Measure Rigidity for Diagonal Actions

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Motivation: QUE

Let $M$ be a compact negatively curved manifold and let $\phi_n$ be $L^2$-normalized eigenfunctions with eigenvalue going to infinity and distributions converging in the sense that $|\phi_n|^2 dv \rightarrow dv$ as $n \rightarrow \infty$. Then Rudnick and Sarnak conjectured $dv = dv_0$. This is commonly referred to as the \underline{Quantum Unique Ergodicity conjecture}.

Figure 1A

Figure 1B

Figure 4

from RECENT PROGRESS ON THE QUANTUM UNIQUE ERGODICITY CONJECTURE
Sarnak, Bulletin of the AMS 2011
Motivation: QUE

Research in the direction of QUE is very active. We focus on the following partial result:

Then (Arithmetic QUE: Lindenstrauss '06, Soundararajan '10)

Suppose $M = \text{SL}_2(\mathbb{Z}) \backslash \mathcal{H}$ is the "modular surface" and $\phi_n$ are $L^2$-normalized joint eigenfunctions of all Hecke-operators and $\Delta$ with diverging eigenvalues.

Then $|\phi_n|_{\text{dvol}} \rightarrow \text{dvol}$ as $n \rightarrow \infty$.

Method of proof

1. Construction of the microlocal lift gives a measure $\mu$ on $T^1M = \text{SL}_2(\mathbb{Z}) \backslash \text{SL}_2(\mathbb{R}) = X$

   invariant under the geodesic flow $A = \{ (e^{it}e^{it}) : t \in \mathbb{R} \}$

2. Establishing Positive entropy of all ergodic components

3. Establishing Recurrence for "Hecke-orbits"

4. Classifying all probability measures $\mu$ on $X$ with the properties 1, 2, & 3

5. Showing $\mu(X) = 1$ using, once more, number theory
Motivation: QUE

What are the Hecke operators on \( \mathcal{M} = \text{SL}_2(\mathbb{Z}) \backslash \Gamma \)?

Let \( p \in \mathbb{N} \) be a prime.

\[
\begin{align*}
\frac{1}{2} & \rightarrow \frac{1}{2} \\
\frac{i}{2} & \rightarrow -\frac{i}{2} \Leftrightarrow \frac{2}{p} \\
\frac{1}{2i} & \rightarrow -\frac{1}{2i} \Leftrightarrow \frac{2+i}{p}
\end{align*}
\]

"Multiplication by \( p \) leads from \( z \in \mathcal{M} \) to \( p+1 \) points in \( \mathcal{M} \)"

Hecke operator

\[
H_p(f) = \frac{1}{\prod_{i=1}^{p-1}} \left( f(pz) \sum_{i=0}^{p-1} f(pz_i) \right)
\]

Hecke orbit

\((p+1)\)-regular tree embedded in \( \mathcal{M} \) for every initial vertex \( z \in \mathcal{M} \).
Short history

Example let $T = \mathbb{R}/\mathbb{Z}$ be the circle group and $T_3(x) = 3x \mod 1$.
Then $T_3$ has lots of complicated (chaotic) orbits and invariant sets.

E.g. $C = \left\{ \sum_{n=1}^{\infty} x_n 3^{-n} : x_n \in \{0,1,2\} \right\}/\mathbb{Z}$

Thus (Furstenberg, 1967)

If $A \subseteq T$ is closed and invariant under $T_2$ and $T_3$, then either $A = T$ or $A \subseteq \mathbb{Q}/\mathbb{Z}$ is finite.

Idea If $A \cap C$ is an accumulation point, then $A = \overline{T}$.

$\log 2, \log 3$ wind over $Q$

$X \log 2 + X \log 3 = \mathbb{R}$

$N \log 2 + N \log 3$ is getting more & more dense to the right $\in \mathbb{R}$
Thm (Rudolph 1990)
If \( \mu \) is a \( T_2 \)- and \( T_3 \)-invariant and ergodic probability measure with \( h_\mu(T_2) > 0 \) (or \( h_\mu(T_3) > 0 \)) then \( \mu = Lebesgue \).

Useful info:
\[
\begin{align*}
  h_\mu(T_2) &= (\dim \mu) \log 2 \\
  h_\mu(T_3) &= (\dim \mu) \log 3
\end{align*}
\]

In other cases the connection between entropy & dimension is less straightforward.

Further conjectures in this direction are due to Furstenberg, Katok-Spatzier, Manzulis.
The old theorems

Thm (Lindenstrauss '06)
Let \( X = \tilde{\Gamma} \setminus \text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R}) \)
for an irreducible lattice \( \tilde{\Gamma} \)
and let \( A = \left\{ \left( \left( \begin{smallmatrix} e^{t_1} & \ast \\ 0 & e^{t_2} \end{smallmatrix} \right) \right)^n : t_1, t_2 \in \mathbb{R} \right\} \)

Thm (Einsiedler, Kottke, Lindenstrauss '06)
let \( X = \mathcal{S}_3(\mathbb{Z}) \setminus \text{SL}_3(\mathbb{R}) \) and let
\( A = \left\{ \left( \begin{smallmatrix} e^{t_1} & e^{t_2} \\ e^{t_2} & e^{t_3} \end{smallmatrix} \right) : t_1 + t_2 + t_3 = 0 \right\} \)

If \( \mu \) is an \( A \)-invariant and ergodic probability measure on \( X \) with
\( \mu(0) > 0 \) for some \( 0 \in A \), then
\( \mu = \text{inv}_X \) is the invariant volume
measure on \( X \).
Motivation: QUE

How to show positive entropy!

Assume zero entropy

\[ \mu(\delta\text{-tube}) \geq \delta \]

geometrically small set

large measure

positive entropy

positive dimension transverse to \( A \)

\[ \mu(\delta\text{-tube}) \leq \delta^k \]

for some \( k > 0 \)

Applying the Oseledec assumption for all \( p < T \)

we find many other \( \delta \)-tubes with a lot of mass.

This way overlaps are forced.

\( x \) needs to be a periodic point.
The oldish theorem

Thm (Emrulder, Lindenstrauss '15)

This gives a partial classification for $A$-invariant and ergodic probability measures on $X=\mathbb{I} \setminus G$, where $G$ is a higher rank semisimple Lie group, $\mathbb{I}$ is an arithmetic lattice, and $A$ is a maximal $\mathbb{R}$-diagonalizable subgroup of $G$.

The conclusion is more difficult to state. Other possibilities can occur; e.g., M. Rees found a cocompact lattice $\mathbb{I}'$ in $\text{SL}_3(\mathbb{R})$ so that $H=\{(0,0,0)\}$ has a closed orbit

$$\mathbb{I}' \cong \text{SL}_2(\mathbb{R}) \times \mathbb{R}^\times$$

\[ X=\mathbb{I}' \setminus \text{SL}_3(\mathbb{R}) \] 8-dim compact manifold
The general idea

$\mu$ is invariant under $A$

$X = T^1 \backslash G$ has stable manifolds for various $\alpha \in A$ and their intersections

- these are orbits of unipotent subgroups $U < G$

$\mu$ is not assumed to be invariant under $U$, but positive entropy for $\alpha \in A$ implies at least something.
The recent theorem

Then (Einsiedler, Lindenstrauss '21+)

let \( X = \mathbb{I} \backslash \text{SL}_2(\mathbb{R})^k \) for some \( k \geq 2 \) and an irreducible arithmetic \( T \).

let \( A \) be a two-parameter \( \mathbb{R} \)-diagonalizable subgroup.

let \( \mu \) be an \( A \)-invariant ergodic probability measure on \( X \) with \( \mu(A) > 0 \) for some \( a \in A \).

If \( X \) is compact, then \( \mu \) is homogeneous, i.e. the invariant measure on a closed orbit of a subgroup.

In general, \( \mu \) is homogeneous or supported on the closed orbit of a solvable subgroup.
New idea

Boshernitzan proved in 1993 Quantitative recurrence results
which we adopt for the study of orbits of stable subgroups.

This simplifies the previous argument for $SL(2, \mathbb{R})$, but becomes increasingly more complicated for other semi-simple groups...
Furstenberg also defined in the same 1967 paper the notion of a joining, which became a fundamental tool for ergodic theory.

Let $A$ be a group acting measure preservingly on two probability spaces $(X_1, m_1)$ and $(X_2, m_2)$. We say $\mu$ is a joining for the two actions if $\mu$ is a measure on $X_1 \times X_2$ that projects to $m_1$ resp. $m_2$ under the coordinate projections and is invariant under the diagonal action of $A$.

Thm (Einsiedler, Lindenstrauss '19)

Joinings between higher rank actions on irreducible arithmetic quotients of semi-simple Lie groups are always homogeneous.