LECTURE 13
MATH 229

LECTURE: PROFESSOR PIERRE SIMON
NOTES: JACKSON VAN DYKE

1. G-flows

We will continue discussing G-flows.

Definition 1. Let X be a G-flow. A *factor* of X is a G-flow Y for which there is a surjective morphism $\pi : X \to Y$, i.e. a continuous map respecting the G action.

The idea is that X is 'at least as complicated' as Y.

Remark 1. Note that if $f : X \to Y$ is a morphism of G flows and Y is minimal, then f is surjective.

Definition 2. A *universal minimal flow* is a minimal G-flow X which has every minimal G-flow as a factor.

Theorem 1. *There is a unique universal minimal G-flow.*

Proof. Interestingly enough, uniqueness is harder to show than existence here. We deal with existence first. Note that if X is a minimal G-flow, then it has a dense orbit of size at most $|G|$, so $|X| \leq 2^{2^{|G|}}$.

Let $(X_i, i < \alpha)$ be an enumeration of all minimal G-flows (up to isomorphism). Take

$$X_* = \prod_{i<\alpha} X_i$$

with the coordinate-wise action of G. Note the product of compact spaces is compact so this is a G-flow. Let $\tilde{X} \subseteq X_*$ be a minimal subflow. Then $\tilde{X}$ admits each X_i as a factor, so by the above remark, it is a universal minimal flow.

The universal property here somehow doesn't have uniqueness built in (as it usually is), so uniqueness isn't so easy here. The formal proof involves introducing new machinery, so we just sketch the proof. It is enough to find some coalescent minimal flow Z. A flow is coalescent if every endomorphism is an automorphism. For a minimal flow this is just saying it is injective since every map to it is already surjective. This is enough because if M and M' are universal minimal flows, and Z is coalescent, then we have a sequence of maps $Z \to M \to M' \to Z$ and the composition has to be an automorphism, and since they're all already surjective, they are all automorphisms. □

Note that G is extremely amenable iff its universal minimal flow is a point. So the universal minimal flow is supposed to be a measure of how far a flow is from being extremely amenable.

Date: March 5, 2019.

2. Back to Fraïssé classes

Let L_0 be a language and let $L = L_0 \cup \{\leq\}$ where $\leq \notin L$. Let K_0 be a Fraïssé class in L_0 and K be a Fraïssé class in L. We want some notion of K being an expansion of K_0.

Definition 3. Say that K is *reasonable* if:

(1) the L_0 reduct of K is K_0, and
(2) if $A \leq B$ in L_0 and A' is an expansion of A in K then there is an expansion B' of B in K such that $A' \leq B'$.

Definition 4. Let M_0 be the Fraïssé limit of K_0. Then a K-*admissible* ordering on M_0 is an expansion of M_0 to L whose age is included in K.

Note that this is not the Fraïssé limit of K, which would not be compact. A K-admissible ordering is naturally an $\operatorname{Aut}(M_0)$-flow, so in particular it is compact.

Definition 5. Say that K has the *ordering property* (relative to K_0) if for every $A \in K_0$, there is $B \in K_0$ such that for any $A', B' \in K$ expanding A and B respectively, we have $A' \subseteq B'$.

Theorem 2 (KPT)**.** *With notation as above, let X_K be the space of K-admissible orderings. Then TFAE:*

(i) K has the Ramsey and ordering properties.
(ii) X_K is the universal minimal $\operatorname{Aut}(M_0)$-flow.

Example 1. The universal minimal flow of $S_\infty = \operatorname{Aut}(\omega, =)$ is the space of linear orders on ω. This was first proved by Glasner-Weiss.

Note that with the assumptions of the above theorem, the universal minimal flow is metrizable. In general the compact space 2^X with X countable is metrizable. The metric is as follows. Fix a bijection $X \simeq \omega$ and then the distance between $(a_i, i < \omega)$ and $(b_i, i < \omega)$ is 2^{-m} where m is maximal such that $a_i = b_i$ for $i < m$.

Extremely amenable means the universal minimal flow is a point, and having metrizable universal minimal flow says that it is still somehow small. For example, discrete groups don't have metrizable universal minimal flows unless they are finite.

Definition 6. Let K_0 be a Fraïssé class. Then $A \in K_0$ has Ramsey degree d if for every $B \in K_0$ there is $C \in K_0$ such that for any coloring $f : \binom{C}{A} \to r$ there is a copy B' of B in C such that f takes at most d values on $\binom{B'}{A}$.

We say K_0 has finite Ramsey degree if for every $A \in K_0$, there is $d < \omega$ such that A has finite Ramsey degree d.

Note that K_0 has finite Ramsey degree for embeddings iff it has finite Ramsey degree for substructures.

Also observe that if there is K which is Ramsey and expands K_0 reasonably by finitely many relation symbols, then K_0 has finite Ramsey degree and it is bounded by the number of expansions of A to a structure in K. The converse is not clear, but it turns out to be true.

Theorem 3 (Zucker)**.** *Let K_0 be a Fraïssé class (homogeneous and locally finite). Then TFAE:*

(i) $\operatorname{Aut}(M_0)$ has metrizable universal minimal flow (where M_0 is the Fraïssé limit of K_0).

(ii) *K_0 has finite Ramsey degree.*
(iii) *There is a Fraisse class K which is Ramsey, is a reasonable expansion of K_0 obtained by adding finitely many relational symbols, (and has the expansion property).*[1]

Note that the universal minimal flow of $\mathbb{Z}$ is a minimal flow of the Stone-Čech compactification $\beta\mathbb{Z}$ which is not metrizable. In general locally compact groups which are not compact have non-metrizable universal minimal flow.

Question 1. If K is a Fraïssé class with ω-categorical limit does it have metrizable universal minimal flow?

The answer is no in general. But it is open in the finitely homogeneous case, i.e. L is finite relational.

2.1. **Hrushovski construction.** We offer an example of a Fraïssé class with ω-categorical limit which does not have metrizable universal minimal flow.

Theorem 4 (Hrushovski)**.** *There exists an ω-categorical graph G which is 2-sparse, i.e. for any finite $A \subseteq G$,*

$$|E(A)| \leq 2|V(A)|$$

and every vertex has infinite degree.

If we remove any one of these conditions this is easy. If we remove the last, we can just take the empty graph. If we remove the ω-categorical condition we can just use a tree. But this is not ω-categorical. We now sketch why this thing doesn't have an ω-categorical Ramsey expansion.

Theorem 5 (Evans)**.** *Such a structure does not have an ω-categorical Ramsey expansion, and hence does not have metrizable universal minimal flow.*

Proof. We will find a universal theory such that no completion of it can be ω-categorical, since we can define distances. The universal theory will be as follows. We will orient the edges with the assumption that there are at most 2 outgoing edges from every vertex. We call this a 2-orientation For example, if we have a tree, we can choose a root, and orient every edge towards this root.

The following essentially follows from the Hall marriage theorem:

Fact 1. *A graph has a 2-orientation iff it is 2-sparse.*

An ω-categorical Ramsey expansion of G has to define a 2-orientation (as this is a universal theory). But this implies there are infinitely many 2-types. The idea is that if we follow the orientations, every point we can reach from a point a is in the algebraic closure of a, and this must end at some point. Write the size of the closure as k. Assume there is some bound, and take a which maximizes this k. a has infinite degree, so it must have some point b not reachable from a which maps into it, which means the closure of b contains the closure of a, but it also contains a so the size of the closure of b is at least $k+1$ which is a contradiction to a having maximal closure. □

[1] This is true with or without this last piece.