Solvability of some integro-differential equations with concentrated sources

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Existence of solutions of nonlocal reaction-diffusion equations: existence of biological species

1. Introduction

One of the hot topics in the Nonlinear Science is the integro-differential equations in Mathematical Biology: nonlocal consumption of resources, intra-specific competition. Also, the nonlocal interaction of neurons.

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + \int_{-\infty}^{\infty} K(x-y)g(w(y)u(y,t))dy + \alpha\delta(x), \quad \alpha \in \mathbb{R}, \tag{1}$$

$\alpha \neq 0$ and $\delta(x)$ is the Dirac delta function from cell population dynamics. Cell genotype is x, cell density as a function of the genotype and time is $u(x,t)$. The evolution of cell density is due to cell proliferation, mutations and cell influx/efflux. The change of genotype due to small random mutations-normal diffusion term. Large mutations is the integral term. $g(w(x)u(x))$ is the rate of cell birth, depends on u, w (density dependent proliferation). $K(x-y)$ is the proportion of newly born cells changing their genotype from y to x, depends on the distance between the genotypes. $\alpha\delta(x)$ is the influx/efflux of cells for different genotypes, singular situation. $w(x)$ is our cut-off function.

$$ex. \quad w(x) = e^{-|x|}, \quad x \in \mathbb{R}.$$

Recall earlier work: V.V., Vitaly Volpert, Springer (2018). Dedicated to the 70th Anniversary of Professor Afraimovich.

$$\frac{\partial u}{\partial t} = -D\left(-\frac{\partial^2}{\partial x^2}\right)^s u + \int_{-\infty}^{\infty} K(x-y)g(u(y,t))dy + f(x), \qquad (2)$$

where $0 < s < \frac{1}{4}$. We proved the existence of a stationary solution in $H^1(\mathbb{R})$. The space variable corresponds to the cell genotype, not the usual physical space. A disease can be caused by 1,2,3,...,100,... genes.

Anomalous diffusion problem with $\left(-\frac{\partial^2}{\partial x^2}\right)^s$: defined via the spectral calculus, namely

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(p) e^{ipx} dp, \quad \left(-\frac{\partial^2}{\partial x^2}\right)^s f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |p|^{2s} \widehat{f}(p) e^{ipx} dp.$$

Anomalous diffusion: plasma physics and turbulence.

B.Carreras, V.Lynch, G.Zaslavsky, Phys. Plasmas (2001).

Surface diffusion.

J.Sancho, A. Lacasta, K.Lindenberg, I.Sokolov, A.Romero, Phys. Rev. Lett. (2004).

Semiconductors.

H.Scher, E.Montroll, Phys. Rev. B (1975).

Physical meaning: the random process occurs with longer jumps in comparison with normal diffusion.

Normal diffusion: finite moments of jump length distribution.

Anomalous diffusion: not the case.

R. Metzler, J. Klafter, Phys. Rep. (2000).

Stationary situation: $\dfrac{\partial u}{\partial t} = 0$, assume the diffusion coefficient $D = 1$.

$$\frac{d^2u}{dx^2} + \int_{-\infty}^{\infty} K(x-y)g(w(y)u(y))dy + \alpha\delta(x) = 0. \tag{3}$$

Set $K(x) = \varepsilon\mathcal{K}(x)$, $\varepsilon \geq 0$ small parameter.

Sobolev norm

$$\|\phi\|^2_{H^1(\mathbb{R})} := \|\phi\|^2_{L^2(\mathbb{R})} + \left\|\frac{d\phi}{dx}\right\|^2_{L^2(\mathbb{R})}.$$

Sobolev inequality in one dimension

$$\|\phi(x)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\sqrt{2}}\|\phi(x)\|_{H^1(\mathbb{R})}. \tag{4}$$

E. Lieb, M.Loss, "Analysis", Providence (1997). Recall the algebraic property of our Sobolev space, follows for instance from (4):

for any $u(x), v(x) \in H^1(\mathbb{R})$

$$\|u(x)v(x)\|_{H^1(\mathbb{R})} \leq c_a \|u(x)\|_{H^1(\mathbb{R})} \|v(x)\|_{H^1(\mathbb{R})}, \tag{5}$$

$c_a > 0$ is a constant.

When the parameter ε vanishes, we obtain the linear Poisson equation with a singular right side

$$-\frac{d^2 u}{dx^2} = \alpha \delta(x), \quad x \in \mathbb{R}. \tag{6}$$

The ramp function

$$R(x) := \begin{cases} x, & x \geq 0 \\ 0, & x < 0. \end{cases} \tag{7}$$

The Heaviside step function

$$H(x) := \begin{cases} 1, & x \geq 0 \\ 0, & x < 0. \end{cases} \tag{8}$$

Clearly

$$\frac{dR(x)}{dx} = H(x), \quad \frac{dH(x)}{dx} = \delta(x).$$

The solution of Poisson equation (6) vanishing at minus infinity:

$$-\alpha R(x). \tag{9}$$

(9) is unbounded, does not belong to $H^1(\mathbb{R})$.

In V.V., Vitaly Volpert, Springer (2018) the unique solution of the Poisson with the minus Laplacian to the fractional power $u_0(x) \in H^1(\mathbb{R})$, bounded via the Sobolev inequality (4) and no need for the cut-off function $w(x)$.

Assume:

$$w(x) : \mathbb{R} \to \mathbb{R}, \quad w(x) \in H^1(\mathbb{R}), \quad w(x)R(x) \quad is \quad nontrivial,$$

$$w(x)R(x) \in H^1(\mathbb{R}), \quad |\alpha| \leq \frac{1}{\|w(x)R(x)\|_{H^1(\mathbb{R})}}.$$

The standard Fourier transform

$$\widehat{\phi}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x) e^{-ipx} dx. \tag{10}$$

Upper bound

$$\|\widehat{\phi}(p)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\sqrt{2\pi}} \|\phi(x)\|_{L^1(\mathbb{R})}. \tag{11}$$

Sobolev norm using Fourier transform (10)

$$\|\phi\|^2_{H^1(\mathbb{R})} = \|\widehat{\phi}(p)\|^2_{L^2(\mathbb{R})} + \|p\widehat{\phi}(p)\|^2_{L^2(\mathbb{R})}. \tag{12}$$

2. Fixed point argument

Seek the resulting solution of the stationary nonlinear problem (3) as

$$u(x) = -\alpha R(x) + u_p(x). \tag{13}$$

Perturbative equation

$$-\frac{d^2 u_p(x)}{dx^2} = \varepsilon \int_{-\infty}^{\infty} \mathcal{K}(x-y) g(w(y)[-\alpha R(y) + u_p(y)]) dy. \tag{14}$$

The Fixed Point argument in a closed ball in our Sobolev space:

$$B_\rho = \{u(x) \in H^1(\mathbb{R}) \mid \|u\|_{H^1(\mathbb{R})} \leq \rho\}, \quad 0 < \rho \leq 1. \tag{15}$$

Seek the solution of (14) as the fixed point of the auxiliary nonlinear problem

$$-\frac{d^2 u(x)}{dx^2} = \varepsilon \int_{-\infty}^{\infty} \mathcal{K}(x-y) g(w(y)[-\alpha R(y) + v(y)]) dy, \tag{16}$$

in ball (15). The non Fredholm operator in the left side of (16)

$$-\frac{d^2}{dx^2} \quad on \quad L^2(\mathbb{R}).$$

Its essential spectrum

$$\sigma_{ess}\left(-\frac{d^2}{dx^2}\right) = [0, +\infty),$$

no bounded inverse. Similar situations, integro-differential equations

V.V., V.Volpert, Doc. Math. (2011),

V.V., V.Volpert, Anal. Math. Phys. (2012).

The fixed point technique to estimate the perturbation to the standing solitary wave

$$\psi(x,t) = \phi(x)e^{i\omega t}$$

of the Nonlinear Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = -\Delta\psi + V(x)\psi + F(|\psi|^2)\psi$$

when small perturbation is applied either to the potential or to the nonlinear term. The Schrödinger operator involved had the Fredholm property.

V.V., Math. Model. Nat. Phenom., (2010).

The operator T_g via the auxiliary nonlinear problem (16), such that $u = T_g v, \quad u$ is a solution.

The existence, stability and bifurcations of the solutions of nonlinear PDEs with Dirac delta function were studied actively in

R. Adami, D. Noja, Math. Model. Nat. Phenom., (2014).

R. Fukuizumi, L. Jeanjean, Discrete Contin. Dyn. Syst., (2008).

J. Holmer, J. Marzuola, M. Zworski, Comm. Math. Phys., (2007).

Our main result is as follows.

Theorem 1. Under our technical assumptions problem (16) defines the map $T_g : B_\rho \to B_\rho$, which is a strict contraction for all $0 < \varepsilon \le \varepsilon^*$ for a certain $\varepsilon^* > 0$. The unique fixed point $u_p(x)$ of the map T_g is the only solution of problem (14) in B_ρ.

The resulting stationary solution of (3) is nontrivial: the parameter $\alpha \neq 0$ and $g(0) = 0$ as assumed.

Proof. Choose arbitrarily $v(x) \in B_\rho$, denote

$$G(x) := g(w(x)[-\alpha R(x) + v(x)]).$$

Apply the standard Fourier transform (10) to (16). Thus

$$\widehat{u}(p) = \varepsilon\sqrt{2\pi}\frac{\widehat{\mathcal{K}}(p)\widehat{G}(p)}{p^2}, \quad p\widehat{u}(p) = \varepsilon\sqrt{2\pi}\frac{\widehat{\mathcal{K}}(p)\widehat{G}(p)}{p}. \tag{17}$$

Positive technical expression

$$Q := \max\left\{\left\|\frac{\widehat{\mathcal{K}}(p)}{p^2}\right\|_{L^\infty(\mathbb{R})}, \quad \left\|\frac{\widehat{\mathcal{K}}(p)}{p}\right\|_{L^\infty(\mathbb{R})}\right\}. \tag{18}$$

Assume about our kernel:

$$\mathcal{K}(x) : \mathbb{R} \to \mathbb{R} \quad is \quad nontrivial, \quad \mathcal{K}(x) \in \mathbb{L}^1(\mathbb{R}), \quad x^2\mathcal{K}(x) \in \mathbb{L}^1(\mathbb{R}).$$

It can be easily established that if

$$\frac{\widehat{\mathcal{K}}(p)}{p^2} \in L^\infty(\mathbb{R}) \quad then \quad \frac{\widehat{\mathcal{K}}(p)}{p} \in L^\infty(\mathbb{R}) \quad as \quad well.$$

We split into the singular and the regular parts as

$$\frac{\widehat{\mathcal{K}}(p)}{p^2} = \frac{\widehat{\mathcal{K}}(p)}{p^2}\chi_{\{|p|\le 1\}} + \frac{\widehat{\mathcal{K}}(p)}{p^2}\chi_{\{|p|>1\}}. \tag{19}$$

For the second term in the right side of (19), using (11)

$$\left|\frac{\widehat{\mathcal{K}}(p)}{p^2}\chi_{\{|p|>1\}}\right| \leq \|\widehat{\mathcal{K}}(p)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\sqrt{2\pi}}\|\mathcal{K}(x)\|_{L^1(\mathbb{R})} < \infty.$$

Let us express

$$\widehat{\mathcal{K}}(p) = \widehat{\mathcal{K}}(0) + p\frac{d\widehat{\mathcal{K}}}{dp}(0) + \int_0^p \left(\int_0^s \frac{d^2\widehat{\mathcal{K}}(q)}{dq^2}dq\right)ds.$$

The first term in the right side of (19), to the leading order

$$\left[\frac{\widehat{\mathcal{K}}(0)}{p^2} + \frac{\frac{d\widehat{\mathcal{K}}}{dp}(0)}{p}\right]\chi_{\{|p|\leq 1\}}. \tag{20}$$

Definition (10) of the standard Fourier transform yields

$$\widehat{\mathcal{K}}(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathcal{K}(x)dx = \frac{1}{\sqrt{2\pi}}(\mathcal{K}(x), 1)_{L^2(\mathbb{R})}, \tag{21}$$

$$\frac{d\widehat{\mathcal{K}}}{dp}(0) = -\frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathcal{K}(x)xdx = -\frac{i}{\sqrt{2\pi}}(\mathcal{K}(x), x)_{L^2(\mathbb{R})}. \tag{22}$$

Formulas (21) and (22) enable us to write expression (20) as

$$\left[\frac{(\mathcal{K}(x),1)_{L^2(\mathbb{R})}}{\sqrt{2\pi}p^2} - i\frac{(\mathcal{K}(x),x)_{L^2(\mathbb{R})}}{\sqrt{2\pi}p}\right]\chi_{\{|p|\le 1\}}. \tag{23}$$

Evidently, (23) is bounded if and only if orthogonality relations

$$(\mathcal{K}(x),1)_{L^2(\mathbb{R})} = 0, \quad (\mathcal{K}(x),x)_{L^2(\mathbb{R})} = 0 \tag{24}$$

hold. Impose (24), then the expression Q defined in (18) is finite.

By means of (17) we estimate

$$|\widehat{u}(p)| \le \varepsilon\sqrt{2\pi}Q|\widehat{G}(p)|, \quad |p\widehat{u}(p)| \le \varepsilon\sqrt{2\pi}Q|\widehat{G}(p)|. \tag{25}$$

Using the expression for the Sobolev norm via the Fourier transform (12) along with (25) we obtain

$$\|u(x)\|_{H^1(\mathbb{R})} \le \varepsilon C \le \rho \tag{26}$$

for all $0 < \varepsilon \le \varepsilon^*$, such that $u(x) \in B_\rho$ as well.

To establish the uniqueness, suppose for some $v(x) \in B_\rho$ there exist two solutions $u_{1,2}(x) \in B_\rho$ of (16). The difference $w(x) = u_1(x) - u_2(x) \in L^2(\mathbb{R})$ solves $-\frac{d^2}{dx^2} w(x) = 0$. No nontrivial square integrable zero modes for $-\frac{d^2}{dx^2}$ on $\mathbb{R}$, $w(x) \equiv 0$. Then (16) defines a map $T_g : B_\rho \to B_\rho$ for all $0 < \varepsilon \leq \varepsilon^*$.

To show that this map is a strict contraction.

Choose arbitrarily $v_{1,2}(x) \in B_\rho$. Then $u_{1,2} := T_g v_{1,2} \in B_\rho$ as well for $0 < \varepsilon \leq \varepsilon^*$.

$$-\frac{d^2}{dx^2} u_1(x) = \varepsilon \int_{-\infty}^{\infty} \mathcal{K}(x-y) g(w(y)[-\alpha R(y) + v_1(y)]) dy,$$

$$-\frac{d^2}{dx^2} u_2(x) = \varepsilon \int_{-\infty}^{\infty} \mathcal{K}(x-y) g(w(y)[-\alpha R(y) + v_2(y)]) dy.$$

Introduce

$G_1(x) := g(w(x)[-\alpha R(x) + v_1(x)])$, $G_2(x) := g(w(x)[-\alpha R(x) + v_2(x)])$.

Apply the standard Fourier transform (10). Arrive at

$$\widehat{u}_1(p) = \varepsilon\sqrt{2\pi}\frac{\widehat{\mathcal{K}}(p)\widehat{G_1}(p)}{p^2}, \quad \widehat{u}_2(p) = \varepsilon\sqrt{2\pi}\frac{\widehat{\mathcal{K}}(p)\widehat{G_2}(p)}{p^2}.$$

Evidently,

$$|\widehat{u}_1(p) - \widehat{u}_2(p)| \le \varepsilon\sqrt{2\pi}Q|\widehat{G_1}(p) - \widehat{G_2}(p)|, \tag{27}$$

$$|p[\widehat{u}_1(p) - \widehat{u}_2(p)]| \le \varepsilon\sqrt{2\pi}Q|\widehat{G_1}(p) - \widehat{G_2}(p)|. \tag{28}$$

Formula (12) for the Sobolev norm via the Fourier transform and (27), (28) help us to estimate the norm

$$\|u_1(x) - u_2(x)\|_{H^1(\mathbb{R})} \le \varepsilon C\|v_1(x) - v_2(x)\|_{H^1(\mathbb{R})}. \tag{29}$$

The constant in the right side of (29) $\varepsilon C < 1$ for all $0 < \varepsilon \le \varepsilon^*$, such that $T_g : B_\rho \to B_\rho$ defined by (16) is strict contraction.

Unique fixed point $u_p(x)$ is the only solution of the perturbative equation (14) in B_ρ.

The resulting solution of the stationary problem (3):

$$u(x) = -\alpha R(x) + u_p(x),$$

where $-\alpha R(x)$ solves our Poisson equation (6) with the Dirac delta function in the right side and vanishes at the negative infinity.

By means of estimate (26) above we have

$$\|u_p(x)\|_{H^1(\mathbb{R})} \leq \varepsilon C,$$

such that

$$\|u_p(x)\|_{H^1(\mathbb{R})} \to 0 \quad as \quad \varepsilon \to 0.$$

Also proved: the cumulative solution $u(x)$ of our stationary, nonlocal equation (3) is continuous in the $H^1(\mathbb{R})$ norm with respect to the nonlinear, continuously differentiable rate of cell birth function $g(z)$.

4. Discussion of the future work

1. To study the convergence of the solutions $u(x,t)$ of the integro-differential equations to the equilibrium.

2. To generalize the results on the existences of the stationary solutions to the case when the normal diffusion is combined with the anomalous diffusion in a single integro-differential equation or a system of coupled integro-differential equations. M.Efendiev, V.V., J. Differential Equations (2021).

3. To perform the iterations of the kernels of integro-differential equations and to show the existence of their stationary solutions in the sense of sequences.

4. To work on the preservation of the nonnegativity of solutions of the systems of parabolic equations. M.Efendiev, V.V., Springer chapters (2021).