Measure Rigidity for Diagonal Actions

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Motivation: QUE

Let M be a compact negatively curved manifold and let ϕ_n be L^2-normalized Δ-eigenfunctions with eigenvalue going to infinity and distributions converging in the sense that $|\phi_n|^2 dvol \xrightarrow[\text{weak}^*]{} d\nu$ as $n \to \infty$. Then Rudnick and Sarnak conjectured $d\nu = dvol$. This is commonly referred to as the Quantum Unique Ergodicity conjecture.

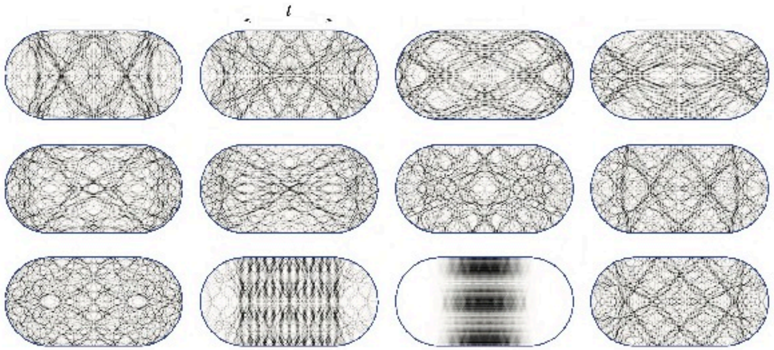


Figure 1S

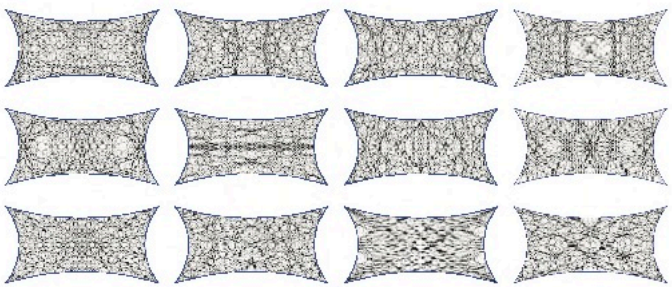


Figure 1B

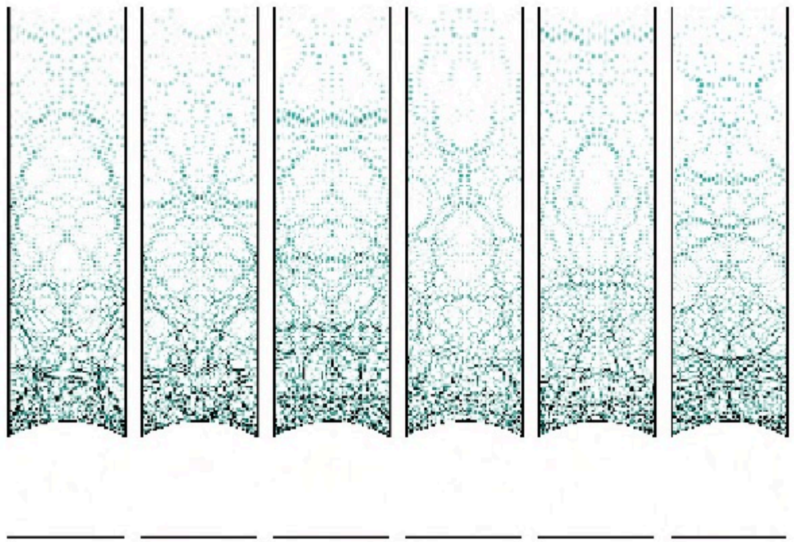


Figure 4

from RECENT PROGRESS ON THE QUANTUM UNIQUE ERGODICITY CONJECTURE
Sarnak, Bulletin of the AMS 2011

Motivation: QUE

Research in the direction of QUE is very active. We focus on the following partial result.

Thm (Arithmetic QUE: Lindenstrauss '06, Soundararajan '10)
Suppose $M = Sl_2(\mathbb{Z})\backslash\mathbb{H}$ is the "modular surface" and ϕ_n are L^2-normalized joint eigenfunctions of all Hecke-operators and Δ with diverging eigenvalues.
Then $|\phi_n|^2 d\mathrm{vol} \to d\mathrm{vol}$ as $n\to\infty$.

Method of proof

1) Construction of the microlocal lift gives a measure μ on $T^1M \cong Sl_2(\mathbb{Z})\backslash Sl_2(\mathbb{R}) = X$ invariant under the geodesic flow $\cong A = \left\{\begin{pmatrix} e^{t/2} & \\ & e^{-t/2}\end{pmatrix} : t\in\mathbb{R}\right\}$

2) Establishing Positive entropy of all ergodic components [Bourgain-Lindenstrauss]

3) Establishing Recurrence for "Hecke-orbits"

4) Classifying all probability measures μ on X with the properties 1), 2) & 3)

[Lindenstrauss] (for 3) and 4))

5) Showing $\mu(X)=1$ using, once more, number theory [Soundararajan]

Motivation: QUE

What are the Hecke-operator on $M = SL_2(\mathbb{Z}) \backslash \mathbb{H}$?

Let $p \in \mathbb{N}$ be a prime.

$$z \overset{?}{\longmapsto} pz$$

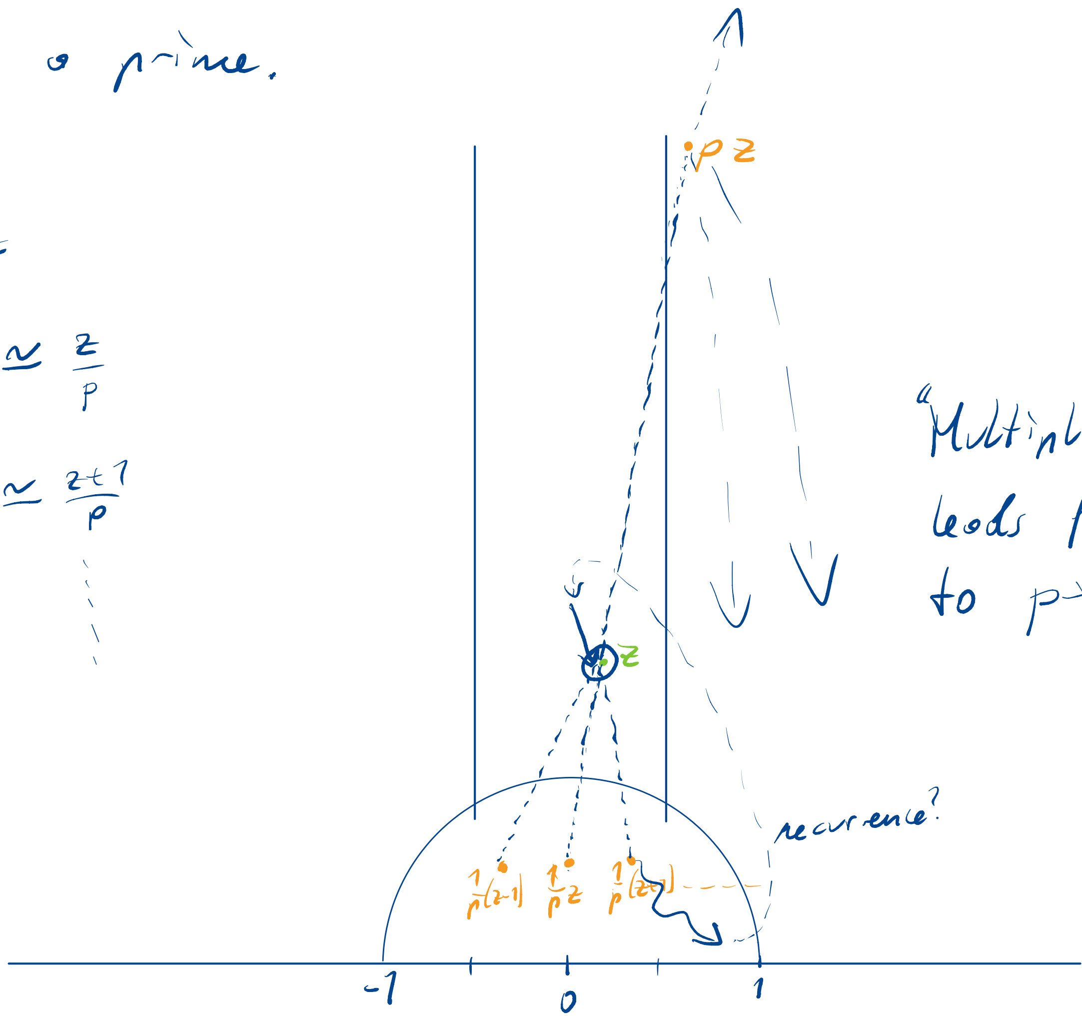
$$\wr$$

$$-\frac{1}{z} \overset{?}{\longmapsto} -\frac{p}{z} \sim \frac{z}{p}$$

$$\wr$$

$$-\frac{1}{z+1} \longmapsto -\frac{p}{z+1} \sim \frac{z+1}{p}$$

$$\vdots$$



"Multiplication by" p leads from $z \in M$ to $p+1$ points in M

Hecke operator

$$H_p(f) = \frac{1}{p+1}\left(f(pz) + \sum_{j=0}^{n-1} f\left(\frac{z+j}{p}\right)\right)$$

Hecke orbit

$(p+1)$-regular tree embedded in M for every initial vertex $z \in M$.

Short history

Example Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the circle group and $T_3(x) = 3x \pmod 1$.

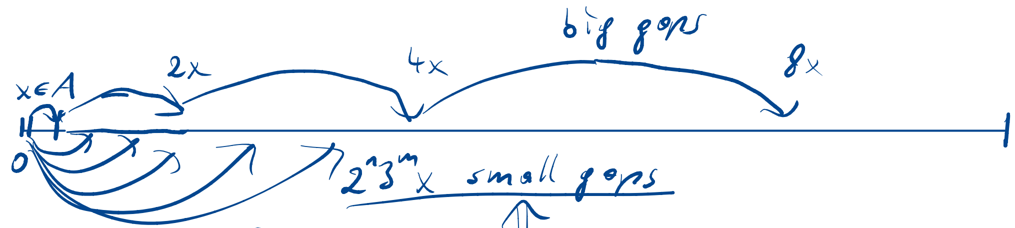
Then T_3 has lots of complicated (chaotic) orbits and invariant sets.

E.g. $C = \left\{ \sum_{n=1}^{\infty} x_n 3^{-n} : x_n \in \{0,2\} \right\} / \mathbb{Z}$

Thm (Furstenberg, 1967)

If $A \subseteq \mathbb{T}$ is closed and invariant under T_2 and T_3, then either $A = \mathbb{T}$ or $A \subseteq \mathbb{Q}/\mathbb{Z}$ is finite.

Idea If $0 \in A$ is an accumulation point, then $A = \mathbb{T}$!



$\log 2, \log 3$ lin. ind. over $\mathbb{Q}$

$\Downarrow$

$\mathbb{Z}\log 2 + \mathbb{Z}\log 3 = \mathbb{R}$

$\Rightarrow$ $\mathbb{N}\log 2 + \mathbb{N}\log 3$ is getting more & more dense to the right in $\mathbb{R}$

Short history

Thm (Rudolph 1990)
If μ is a T_2- and T_3-invariant and ergodic probability measure with $h_\mu(T_2)>0$ (or $h_\mu(T_3)>0$) then $\mu=$ Lebesgue.

positive entropy for all erg. comp.

$\Downarrow$

$\boxed{\mu(B_r(x)) \ll r^\delta \quad \text{a.s. for some fixed } \delta>0}$

Useful info: $h_\mu(T_2) = (\dim \mu) \log 2$

$h_\mu(T_3) = (\dim \mu) \log 3$

In other cases the connection between entropy & dimension is less straight forward.

Further conjectures in this direction are due to

Furstenberg, Katok-Spatzier, Margulis

The old theorems

Thm (Lindenstrauss '06)

Let $X = \Gamma \backslash SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$ for an irreducible lattice Γ and let $A = \left\{ \left(\begin{pmatrix} e^{t_1} & \\ & e^{-t_1} \end{pmatrix}, \begin{pmatrix} e^{t_2} & \\ & e^{-t_2} \end{pmatrix} \right) : t_1, t_2 \in \mathbb{R} \right\}$

Thm (Einsiedler, Katok, Lindenstrauss '06)

Let $X = SL_3(\mathbb{Z}) \backslash SL_3(\mathbb{R})$ and let $A = \left\{ \begin{pmatrix} e^{t_1} & & \\ & e^{t_2} & \\ & & e^{t_3} \end{pmatrix} : t_1 + t_2 + t_3 = 0 \right\}$

If μ is an A-invariant and ergodic probability measure on X with $h_\mu(a) > 0$ for some $a \in A$, then $\mu = m_X$ is the invariant volume measure on X.

Motivation: QUE

How to show positive entropy?

positive entropy

$\Updownarrow$

positive dimension transverse to A

$\Updownarrow$

$\mu(\delta\text{-tube}) \ll \delta^{\kappa}$ for some $\kappa > 0$

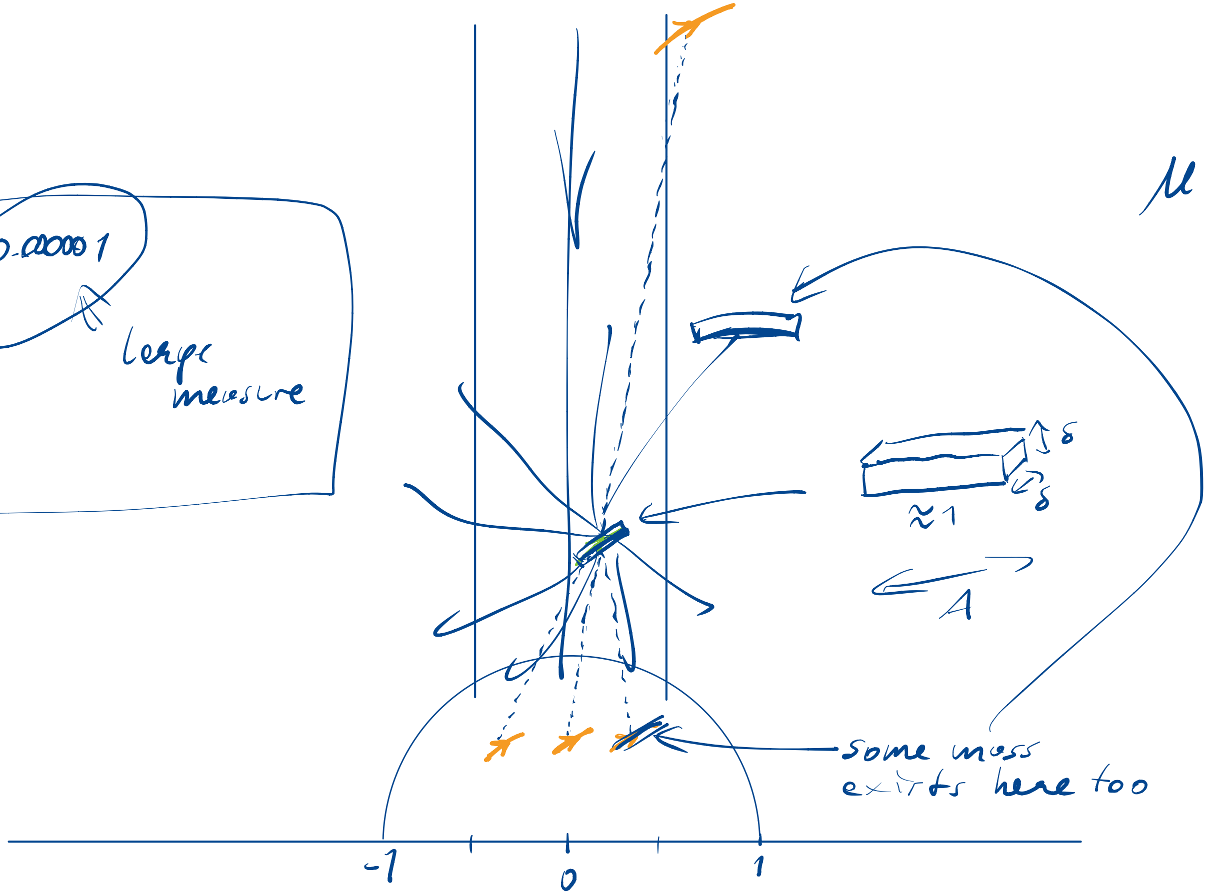
Assume zero entropy

$\Downarrow$

$\boxed{\mu(\delta\text{-tube}) \geq \delta^{0.00001}}$

geometrically small set

large measure



Applying the Hecke-assumption for all $p < T$ we find many other δ-tubes with a lot of mass.

$\Longrightarrow$ This way overlaps are forced.

$\Longrightarrow$ x needs to be a periodic point.

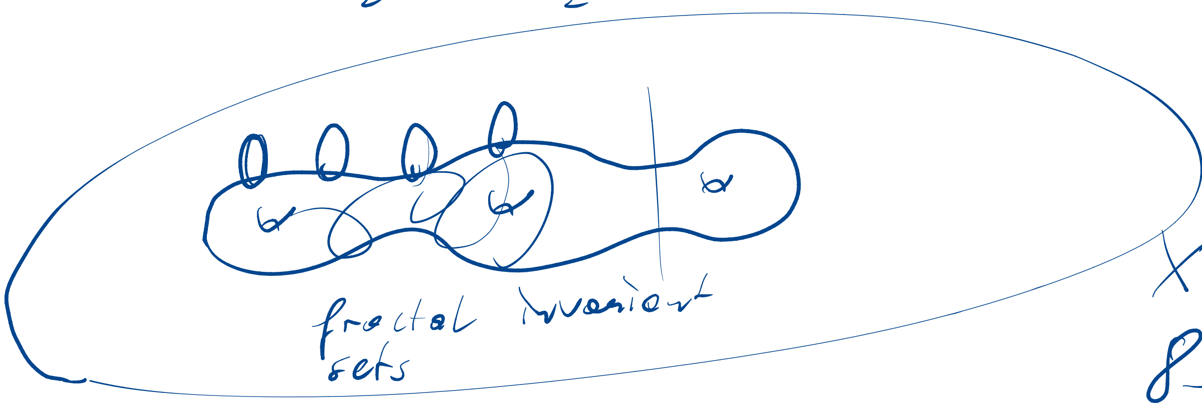
The oldish theorem

Thm (Einsiedler, Lindenstrauss '15)

This gives a partial classification for A-invariant and ergodic probability measures on $X = \Gamma \backslash G$, where G is a higher rank semisimple Lie group, Γ is an arithmetic lattice, and A is a <u>maximal</u> $\mathbb{R}$-diagonalizable subgroup of G.

The conclusion is more difficult to state. Other possibilities can occur: E.g. M. Rees found a cocompact lattice Γ in $Sl_3(\mathbb{R})$ so that $H = \left\{ \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} \right\}$ has a closed orbit

$$\cong Gl_2(\mathbb{R}) \cong Sl_2(\mathbb{R}) \times \mathbb{R}^\times$$



$X = \Gamma \backslash Sl_3(\mathbb{R})$

8-dim compact manifold

<u>The general idea</u>

μ is invariant under A

$X = \Gamma \backslash G$ has stable manifolds for various $a \in A$ and their intersections

– these are orbits of unipotent subgroups $U < G$

μ is <u>not</u> assumed to be invariant under U, but positive entropy

for $a \in A$ implies at least something...

The recent theorem

Thm (Einsiedler, Lindenstrauss '21+)
Let $X = \Gamma \backslash SL_2(\mathbb{R})^k$ for some $k \geq 2$ and an irreducible arithmetic Γ.
Let A be a two-parameter $\mathbb{R}$-diagonalizable subgroup.
Let μ be an A-inv. & ergodic probability measure on X
with $h_\mu(a) > 0$ for some $a \in A$.

- If X is compact, then μ is homogeneous i.e. the invariant measure on a closed orbit of a subgroup.
- In general, μ is homogeneous or supported on the closed orbit of a solvable subgroup.

New idea

Boshernitzan proved in 1993 Quantitative recurrence results, which we adapt for the study of orbits of stable subgroups.

This simplifies the previous argument for $Sl_2(\mathbb{R})^2$, but becomes increasingly more complicated for other semisimple groups....

The joining theorem

Furstenberg also defined in the same 1967 paper the notion of a joining, which became a fundamental tool for ergodic theory.

Let A be a group acting measure preservingly on two probability spaces (X_1, m_1) and (X_2, m_2). We say μ is a <u>joining</u> for the two actions if μ is a measure on $X_1 \times X_2$ that projects to m_1 resp. m_2 under the coordinate projections and is invariant under the diagonal action of A.

Thm (Einsiedler, Lindenstrauss '19)

Joinings between higher rank actions on irreducible arithmetic quotients of semi-simple Lie groups are always homogeneous.