BACKGROUND

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I missed the first lecture, and have never taken a course in mathematical logic, so this is my attempt at learning some of the basics, and hopefully it will simultaneously prepare me for the second lecture.

1. Languages and structures

Definition 1. A language $\mathcal{L}$ consists of the following data:

(1) A set F, called the set of function symbols, and a positive integer n_f for every $f \in F$,
(2) a set R, called the set of relation symbols, and a positive integer n_r for every $r \in R$,
(3) a set C, called the set of constant symbols.

Remark 1. The number n_f should be thought of as the number of 'arguments' of f. The number n_r should be thought of as telling us that r is an n_r-ary relation.

Definition 2. An $\mathcal{L}$-structure, $\mathcal{M}$, consists of the following data:

(1) A set M, called the underlying set of $\mathcal{M}$,
(2) a function $f^{\mathcal{M}} : M^{n_f} \to M$ for each $f \in F$,
(3) a set $r^{\mathcal{M}} \subseteq M^{n_R}$ for every $r \in R$,
(4) and an element $c^{\mathcal{M}} \in M$ for every $c \in C$.

The function $f^{\mathcal{M}}$, the set $r^{\mathcal{M}}$, and the element $c^{\mathcal{M}}$ are referred to as the *interpretation* of the symbols f, r, and c.

Definition 3. Let $\mathcal{L}$ be some language, and $\mathcal{M}$ and $\mathcal{N}$ be two $\mathcal{L}$-structures. A map of the underlying sets $\alpha : M \to N$ is a morphism of $\mathcal{L}$-structures iff it preserves the interpretation of the symbols in the sense that

(1) $\alpha\left(f^{\mathcal{M}}\left(m_1, \cdots, m_{n_f}\right)\right) = f^{\mathcal{N}}\left(\alpha\left(m_1\right), \cdots, \alpha\left(m_{n_f}\right)\right)$ for all $f \in F$ and $m_i \in M$,
(2) $\left(m_1, \cdots, m_{n_r}\right) \in r^{\mathcal{M}}$ iff $\left(\alpha\left(m_1\right), \alpha\left(m_2\right), \cdots, \alpha\left(m_{n_r}\right)\right) \in r^{\mathcal{N}}$ for all $r \in R$ and $m_i \in M$, and
(3) $\alpha\left(c^{\mathcal{M}}\right) = c^{\mathcal{N}}$ for all $c \in C$.

Definition 4. A *word in the language* $\mathcal{L}$ is a string of symbols consisting of:

(1) Symbols of $\mathcal{L}$,
(2) variable symbols $v_1, v_2, \cdots$,
(3) the equality symbol $=$,
(4) the Boolean connectives $\vee$, $\wedge$, and $\neg$,
(5) the quantifiers $\exists$ and $\forall$, and
(6) parentheses (,).

Date: January 23, 2019.

Definition 5. The set of $\mathcal{L}$-terms is the smallest set T such that

(1) $C \subseteq T$,
(2) $v_i \in T$ for all i,
(3) if $t_1, \cdots t_{n_f} \in T$ then $f\left(t_1, \cdots, t_{n_f}\right) \in T$ for all $f \in F$.

Remark 2. A term is supposed to be thought of as a word that can be interpreted as an element of an $\mathcal{L}$-structure, possibly after substitution of variables.

Let $t \in T$ be built from variables $v_{i_1}, \cdots, v_{i_m}$. We want to view t as a function $t^{\mathcal{M}} : M^m \to M$. Inductively define any term $s \in T$ to act on $\overline{a} = (a_{i_1}, \cdots, a_{i_m})$ as follows:

(1) If s is the constant symbol c, then $s^{\mathcal{M}}(\overline{a}) = c^{\mathcal{M}}$.
(2) If s the variable v_{i_j}, then $s^{\mathcal{M}}(\overline{a}) = a_{i_j}$.
(3) If s is the term $f\left(t_1, \cdots, t_{n_f}\right)$ for some function symbol f of $\mathcal{L}$, and the t_{n_i} are terms, then

$$s^{\mathcal{M}}(\overline{a}) = f^{\mathcal{M}}\left(t_1^{\mathcal{M}}(\overline{a}), \cdots, t_{n_f}^{\mathcal{M}}(\overline{a})\right) .$$

Definition 6. An atomic formula is of one of the following forms:

(1) $t_1 = t_2$, for $t_1, t_2 \in T$
(2) $r(t_1, \cdots, t_{n_r})$ for $r \in R$, and $t_i \in T$.

Definition 7. The set of formulae for a language $\mathcal{L}$ is the smallest set containing all atomic formulae such that

(1) if φ is a formula then $\neg\varphi$ is a formula,
(2) if φ and ψ are formulas then $\varphi \vee \psi$ and $\varphi \wedge \psi$ are formulae, and
(3) if φ is a formula, then $\forall v_i \varphi$ and $\exists v_i \varphi$ are formulas.

Definition 8. We say a variable is *free* inside a formula φ if it appears outside of the quantifiers $\forall v$ or $\exists v$. Otherwise we say it is *bound.* A formula φ is a *sentence* iff it has only bound variables.

The point is that $\mathcal{L}$-sentences must be true or false in any given $\mathcal{L}$-structure. On the other hand, if a formula φ has n free variables, then it is somehow expressing a property of elements of M^n, so we want a notion of whether or not such a formula is true when we substitute things in for the free variables. We will construct this notion inductively.

Definition 9. Let φ be a formula with free variables from $(v_{i_1}, \cdots, v_{i_m})$, and write $\overline{a} = (a_{i_1}, \cdots, a_{i_m}) \in M^m$. Then define $M \models \varphi(\overline{a})$ as follows:[1]

(1) If φ is $t_1 = t_2$, then $M \models \varphi(\overline{a})$ if $t_1^{\mathcal{M}}(\overline{a}) = t_2^{\mathcal{M}}(\overline{a})$.
(2) If φ is $r(t_1, \cdots, t_{n_r})$ then $\mathcal{M} \models \varphi(\overline{a})$ if $\left(t_1^{\mathcal{M}}(\overline{a}), \cdots t_{n_r}^{\mathcal{M}}(\overline{a})\right) \in R^{\mathcal{M}}$
(3) If φ is $\neg\psi$, then $\mathcal{M} \models \varphi$ if $\mathcal{M} \not\models \psi$.
(4) If φ is $\psi \wedge \theta$, then $\mathcal{M} \models \varphi$ if $\mathcal{M} \models \psi$ and $\mathcal{M} \models \theta$.
(5) If φ is $\psi \vee \theta$, then $\mathcal{M} \models \varphi$ if $\mathcal{M} \models \psi$ or $\mathcal{M} \models \theta$.
(6) If φ is $\exists v_j \psi(\overline{v}, v_j)$, then $\mathcal{M} \models \varphi$ if there exists $b \in M$ such that $\mathcal{M} \models \psi(\overline{a}, b)$.
(7) If φ is $\forall v_j \psi(\overline{v}, v_j)$, then $\mathcal{M} \models \varphi$ if for all $b \in M$ $\mathcal{M} \models \psi(\overline{a}, b)$.

[1]We say that $\varphi(\overline{a})$ satisfies M, or $\varphi(\overline{a})$ is true in M.

2. Theories, models, definable sets

Definition 10. Let $\mathcal{L}$ be a first-order language, then an $\mathcal{L}$-theory T is a set of $\mathcal{L}$-sentences. We say that $\mathcal{M}$ is a *model* of T if $\mathcal{M} \models \varphi$ for all $\varphi \in T$. A theory is *satisfiable* if there exists some model of it.

Definition 11. We say that two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$ are *elementarily equivalent* if we have that $\mathcal{M} \models \varphi$ iff $\mathcal{N} \models \varphi$ for every $\mathcal{L}$-sentence φ.

Let $\mathcal{L}$ be a first-order language, and T an $\mathcal{L}$ theory. Any formula $\varphi(x_1, \cdots, x_n)$ which has free variables only among the x_i, together with a model $\mathcal{M}$ of T, evaluates to a truth value at any point $a = (a_1, \cdots, a_n) \in M^n$. Therefore it determines a subset

$$\varphi(M) = \{a \in M^n \mid \varphi(a)\} \ .$$

Two formulas φ and ψ with the same number of free variables are equivalent if $\varphi(\mathcal{M}) = \psi(\mathcal{M})$ for every model $\mathcal{M}$ of T.

Definition 12. A set X is definable if it is the equivalence class of formulas in $\mathcal{L}$ under this relation.

Remark 3. If we consider the category $\mathcal{M}_{el}(\mathcal{L})$ of $\mathcal{L}$-structures and elementary embeddings, and its full subcategory $\mathcal{M}_{el}(T)$ whose objects are models of T, then definable set for a theory T is a functor $\mathcal{M}_{el}(T) \to \textbf{Set}$ which is of the form $\mathcal{M} \to \varphi(\mathcal{M})$ for some $\mathcal{L}$-formula φ.

The idea is that (as long as we have no parameters) a set $A \subseteq M^n$ is definable in $\mathcal{M}$ iff there exists a formula φ such that

$$\overline{a} \in A \qquad \Longleftrightarrow \qquad \mathcal{M} \models \varphi(\overline{a})$$