LECTURE 12
MATH 229

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The combinatorial portion of the course is now more-or-less over. Now we will start discussing the automorphism group. This will deal with topological dynamics and descriptive set theory.

1. Kechris-Pestov-Todorcevic correspondence

Let M be a homogeneous structure (locally finite, or just relational). Recall that M is Ramsey iff for every universal theory T' in a language $L' \supseteq L$ consistent with $\mathrm{Th}(M)$, M has a definable expansion to a model of T'.

Take such a T', and let M' be some expansion of M to a model of T' in $L' \supseteq L$. Now we have an action $\mathrm{Aut}(M) \circlearrowright M'$. Another way of thinking of this is the following. For simplicity say $L' = L \cup \{R\}$ for R some k-ary relation. Then an expansion of M to L' can be thought of as an element of $2^{(M)^k}$. Note that the set S of elements of $2^{(M)^k}$ corresponding to models of T' is closed in the product topology. In particular, it is compact.

Now we have an action $\mathrm{Aut}(M) \circlearrowright 2^{(M)^k}$ and then the action we really care about is the induced action $\mathrm{Aut}(M) \circlearrowright S$. This is well defined because the forbidden conditions defining S are invariant under automorphisms. Furthermore this action is continuous, i.e. the map

$$\mathrm{Aut}(M) \times 2^{(M)^k} \to 2^{(M)^k}$$

is continuous. To see this let $V \subseteq 2^{(M)^k}$ be a basic open, i.e. it depends only on a finite set $\overline{a} \in M$. If $g \cdot \overline{n} \in V$, then for any $\overline{n}'$ which coincides with $\overline{n}$ on $g^{-1}(\overline{a})$ and for any h such that $h|_{g^{-1}(\overline{a})} = g|_{g^{-1}(\overline{a})}$ we have $h \cdot \overline{n}' \in V$.

So we have a continuous action on a compact space S. Now we can restate the criterion from above as follows:

M is Ramsey iff for all T', the action $\mathrm{Aut}(M) \circlearrowright S$ has a fixed point.

Now we might wonder what is so special about S, and as it turns out the only special thing is that it's compact. To see this we need some definitions.

Definition 1. Let G be a topological group. Then a G-flow is a compact Hausdorff space X along with a continuous action $G \circlearrowright X$.

Theorem 1 (KPT)**.** *If M is as above, then M is Ramsey iff all $\mathrm{Aut}(M)$ flows have a fixed point.*

Proof. ($\Longleftarrow$): See above.

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($\implies$): Assume M is Ramsey and all substructures are rigid.[1] Let $G = \text{Aut}(M)$, and let X be a G-flow. Assume X does not have a fixed point. Now we want to use compactness to find finitely many elements of the group which move all of the points. First we recall the following lemma:

Lemma 1 (Urysohn)**.** *If X is compact Hausdorff, and $A, B \subseteq X$ are two disjoint, closed subspaces, then there is a continuous function $f : X \to [0,1]$ such that $f|_A = 0$ and $f|_B = 1$.*

By assumption, for all $x \in X$, there is $f : X \to [0,2]$ and $g \in G$ such that $f(x) = 0$ and $f(gx) = 2$. Given such f and g, the set:

$$U_{f,g} = \{x \in X \mid |f(x) - f(gx)| > 1\}$$

is open. By compactness, there are $f_1, \cdots, f_n$ and $g_1, \cdots, g_n \in G$ such that for all $x \in X$ there is $i \leq n$ such that

$$|f_i(x) - f_i(g_i x)| > 1 \ .$$

Define

$$\begin{aligned} X &\xrightarrow{\quad F \quad} \mathbb{R}^2 \\ x &\longmapsto (f_1(x), \cdots, f_n(x)) \end{aligned}$$

so for all $x \in X$ there exists $i \leq n$ such that

$$\|F(x) - F(g_i x)\|_\infty > 1 \ .$$

Now notice that F is continuous and X is compact, so the image of F in $\mathbb{R}^n$ is compact. So now we can discretize the image into finitely many blocks each with diameter less than 1/3. I.e. we can write

$$F(X) = Y_1 \cup \cdots \cup Y_n$$

where the Y_i have diameter $< 1/3$.

Now we claim there is some neighborhood $V \subseteq G$ of the identity such that for all $g \in V$ and $x \in X$

$$\|F(x) - F(gx)\| < 1/3 \ .$$

This is because F is continuous and X is compact. There is a finite[2] substructure $A \subseteq M$ such that

$$\{g \in G \mid \ g|_A = \text{id}|_A\} \subseteq V \ . \tag{1}$$

In particular, if $g|_A = h|_A$, then $\left\|F(x) - F\left(h^{-1}gx\right)\right\| < 1/3$, so for all $x \in X$,

$$\left\|F\left(g^{-1}x\right) - F\left(h^{-1}x\right)\right\| < \frac{1}{3} \ .$$

The idea is that if two things do the same thing on A, then their image is very close. So they almost do the same thing even after applying F.

Fix $x_0 \in X$, and consider a copy A' of A in M. Now we want to color these copies of A. Let $c(A')$ be the minimal $j \leq m$ for which there is $g \in G$ such that $g(A) = A'$ with $F\left(g^{-1}x_0\right) \in Y_j$. What we're really coloring is cosets of (1) (which is equivalent to a coloring of copies of A in M).

[1] This is certainly true if we have a linear order.

[2] We are using the locally finite condition to get this to be finite.

Now we want to construct some substructure B which contradicts Ramsey. So let B be the structure generated by

$$A \cup \bigcup_{i \leq n} g_i^{-1}(A) \ .$$

Since M is Ramsey, there is $B' \subseteq M$ a copy of B such that all copies of A in B' have the same color j.

Let $g_* \in G$ map B to B', and let $x_* = g_*^{-1}(x_0)$. Now we show that

$$\|F(g_i x_*) - F(x_*)\| < 1$$

for all $i \leq n$. So somehow this x_* is not moved much. This will be the contradiction. Set $g_0 = \text{id}$. For $i \leq n$ we have that $g_* g_i^{-1}(A)$ is a copy of A in B', so it has color j. By the definition of the color, there is some $h_i \in G$ such that

$$h_i|_A = g_* g_i^{-1}\big|_A \qquad\qquad F\left(h_i^{-1} x_0\right) \in Y_j \ .$$

So now we just have to compute. We know that

$$1/3 > \left\|F\left(h_i^{-1} x_0\right) - F\left(g_i g_*^{-1} x_0\right)\right\| = \left\|F\left(h_i^{-1} x_0 - F\left(g_i x_*\right)\right)\right\| \ . \tag{2}$$

Now we have that:

$$\begin{aligned} \|F(g_i x_*) - F(x_*)\| \leq{}& \left\|F(g_i x_*) - F\left(h_i^{-1} x_i\right)\right\| \\ &+ \left\|F\left(h_i^{-1} x_0\right) - F\left(h_0^{-1} x_0\right)\right\| + \left\|F\left(h_0^{-1} x_0\right) - F(x_*)\right\| \ . \end{aligned} \tag{3}$$

On the RHS of (3), the first and third terms are less than 1/3 because of (2), and the middle term is less than 1/3 since they are both in Y_j. So the entire RHS is less than 1. □

1.1. **Some definitions.** Let G be a topological group.

Definition 2. A topological group G is *extremely amenable* if all G-flows have a fixed point.

Definition 3. A G-flow X is minimal if every orbit is dense. Equivalently there is no proper non-empty G-invariant closed $Y \subseteq X$. Such a Y is called a subflow.

Remark 1. Every G-flow has a minimal subflow (using Zorn's lemma) so G is extremely amenable iff its only minimal flow is $\{\text{pt}\}$.

Next time we will define a universal flow which measures how far you are from being extremely amenable, and somehow tells us something about a minimal Ramsey expansion.