LECTURE 10
MATH 229

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We will continue with the partite construction, and just as we did for EPPA, prove the Ramsey property for various structures. We will see that the argument for Ramsey statements is easier to adapt, and indeed we know more Ramsey statements than we do EPPA statements.

1. K_n-FREE GRAPHS

Theorem 1 (Nešetřil-Rödl)**.** *the class of finite ordered K_n-free graphs is Ramsey.*

Proof. Just as with EPPA, we will start with a Ramsey object for a larger class, i.e. all ordered graphs, and fix it by removing some edges to get rid of K_ns. Let A, B be finite ordered K_n-free graphs. Let $r < \omega$. First take C_0 to be such that $C_0 \to (B)^A_r$ just as ordered graphs. The issue is this might have K_ns.

We build a sequence of $d+1$ pictures P_i where $d = \left|\binom{C_0}{A}\right|$. Enumerate the copies of A in C_0 as

$$\binom{C_0}{A} = \{A_1, \cdots, A_d\} \ .$$

First we build the 0th picture, P_0, which is a disjoint (free) union of copies of B in C_0 (as partite graphs). In particular, we have a homomorphism $P_0 \to C_0$ which respects partitions.

We now build the P_i inductively. Assume we have built the picture P_{i-1}. Consider A_i, and let B_i be the preimage $\pi^{-1}(A_i)$ where $\pi : P_{i-1} \to C_0$ is the canonical homomorphism. The partite lemma applied to A_i, B_i gives us C_i such that $C_i \to (B_i)^{A_i}_r$ as partite graphs. Note that B_i has a homomorphism to A_i and the partite lemma preserves this property, i.e. it does not put edges between two parts if there is no such edge in B_i. Hence C_i is K_n-free. Now we build P_i as follows. For every copy of B_i in C_i we extend this to a copy of P_{i-1}, and then amalgamate those freely over the B_i. This does not create K_ns, and we still have a homomorphism $\pi : P_i \to C_0$.

Then the statement is that P_d is our Ramsey object. As before we show by downward induction that if $\binom{P_d}{A}$ is r-colored, there is a copy of P_0 in P_d in which any two copies of A with the same projection to C_0 have the same color. This gives us a coloring of $\binom{C_0}{A}$. By the Ramsey property of C_0, we find a homogeneous copy of B. The corresponding copy of B in P_0 is homogeneous. □

Remark 1. This is somehow the same as the proof for ordered graphs from last lecture, except instead of starting with the usual Ramsey theorem which gives

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$N \to (|B|)_r^{|A|}$, we now start with $C_0 \to (B)_r^A$ as ordered graphs. But the rest of the steps are effectively the same.

2. Irreducible structures

In fact the techniques of the previous proof work for any structures which behave in a similar way to ordered graphs. This notion is captured by the following more general statement.

Definition 1. A relational structure is *irreducible* if any two distinct elements belong to some relation.

Let $\mathcal{F}$ be a family of finite irreducible structures in a relational language $\mathcal{L}$. Let $\mathrm{Forb}\,(\mathcal{F})$ be the class of $\mathcal{L}$-structures which omit all structures which have members of $\mathcal{F}$ as induced substructures. Then the same proof as above will give that the class of ordered expansions of members of $\mathrm{Forb}\,(\mathcal{F})$ is Ramsey.

3. Metric spaces

The class of metric spaces is not covered by the previous theorem because bad n-cycles for $n \geq 4$ are not irreducible. The point is that somehow for 4-cycles (and larger), opposite vertices have no relation, but we want to sometimes forbid this to get the class of metric spaces, so metric spaces can't be defined only by forbidding things. Nonetheless we have the following:

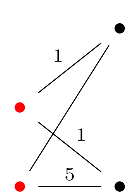
Theorem 2. *The class of finite ordered metric spaces is Ramsey.*

Proof. Let A, B be finite ordered metric spaces, and $r < \omega$. Let

$$n = \frac{\max\ \text{length in } B}{\min\ \text{length in } B} + 1\ .$$

This is the maximal size of a bad cycle. First, build C_0 so that $C_0 \to (B)_r^A$ as colored ordered graphs and all distances in C_0 appear in B.

We now execute the construction of the previous proof $n-2$ times[1] to obtain C_0, C_1 , $\cdots$, C_{n-2} with each new structure acting as the base, i.e. we have homomorphisms $\pi_i : C_i \to C_{i-1}$. Now we claim by induction that C_i doesn't have bad cycles of size $i+2$. Let us prove this for i. C_i is built as an increasing union of pictures $P_0, \cdots, P_d$. P_0 is a disjoint union of copies of B since it has no bad cycle at all. We now build P_i assuming we have built P_{i-1}. Note that the homomorphic image of a bad cycle at least contains a bad cycle. So in π^{-1} of a copy of A in C_{i-1} there is no bad cycle. This property of having a homomorphism is preserved, so in particular applying the partite lemma does not create bad cycles. But now the free amalgamation of things without bad cycles might have a bad cycle. For example, if we amalgamate over the red points:



[1]One might expect this to take n times, but it only takes $n-2$ because already C_0 won't have bad 2-cycles.

we get a bad cycle. But as it turns out this can only happen if we already have a bad triangle. I.e. the general claim is that every projection of a bad cycle K contains a bad cycle of smaller size. This is because of the following. Assume we create a bad cycle by amalgamating two copies of P_{i-1}. Any bad cycle has to have points in both copies by the induction hypothesis. If the projection π is not injective on K we get a smaller bad cycle in the base C_{i-1}, so we can assume the projection to the base is injective. Now notice that K must have at least two vertices which both project to A_i with no distance defined. But since A_i is complete, we must add a distance when we project. But this means we have broken the bad cycle into two pieces, one of which must therefore be a smaller bad cycle.

This means C_{n-2} has no bad cycle at all, and now we complete this arbitrarily to an ordered metric space, and we are done. □

4. Locally finite structures

We now state a general definition which captures the features of metric spaces which we took advantage of in the previous proof.

Recall a homomorphism is a map which sends relations to relations, but it might not be injective, and might miss some relations. As usual, an embedding is injective and preserves relations in both directions, i.e. an isomorphism with its image.

Definition 2. A *homomorphism-embedding* $f : A \to B$ is a homomorphism whose restriction to any irreducible substructure of A is an embedding.

Definition 3. A *completion* B of A is an irreducible structure B with a homomorphism-embedding $f : A \to B$. A *strong completion* is a completion such that the homomorphism-embedding is injective.

Definition 4. Let $\mathcal{R}$ be a class of finite irreducible structures. Then $\mathcal{K} \subseteq \mathcal{R}$ is a *locally finite* subclass if for every $C_0 \in \mathcal{R}$ there is some integer[2] $n = n(C_0)$ such that any structure C has a strong $\mathcal{K}$-completion provided that:

- C_0 is a completion of C
- every substructure of size at most n has a strong $\mathcal{K}$-completion.

Theorem 3. *Let $\mathcal{R}$ be a Ramsey class of irreducible finite $\mathcal{L}$-structures and let $\mathcal{K}$ be a hereditary locally finite subclass of $\mathcal{R}$ with strong[3] amalgamation. Then $\mathcal{K}$ is homogeneous.*

[2] This is supposed to be the maximal size of a bad substructure.

[3] This just means disjoint.