consider long orbits on large graphs.

- In the semiclassical limit half of PPO with a 2-encounter will have encounter length zero, as the probability to follow the orbit at the initial encounter vertex is $1/2$.
- As the graph is mixing the proportion of orbits with 3-encounters is vanishingly small compared to 2-encounters.
- Let $\mathcal{P}_N^n$ denote the set of primitive pseudo orbits length n with N encounters. Then $|\widehat{\mathcal{P}}_N^n| \approx 2^{-N} |\mathcal{P}_N^n|$.

$$\langle |a_n|^2 \rangle = 2^{-n} \left(|\mathcal{P}_0^n| + \sum_{N=1}^{n} 2^N \, |\widehat{\mathcal{P}}_N^n| \right) \approx 2^{-n} \sum_{N=0}^{n} |\mathcal{P}_N^n| = 2^{-n} \, |\mathcal{P}^n|$$

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Future directions

- Examples were binary graphs where we use a connection to Lyndon words to count primitive pseudo orbits.
- Are there other families of 4-regular graphs where pseudo orbits can be counted?
- Does the result extend to k-regular graphs? Partial results appeared in Tori's thesis.
- Can the cancellation scheme be applied in other quantum chaotic systems?

J.M. Harrison and T. Hudgins, "Periodic-orbit evaluation of a spectral statistic of quantum graphs without the semiclassical limit," arXiv:2101.00006

J.M. Harrison and T. Hudgins, "Complete dynamical evaluation of the characteristic polynomial of binary quantum graphs," arXiv:2011.05213