LECTURE 16
MATH 229

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1. Descriptive set theory

We will continue with our crash course in descriptive set theory.

1.1. Polish spaces.

Corollary 1. *Polish spaces are precisely the G_δ subspaces of W.*

Then we have an analogous result:

Proposition 1.
- *Every perfect[1] Polish space has a subspace homeomorphic to the cantor set C.*
- *If X is a compact polish space then there is a co-continuous surjective map $C \to X$. The idea is that the cantor set is universal among compact polish spaces.*

Sketch proof. Let X be perfect Polish. Pick two points, pick two disjoint open sets containing them. These are not isolated so we can break these into two again. Now we can continue this process with decreasing radius of the balls going to 0. Then we get a homeomorphic embedding.

In the second part, since X is compact we can cover it by finitely many balls possibly not disjoint. And then we're naturally constructing a surjective map from the finite tree to X, and then we check that this is just homeomorphic to C. □

A compact space X is polish iff metrizable iff second countable. Note this is not equivalent to separable.

Example 1 (Counter-example)**.** $[0,1]^{[0,1]}$ is compact, separable, and not second countable.

Let X be Polish. Define $K(X)$ to consist of the compact subspaces of X with the Vietoros topology. The base is given by

$$\{K \subseteq X \mid \exists U_0, \cdots, U_n \subseteq X \text{ s.t. } K \subseteq U_0, K \cap U_1 \neq \emptyset, \cdots, K \cap U_n \neq \emptyset\} \ .$$

This is metrizable. In particular this is given by the Hausdorff metric. Define

$$d_H(K, L) = \begin{cases} 0 & K = L = \emptyset \\ 1 & K = \emptyset \neq L \text{or} L = \emptyset \neq K \\ \max\{\delta(K, L), \delta(L, K)\} & K, L \neq \emptyset \end{cases}$$

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[1]Recall this means it has no isolated points.

where

$$\delta(K,L) \coloneqq \sup_{x\in K} d(x,L) \ .$$

Then it is a fact that this metric gives the above topology. In fact, if $D \subseteq X$ is dense,

$$K_f(D) = \{K \subseteq D, K \text{ finite}\}$$

is dense in $K(X)$, so $K(X)$ is Polish.

1.2. **Baire category.** Let X be a topological space.

Definition 1. A set $A \subseteq X$ is *nowhere dense* if $\overline{A}$ has empty interior.

(1) A set $A \subseteq X$ is *meager* (a first category) if $A = \cup A_n$ where A_n is nowhere dense (equivalently, $A \subseteq \cup F_n$, $F_n \subseteq X$ closed with empty interior).
(2) $A \subseteq X$ is *comeager* if $X \setminus A$ is meager[2] (equivalently A contains a countable intersection of dense opens).

Definition 2. The space X is *Baire* if any of the following equivalent conditions are satisfied:

(i) Every nonempty open set is non-meager.
(ii) A meager set has empty interior.
(iii) Every comeager set is dense.
(iv) Intersection of countably many dense open sets is dense.

Theorem 1. *Every completely metrizable space is Baire. Every locally compact Hausdorff space is Baire.*

Proof. Consider a countable intersection of dense open sets

$$\bigcap_{n<\omega} U_n \ .$$

Take a ball in U_1 of radius < 1. Then take a ball of radius $< 1/2$ in U_2, and continue in this fashion. Then what we get is a point in the intersection of the open sets. □

In a Baire space, meager sets form a σ-ideal. This means the collection of these is closed under taking subsets, countable unions, and doesn't contain the whole space.[3] The idea is that this gives a notion of smallness. Meager sets somehow have measure 0, non-meager somehow have positive measure, and co-meager sets have measure 1.

We will say that a property holds *generically* if it holds on a comeager set. We write $\forall^* x P(x)$ to mean that P holds generically, and $\exists^* x P(x)$ means $\{x \mid P(x)\}$ is non-meager.

Definition 3. A set $A \subseteq X$ has the Baire property (BP) if $A \triangle U$ is meager for some open U. The notation is that $A =^* U$.

Proposition 2. *The class of subsets of X which have BP is a σ-algebra.*

[2] This is also called residual.

[3] Recall this is different from a σ-algebra because, first of all, a σ-algebra must contain the whole space, and it is closed under countable unions, but also complements, which implies it is closed under countable intersections as well.

Proof. Recall this means it is closed under complements, countable unions, and countable intersections. Note that for open U we have $U =^* \overline{U}$ and for closed F we have $F =^* \operatorname{int}(F)$. If A has BP, then $A =^* U$ for some U, and then

$$X \setminus A =^* X \setminus U =^* \operatorname{int}(X \setminus U) \ .$$

If $A_n =^* U_n$ then

$$\bigcup_n A_n =^* \bigcup_n U_n$$

so we are done. □

Notice that every Borel set has BP (recall the Borel sets comprise the smallest σ-algebra containing all open sets).

Lemma 1. *If A has BP then we can write*

$$G \subseteq A \subseteq F$$

such that G is a G_δ set, and F is an F_σ set such that $F \setminus G$ is meager.

Proof. $A =^* U$ for some open U, which means $A \triangle U \subseteq F$, where F is closed and meager. So $G = U \setminus F$ is G_δ, $G \subseteq A$, and $A \setminus G \subseteq F$ is meager. Now we can apply this to the complement $X \setminus A$ to get the F_σ set. □

Definition 4. A function $f : X \to Y$ is Baire-measurable if the preimage of any open set has BP.

Now we have some sort of Fubini property.

Theorem 2 (Kuratowski-Ulam)**.** *Let X and Y be second countable topological spaces. If $A \subseteq X \times Y$ then*

- *(i)* $\forall^* x \ A_x = \{y : A(x, y)\}$ *has BP.*
- *(ii)* A *is meager iff* $\forall^* x, A_x$ *is meager.*
- *(iii)* A *is comeager iff* $\forall^* x A_x$ *is comeager.*

We said this is some sort of Fubini property because this can be rewritten as

$$\forall (x, y) \, A(x, y) \iff \forall^* x \forall^* y A(x, y) \iff \forall^* y \forall^* x A(x, y) \ .$$

Theorem 3. *If X is perfect Polish and $E \subseteq X^2$ is a meager equivalence relation then*

$$|X/E| = 2^{\aleph_0}$$

and in fact there is a cantor set $C \subseteq X$ which is transversal.

1.3. **Analytic sets.**

Definition 5. If X is Polish, a subset $A \subseteq X$ is *analytic* if there is Y Polish and $f : Y \to X$ continuous with $f(Y) = A$.

We say $A \subseteq X$ is *co-analytic* if $X \setminus A$ is analytic.

Theorem 4. *If X is Polish, then*

- *Every analytic (and hence co-analytic) set has BP.*
- *A set $A \subseteq X$ is Borel iff it is both analytic and co-analytic.*

2. Polish groups

Recall the following. A topological group is a group G equipped with a topology so that the map $G^2 \to G$ which sends $(x,y) \mapsto xy^{-1}$ is continuous. If G is a topological group and $H \subseteq G$ is an open subgroup, then H is closed.

Definition 6. A *Polish group* is a topological group whose topology is Polish.

Theorem 5 (Birkhoff-Kakutani)**.** *Let G be a topological group. Then G is metrizable iff there is a countable basis of a neighborhood of 1. In this case there is a left-invariant compatible metric.*

To be clear, left-invariant means that $d(x,y) = d(gx, gy)$.

Remark 1. It is important that we define a Polish group as a topological space rather than a metric space. Since it is polish there is always a complete metric, and then by the theorem there is always this second left-invariant metric. But in general there is no complete invariant metric. If it is locally compact this does exist (like for $\mathbb{R}$).

Proposition 3. *Let G be a Polish group and let $H \leq G$. Then H is Polish iff H is G_δ. This is also equivalent to H being closed. In this case G/H is a Polish space, and in particular a Polish group if $H \trianglelefteq G$.*

Proof. We prove only the second ($\iff$). Let $H \leq G$ be G_δ and assume $G = \overline{H}$. Then H is a dense G_δ set, and so is every coset of H. However we cannot have two disjoint comeager sets which means there is only one coset so $H = G$. □

Theorem 6 (Pettis)**.** *Let G be a topological group with non-meager $A \subseteq G$ having BP. Then $A^{-1}A$ contains a neighborhood of 1.*

Proof. We know that for some open U we have $A =^* U$. Then we have $g \in G$ and $1 \in V$ for V open such that $gVV^{-1} \subseteq U$. Therefore $gV \subseteq U \cap Uh$ for $h \in V$. Then we claim $V \subseteq A^{-1}A$. Let $h \in V$. Then we have

$$(U \cap Uh) \triangle (A \cap Ah) \subseteq (A \triangle A) \cup ((U \triangle A) h)$$

which means

$$A \cap Ah =^* U \cap Uh$$

and now $U \cap Uh$ contains a neighborhood of the identity, so it is nonempty, and even has nonempty interior so $A \cap Ah \neq \emptyset$ as well.

Let $x \in A \cap Ah$. So $x \in A$ and $x = yh$ for some $y \in A$. Then $y^{-1}x = h \in A^{-1}A$. □