LECTURE 20
MATH 229

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We will continue trying to prove the sip via proving ample generics. What we're aiming at, which we may or may not get to today is a characterization of having ample generics.

Recall last week we had a criterion for having a dense conjugacy class. Now we want to upgrade this to a criterion for having a comeager class.

1. Comeager conjugacy class

1.1. Topological lemmas.

Fix $G \circlearrowleft X$ polish.

Definition 1. We say $x \in X$ is *G-big* if $Ux \subseteq X$ is somewhere dense for any $1 \in U \subseteq G$, i.e. $\operatorname{int}\left(\overline{Ux}\right) \neq \emptyset$ for any such U.

This will turn out to be equivalent to having comeager orbit.

Fix a left-invariant metric on G. Recall that since G is polish it has a complete metric and a left-invariant metric. It is somehow rare that these coincide. This is impossible for S_∞, and in general if the left-invariant metric is not also right-invariant it cannot be complete. In any case, we can take the left-invariant metric d and then

$$d(x,y) = d\left(x^{-1}, y^{-1}\right)$$

is complete.

For $x \in X$, let

$$(x)_{<\epsilon} = B_\epsilon(1) \cdot x$$

be the open ball of radius ϵ (wrt d) around 1 in G. Similarly if $V \subseteq X$,

$$(V)_{<\epsilon} = B_\epsilon(1) \cdot V \ .$$

Lemma 1. *(1) For all $A \subseteq X$, $\epsilon > 0$,*

$$\left(\overline{A}\right)_{<\epsilon} \subseteq \overline{(A)_{<\epsilon}}$$

and

$$\left(\operatorname{int}\left(\overline{A}\right)\right)_{<\epsilon} \subseteq \operatorname{int}\left(\overline{(A)_{<\epsilon}}\right) \ .$$

(2) If $x \in X$ is G-big, then

$$x \in \operatorname{int}\left(\overline{(x)_{<\epsilon}}\right)$$

for all $\epsilon > 0$.

Proof. (1): Let $x \in \left(\overline{A}\right)_{<\epsilon}$ and $U \ni x$. Then we need to show U intersects $\left(\overline{A}\right)_{<\epsilon}$. This is equivalent to showing that $(U)_{<\epsilon}$ intersects $\overline{A}$. In fact, we have a distance function on X, which is

$$\overline{d}(x,y) = \inf\left(d(g,1) \mid gx = y\right)$$

and $(A)_{<\epsilon}$ is an ϵ-neighborhood for this distance. In particular we have $\overline{d}(x,y) = \overline{d}(y,x)$ since $d(g,1) = d\left(1,g^{-1}\right)$. Now we certainly have $(U)_{<\epsilon} \cap \overline{A} \neq \emptyset$, and being open this means,

$$(U)_{<\epsilon} \cap A \neq \emptyset$$

which is equivalent to

$$U \cap (A)_{<\epsilon} \neq \emptyset$$

so we are done.

We skip the next part.

(2): Let $x \in X$ be G-big. Then $\operatorname{int}\left(\overline{(x)_{<\epsilon}}\right) \neq \emptyset$, so

$$V \cap (x)_{<\epsilon} \neq \emptyset$$

so $x \in (V)_{<\epsilon}$ and

$$(V)_{<\epsilon} \subseteq \left(\operatorname{int}\left(\overline{(x)_{<\epsilon}}\right)\right)_{<\epsilon} \subseteq^{1} \operatorname{int}\left(\overline{(x)_{<\epsilon}}\right)$$

□

Definition 2. A set $W \subseteq X$ is ϵ-small if for any two nonempty open subsets $U_1 U_2 \subseteq W$ we have

$$(U_1)_{<\epsilon} \cap U_2 \neq \emptyset\ .$$

Lemma 2. *Suppose that for every $\epsilon > 0$ the union of all ϵ-small open subsets of X is dense. Then the set of G-big $x \in X$ is comeager.*

Proof. For $\{V_n \mid n < \omega\}$ a countable basis of X. For all $n, m < \omega$, by hypothesis, V_n intersects some $1/m$-small open set. Let $\emptyset \neq W_{n,m} \subseteq V_n$ be $1/m$ small. For $m < \omega$, let

$$W_m = \bigcup_{n<\omega} W_{n,m}\ .$$

Then W_m is open, dense in X. Let

$$D_{n,m} = \bigcap \left\{ (V_k)_{<1/m} \mid V_k \subseteq W_{n,m} \right\}\ .$$

Then $D_{n,m}$ is comeager in X in $W_{n,m}$, as each $(V_k)_{<1/m}$ is dense open in $W_{n,m}$. Let

$$D_m = \bigcup_{n<\omega} D_{n,m}$$

so $D_m \subseteq W_m$ and D_m is comeager in W_m. Since W_m is dense open we have that D_m is comeager in X. Finally, let

$$D = \bigcap_{m<\omega} D_m$$

so D is comeager.

Finally we show if $x \in D$, then x is G-big. Fix $\epsilon > 0$, then we want to show $\operatorname{int}\left(\overline{(x)_{<\epsilon}}\right) \neq \emptyset$. Take $\epsilon = 1/m$, $x \in D_{n,m}$ for some $n < \omega$. So $x \in (V_k)_{<1/m}$ for all k such that $V_k \subseteq W_{n,m}$, so in fact

$$V_k \cap (x)_{<1/m} \neq \emptyset$$

[1] By (1).

for all k such that $V_k \subseteq W_{n,m}$. Therefore $(x)_{<1/m}$ is dense in $W_{n,m}$ so $W_{n,m} \subseteq \operatorname{int}\left(\overline{(x)_{1/m}}\right)$. □

Lemma 3. *Let $x \in X$ be G-big. Then the action is topologically transitive iff $G \cdot x$ is dense.*

Proof. ($\Longleftarrow$): Clear.

($\Longrightarrow$): Let $V = \operatorname{int}\left(\overline{G \cdot x}\right)$. x being G-big means $U \cdot x$ is somewhere dense, i.e. $G \cdot x$ is somewhere dense, i.e. V is nonempty. Then V is G-invariant, but now the action is topologically transitive, so for every U there is V such that $g(V \cap U) \neq \emptyset$, so V is dense, so $G \cdot x$ is dense. □

The conclusion is the following.

Lemma 4. *Suppose $G \circlearrowright X$ is topologically transitive. Then TFAE:*

(i) For every ϵ the union of ϵ-small open sets is dense.
(ii) The set of G-big points is comeager.
(iii) There is a G-big point.

Proof. $(iii) \implies (i)$: Let $x \in X$ be G-big. Note that for any $y \in X$, $\overline{(y)_{<\epsilon}}$ is 2ϵ small because of the following. For $U, V \subseteq \overline{(y)_{<\epsilon}}$ then U, V both intersect $(y)_{<\epsilon}$ so their distance is at most 2ϵ.

Note also that if x is G-big, so is any point in any point in $G \cdot x$. By assumption $G \cdot x$ is dense, so

$$\bigcup_{y \in G \cdot x} \operatorname{int}\left(\overline{(y)_{\epsilon/2}}\right)$$

is dense in X. □

So now we've found a collection of dense points, and now we see when they are actually an orbit.

Lemma 5. *If $x, y \in X$ are both G-big and $y \in \operatorname{int}\left(\overline{G \cdot x}\right)$, then $Y \in G \cdot x$.*

Proof. Let x, y be G-big, $y \in \operatorname{int}\left(\overline{G \cdot x}\right)$. We will construct points $g_n \in G$ and open symmetric neighborhoods $1 \in U_n \subseteq G$ such that

(1) $g_{2n}^{-1} y \in \operatorname{int}\left(\overline{U_{2n} x}\right)$
(2) $g_{2n+1} \in g_{2n} U_{2n}$
(3) $g_{2n+1} x \in \operatorname{int}\left(\overline{U_{2n+1} y}\right)$
(4) $g_{2n+2} \in U_{2n+1} g_{2n+1}$
(5) $g_{2n}^{-1} U_{2n+1} g_{2n} \subseteq U_{2n}$
(6) $g_{2n+1} U_{2n+2} g_{2n+1}^{-1} \subseteq U_{2n+1}$
(7) For $n \geq 1$, if $h \in U_n^3 \supseteq U_n$, then $d(1, n) < 2^{-(n+1)}$.
(8) The set $\operatorname{int}\left(\overline{U_{2n+1} y}\right)$ is included in a closed ball of radius $< 1/n$ around g.

The last two are somehow saying U_n is small, and the others are somehow saying that we are getting closer. If we succeed, then $(g_n)_{n<\omega}$ is a Cauchy sequence with limit g_* and items 3 and 8 imply that $g_* x = y$. To show this is Cauchy it is enough to show

$$d(g_n, g_{n+1}) < 2^{-n} \qquad d\left(g_n^{-1}, g_{n+1}^{-1}\right) < 2^{-n}.$$

Say $n = 2k+2$. Then $g_n^{-1} g_{n+1} \in U_n$ by item 2 so

$$d(g_n, g_{n+1}) = d\left(1, g_n^{-1} g_{n+1}\right) < 2^{-(n+1)}$$

by item 7. Then

$$g_n g_{n+1}^{-1} = g_n \left(g_{n+1}^{-1} g_n\right) g_n^{-1}$$

and

$$g_n = u' g_{n+1}$$

for $u' \in U_{n-1}$ by item 4. So

$$g_n (g_{n+1})^{-1} = u' g_{n-1} \left(g_{n+1}^{-1} g_n\right) (g_{n-1})^{-1} (u')^{-1} \ .$$

Then we know $g_{n+1}^{-1} g_n \in U_n$, and by item 6 we have $g_{n-1}\left(g_{n+1}^{-1} g_n\right)(g_{n-1})^{-1} \in U_{n-1}$ and therefore $g_n g_{n+1}^{-1} \in U_{n-1}^3$. So finally

$$d\left(g_n^{-1}, g_{n+1}^{-1}\right) = d\left(1, g_n g_{n+1}^{-1}\right) < 2^{-n} \ .$$

The argument is similar for n odd, but we would use items 2 and 5 instead of items 3 and 6.

It remains to construct the g_ns and U_ns. Set $g_0 = 1$ and $U_0 = G$. Note item 1 holds by assumption. Now assume we have g_{2n} and U_{2n}. For any open V,

$$y \in g_{2n} \cdot \operatorname{int}\left(\overline{U_{2n} x}\right) \cap \operatorname{int}\left(\overline{V y}\right)$$

so

$$\operatorname{int}\left(\overline{V y}\right) \cap g_{2n} \overline{U_{2n} x} \neq \emptyset$$

so

$$\operatorname{int}\left(\overline{V y}\right) \cap g_{2n} U_{2n} x \neq \emptyset \ .$$

Now take U_{2n+1} so that items 5, 7 and 8 hold and pick $g_{2n+1} \in g_{2n} U_{2n}$ such that $g_{2n+1} x \in \operatorname{int}\left(\overline{U_{2n+1} y}\right)$, i.e. items 2 and 3 hold. This can be done for any choice of U_{2n+1}.

Now we take U_{2n+2} and g_{2n+2} similarly using the fact that x is G-big. $\square$