LECTURE 21
MATH 229

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Recall we had all these lemmas last time. Now we have the following corollary:

Corollary 1. *If $G \circlearrowright X$ is topologically transitive, then the set of G-big elements is either empty or is one orbit.*

Proof. This follows almost immediately. We have seen that as long as the action is topologically transitive (some dense orbit) then x being G-big implies $G \cdot x$ is dense. Then this means any y which is G-big is in $\operatorname{int}\left(\overline{G \cdot x}\right)$ so is in $G \cdot x$ by the lemma. □

Corollary 2. *If $G \circlearrowright X$ is topologically transitive and $x \in X$ then TFAE:*

(i) x is G-big,
(ii) $G \cdot x$ is comeager,
(iii) $G \cdot x$ is not meager.

Proof. $(i) \implies (ii)$: This follows directly from previous results.

$(ii) \implies (iii)$: This is clear.

$(iii) \implies (i)$: This is just a calculation. Assume $G \cdot x$ is not meager and let $1 \in U \subseteq G$. So now we want to show that $U \cdot x$ is somewhere dense. Write

$$G = \bigcup_{n<\omega} g_n U$$

so

$$G \cdot x = \bigcup_{n<\omega} g_n U \cdot x$$

but this means for some n $g_n U \cdot x$ is somewhere dense, and hence $U \cdot x$ is somewhere dense and we are done. □

Theorem 1. *Let $\mathcal{K}$ be a Fraïssé class with limit M, $G = \operatorname{Aut}(M)$, $n < \omega$. Then TFAE:*

(i) $\mathcal{K}(n)$ has EAP and JEP.
(ii) There is a generic tuple in G^n.

Proof. $(i) \implies (ii)$: Assuming (i) the JEP implies that the action $G \circlearrowright G^n$ by conjugation is topologically transitive by a previous theorem. We need to show that for all $\epsilon > 0$ the union of all ϵ-small open sets is dense.

So let $\epsilon > 0$, $U \subseteq G^n$ open. We can assume U is of the form

$$U = U_{\overline{f}} = \left\{ \overline{\sigma} \in G^n \mid \sigma_i \supseteq f_i \right\} .$$

Let $A_0 \in \mathcal{K}$ contain the domain and range of the f_is so that $\left(A_0, \overline{f}\right) \in \mathcal{K}(n)$. Now increase A_0 to $M \supseteq A \supseteq A_0$ such that for all $\tau \in \operatorname{Stab}(A)$ we have

$$\operatorname{dist}(\tau, 1) < \epsilon .$$

Then we still have $\left(A, \overline{f}\right) \in \mathcal{K}(n)$.

Now by EAP there is

$$\left(A, \overline{f}\right) \subseteq (B, \overline{g})$$

definite on $\left(A, \overline{f}\right)$. So $U_{\overline{g}} \subseteq U_{\overline{f}} = U$. Then the claim is the following:

Claim 1. $U_{\overline{g}}$ is ϵ-small.

But we have actually already shown this. We gave a characterization of what it means to have this EAP property. For $V_1, V_2 \subseteq U_g$, by the previous lemma we have that

$$\operatorname{Stab}(A) \cdot V_1 \cap V_2 \neq \emptyset \ ,$$

and hence $(V_1)_{<\epsilon} \cap V_2 \neq \emptyset$ which is exactly what we want.

$(ii) \implies (i)$: We have JEP since there is a dense orbit. We now show EAP. Consider $\left(A, \overline{f}\right)$. Then choose $\epsilon > 0$ such that

$$\operatorname{Stab}(A) \supseteq B_\epsilon(1) \ .$$

Let $U_{\overline{g}} \subseteq U_{\overline{f}}$ be ϵ-small corresponding to some $(B, \overline{g})$, $A \subseteq B$. This is definite on A. ∎

It remains to prove/give a criterion for EAP.

Proposition 1. *The class of graphs has EAP.*

Proof. Let $\left(A, \overline{f}\right) \in \mathcal{K}(n)$. Take B an EPPA witness for A, and extend each f_i to $g_i \in \operatorname{Aut}(B)$. Then $(B, \overline{g})$ is definite over $\left(A, \overline{f}\right)$. If

$$\begin{array}{ccc} & & \left(C_1, \overline{h_1}\right) \\ & \nearrow & \\ (B, \overline{g}) & & \\ & \searrow & \\ & & \left(C_2, \overline{h_2}\right) \end{array}$$

let

$$D = C_1 \amalg C_2$$

with no edges between $C_1 \setminus B$ and $C_2 \setminus B$, $k_i = (h_1)_i \cup (h_2)_i$. Then

$$\begin{array}{ccccc} & & \left(C_1, \overline{h_1}\right) & & \\ & \nearrow & & \searrow & \\ (B, \overline{g}) & & & & \left(D, \overline{k}\right) \\ & \searrow & & \nearrow & \\ & & \left(C_2, \overline{h_2}\right) & & \end{array} .$$

∎

This argument shows ample generics, and therefore sip, for (almost) all classes for which we have proved EPPA. It fails for, say, DLO. However, it is true that DLO has sip, but by a completely different argument.

Remark 1. Sip is open for:

- random ordered graph,
- generic tournament (EPPA is not known)

Original proof of sip for S_∞; Dixon, Neumann, Thomas (1985). Let $G \leq S_\infty = \operatorname{Sym}\Omega$ (where Ω is a countable set) of index $< 2^{\aleph_0}$. We want to show there is a finite $A_0 \subseteq \Omega$ such that

$$G_{(A_0)} \leq G \leq G_{\{A_0\}} \ .$$

Lemma 1. *Let $\Gamma_1, \Gamma_2 \subseteq \Omega$ be infinite such that $\Gamma_1 \cap \Gamma_2$ is infinite and write*

$$\operatorname{Sym}(\Gamma_1) = G_{(\Omega \setminus \Gamma_1)} \ .$$

Then

$$\langle \operatorname{Sym}(\Gamma_1), \operatorname{Sym}(\Gamma_2) \rangle = \operatorname{Sym}(\Gamma_1 \cup \Gamma_2)$$

Proof. Exercise. □

Claim 2. There is a moiety[1] $\Sigma \subseteq \Omega$ such that $\operatorname{Sym}(\Sigma) \leq G$.

Proof. Let $(\Sigma_i, i < \omega)$ be disjoint moieties partitioning Ω. Let $S_i = \operatorname{Sym}(\Sigma_i) \leq S_\infty$. Let $H \leq G$ be

$$H = \{g \in G \mid \forall i, g\Sigma_i = \Sigma_i\} \ .$$

Then $H \leq \prod S_i$ has index less than $2^{\aleph_0}$ (since G has index less than this and $H = G \cap (\prod S_i)$). Then it follows that for some i, the image of the projection $H \to S_i$ is equal to S_i.

Now consider $K = G \cap \operatorname{Sym}\Sigma_i$.

Claim 3. $K \trianglelefteq \operatorname{Sym}(\Sigma_i)$.

Let $\sigma \in K, \tau \in \operatorname{Sym}\Sigma_i$. Then we want to show $\tau\sigma\tau^{-1} \in K$. Now let $h \in H$ which projects to $\tau \in S_i$. Then

$$h\sigma h^{-1} = \tau\sigma\tau^{-1} \in K \ .$$

Fact 1. *The normal subgroups of S_∞ are 0, S_∞, finitary permutations, and finitary alternating permutations.*

But there is only one group here with index less than $2^{\aleph_0}$ is S_∞, so $K = S_\infty$. □

Now note that there is an almost disjoint family of $\left(A_j, j < 2^{\aleph_0}\right)$ of moieties of Σ_i. Then for each $j < 2^{\aleph_0}$, let $g_j \in S_\infty$ be an involution which exchanges A_j and $\Omega \setminus \Sigma_i$, and fixes $\Omega \setminus A_j$. Now we can find $j, k < 2^{\aleph_0}$ such that

$$g_j^{-1} g_k \in G \ .$$

Now we have A_j and A_j which might have finite intersection. Then this group element takes an element of A_k, throws is out to $\Omega \setminus \Sigma_i$, brings it back into A_j, and throws it back out. Specifically, we claim that

$$g_j^{-1} g_k(\Sigma_i) \supseteq (A_j \setminus A_k) \cup g_j(A_j \setminus A_k) \ .$$

[1] This means Σ and $\Omega \setminus \Sigma$ are infinite.

The first set is infinite, and the second is the complement minus finitely many points. Now let B_0 be finite such that

$$\Omega \setminus \Sigma_i = g_j \left(A_j \setminus A_k \right) \cup B_0 \ .$$

Now apply the lemma to Σ_i and $g_i^{-1} g_k \left(\Sigma_i \right)$ to get

$$\mathrm{Stab} \left(B_0 \right) \leq G$$

and we are basically done. ■

This is the proof that would be generalized to DLO, but the first lemma doesn't quite apply, so we would need to somehow leverage the product argument much more.