LECTURE 17
MATH 229

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1. Polish groups

Recall we proved the following theorem last time:

Theorem 1 (Pettis)**.** *Let G be a topological group with non-meager $A \subseteq G$ having BP. Then $A^{-1}A$ contains a neighborhood of* 1.

Maps between Polish groups tend to be automatically continuous.

Corollary 1. *A Baire measurable homomorphism between Polish groups is continuous.*

Proof. Let $\varphi : G \to H$ be Baire measurable. It is enough to show that the preimage of an open neighborhood of the identity in h is open. So fix an open $1 \in U \subseteq H$. Since it is a topological group, we can find some open $1 \in V \subseteq H$ such that $VV^{-1} \subseteq U$. Let $\{h_n\}_{n<\omega}$ dense in H so

$$\bigcup_{n<\omega} (h_n V) = H \ .$$

Then we have that

$$\bigcup_{n<\omega} \varphi^{-1} (h_n V) = G \ .$$

We know that for some n $\varphi^{-1}(h_n V)$ is non-meager, and since φ is Baire measurable these have the Baire property. Now we can apply theorem 1 to get that

$$\varphi^{-1} (h_n V)^{-1} \varphi^{-1} (h_n V)$$

contains a neighborhood of 1. Now we have to check this is contained in $\varphi^{-1}\left(V^{-1}V\right) \subseteq \varphi^{-1}(U)$ which follows from the fact that φ is a morphism. So we are done. □

Corollary 2. *Let G be a Polish group. If $H \leq G$ is non-meager and has BP, then it is open (and hence clopen).*

Proof. This follows immediately because $H^{-1}H = H$. □

Note that by a previous theorem we have that if H is meager then

$$|G/H| \geq 2^{\aleph_0} \ .$$

Therefore a subgroup with the BP is either meager or non-meager. If it is non-meager it is open, and if it is meager it has large index. So the sip follows for subgroups with BP. Hence to find a counterexample to sip we need the axiom of choice.

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Exercise 1. Sip is invariant under group (algebraic) isomorphisms.

We are really only interested in automorphism groups of first-order structures. As it turns out there is a characterization of such groups inside all Polish groups.

Theorem 2. *Let G be a Polish group. Then TFAE:*

(i) G is isomorphic to a closed subgroup of S_∞.
(ii) G is non-archimedean, i.e. G has a basis of a neighborhood of 1 consisting of subgroups.
(iii) G has a left-invariant compatible ultrametric.[1]

Exercise 2. Show that S_∞ does not have a complete, compatible, left-invariant metric.

1.1. Action of Polish groups.

Proposition 1. *Let G be a Polish group and X a metrizable space. If we have a map $a : G \times X \to X$ which is separably continuous then it is continuous.*

If $G \circlearrowright X$ (i.e. we have a continuous map $G \times X \to X$) then for $x \in X$ the stabilizer G_x is closed (by continuity) and G/G_x is a Polish space and the canonical map

$$G/G_x \to G \cdot x$$

which sends $[g] \mapsto g \cdot x$ is continuous.

Theorem 3 (Effros)**.** *Let G be a polish group and X be a polish space. For $G \circlearrowright X$ TFAE:*

(i) $G/G_x \to G \cdot x$ is a homeomorphism,
(ii) $G \cdot x$ is not meager (in its relative topology), and
(iii) $G \cdot x$ is G_δ in X.

Remark 1. $G \cdot x$ not being meager in X certainly implies that it is not meager in its relative topology. The converse is false. For example consider a countable discrete set where X has no isolated points.

Proof of remark. Assume $G \cdot x$ is meager in its relative topology. Then we can write:

$$G \cdot x = \bigcup F_n$$

where $F_m n \subseteq G \cdot x$ is relatively closed of empty interior. Set $F_n' = \overline{F_n}$ in X. If $\operatorname{int} F_n' = \emptyset$ then $G \cdot x$ is meager in X. If not, then its intersection $\int (F_n') \cap F_n \neq \emptyset$ so it is relatively open so we are done. □

2. Ample generics

First we will find a criterion for determining the sip, and then we will find some criterion for the criterion which will be hard.

Definition 1. Let $G \circlearrowright X$, both Polish. Then we have a natural action $G \circlearrowright X^n$. Say that $\overline{a} \in X^n$ is *generic* if $G \cdot \overline{a}$ is comeager in X^n. We say that the action has *ample generics* if there is a generic $\overline{a} \in X^n$ for each $n < \omega$. We say that G itself has *ample generics* if the action on itself by conjugation has ample generics.

[1] This means $d(u,v) \leq \max\{d(u,x), d(x,v)\}$ instead of the sum.

Note that there can be at most one generic orbit since otherwise there would be two disjoint comeager subsets.

Example 1. Consider $G = S_\infty$ acting on itself by conjugation. We claim this action has a comeager orbit. First take $W_0 \subseteq G$ to be the set of permutations with no infinite orbit, i.e.

$$W_0 = \bigcap V_n$$

where V_n is the set of all permutations for which the orbit of n is finite. Note the V_ns are open and dense. Similarly the set W_1 of $\sigma \in G$ which have infinitely many orbits of size k for each k is dense G_δ and if we intersect $W_0 \cap W_1$ we get a dense G_δ set which is the comeager orbit.

Exercise 3. Show that S_∞ has ample generics.

Now we want to show that ample generics implies sip. Throughout we have $G \circlearrowright X$ with ample generics.

Lemma 1. *Let $A, B \subseteq X$ with A non-meager and B not meager in any open set (in particular B is dense). Let $x_0 \in X^n$ be generic and $1 \in V \subseteq G$. Then there are $y_0 \in A$, $y_1 \in B$, and $h \in V$ such that (x_0, y_0), (x_0, y_1) are generic in X^{n+1} and*

$$h\left(x_0, y_0\right) = \left(x_0, y_1\right) \ .$$

Proof. Let $C \subseteq X^{n+1}$ be the comeager orbit. For $z \in X^n$ let

$$C_z = \{x \in X \mid (z, x) \in C\} \ .$$

Note that $C_{gz} = gC_z$ since $(gz, x) = g\left(z, g^{-1}x\right) \in C$ so $g^{-1}x \in C_z$ and $x \in gC_z$. Also note that if C_z is comeager then C_{gz} is comeager. By Kuratowski-Ulam

$$\{z \in X^n \,|\, C_z \text{ comeager}\}$$

is comeager so for z generic, C_z is comeager. In particular there is $y_0 \in A$ such that $(x_0, y_0) \in C$. Notice that $G_{x_0} \cdot y_0 = C_{x_0}$ is comeager. By theorem 3, the map $gG_{x_0} \mapsto g \cdot y_0$ is open. Let $U \subseteq X$ be open such that

$$U \cap G_{x_0} \cdot y_0 = \left(G_{x_0} \cap V\right) \cdot y_0 \ .$$

Now we know B is not meager in U and $G_{x_0} y_0$ is comeager in U so they must intersect, i.e. $B \cap U \cap G_{x_0} y_0 \neq \emptyset$ so we can find such an h. $\square$