LECTURE 22
MATH 229

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1. Model theoretic properties of structures

Today we will mostly state things and not prove much, and then we will do the NIP case in more detail.

1.1. **Counting substructures.**

To motivate the introduction we start with the combinatorial question from last time of counting the number of substructures.

Assume the language is relational. Let M be a ω-categorical and $G = \mathrm{Aut}(M)$. Define $f_n(M)$ to be the number of substructures of M of size n up to isomorphism. We could also define some other function (which we will not be using) that we just mention. First $F_n(M)$ is the number of n-types over $\emptyset$, and $\tilde{F}_n(M)$ is the number of n-types of pairwise distinct elements. The difference between the two is somehow that in f we somehow don't have an ordering on the things we are counting so we have

$$f_n(M) \leq \tilde{F}_n(M) < F_n(M) \ .$$

If substructures are rigid (e.g. we have a linear order) then

$$f_n(M) = \frac{\tilde{F}_n(M)}{n!} \ .$$

The problem of $\tilde{F}_n(M)$ is that this thing somehow grows too quickly whereas f_n has a slower growth rate so it's somehow more interesting to the combinatorialists.

Example 1. For DLO, $f_n(M) = 1$.

Example 2. For M the random graph we roughly have that

$$f_n(M) \approx F_n(M) \approx 2^{n^2}$$

since this is somehow asking for graphs with n vertices.

Example 3. Let $L = \{\leq\}$ be a partial order then a tree is when this is actually a linear order. Then $f_n(M)$ is the number of trees on n vertices, so it's c^n for some constant c which is between $2 < c < 3$. This is some kind of Catalan number. Note that this is smaller than a factorial.

If we take two linear orders $L = \{\leq_1, \leq_2\}$ then $f_n(M) = n!$. This is why these things are sometimes called permutation structures.

So we have seen constant growth rates, exponential, and factorial growth rates. Now we have some basic facts.

Fact 1 (Cameron)**.** *(i) f_n is non-decreasing.*

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(ii) *f_n can grow as fact as one wants (with an infinite language).*
(iii) *If the language has max arity k and M is homogeneous in L then $f_n(M) = \mathcal{O}\left(e^{n^k}\right)$.*

Question 1. Understand structures with slow growth rate of $f_n(M)$.

Fact 2 (Cameron)**.** *If $f_n(M) = 1$ for all n then M is one of the five reducts of DLO, i.e. one of the following:*

(i) *($\leftarrow$): DLO*
(ii) *($\leftrightarrow$): Betweenness relation $B(x, y, z)$*
(iii) *($\circlearrowleft$): Circular order $C(x, y, z)$*
(iv) *($\oplus$): Separation relation $S(x, y, u, r)$*
(v) *Pure equality*

Remark 1. This class is a priori included in the reducts of DLO (since f_n only decreases when one takes reducts) so this fact was shown by showing the other inclusion manually on each reduct.

Fact 3 (Macpherson)**.** *If M is primitive, either $f_n(M) = 1$ for all n or*

$$f_n(M) \geq \frac{c^n}{p(n)}$$

for $c = 2^{1/5} \approx 1.148$ and p is some polynomial depending on M.

Conjecture 1. *In fact, it is true with $c = 2$.*

Example 4. Note that 2 is possible. Take a circular order and add a relation where from every point we add an arrow to the other half of the circle. This should be thought of as a topological 2-cover of the circle.

1.2. **Model theoretical notions.** There is a model theoretic property (NIP) which is implied by a small growth rate of f_n.

Definition 1. A formula $\varphi(\overline{x}, \overline{y})$ has IP (in M) if for all $n < \omega$ we can find tuples

$$(\overline{a}_i \mid i < n) \qquad \left(\overline{b}_j \mid j \in \mathcal{P}(n)\right)$$

in M such that

$$M \models \varphi\left(\overline{a}_i, \overline{b}_j\right) \iff i \in j .$$

Definition 2. We say M is NIP if no formula has IP.

So IP somehow gives us something complicated and it has NIP if this doesn't happen.

Example 5. The graph relation $R(x, y)$ has IP in M the random graph.

Example 6. Higher order things like DLO and trees are NIP. NIP formulas are closed under Boolean combinations.

Lemma 1 (Sauer-Shelah)**.** *If $\varphi(\overline{x}, \overline{y})$ is NIP then for finite $A \subseteq M^{\overline{y}}$ there is some k such that*

$$\left|S_\varphi(A)\right| = \mathcal{O}\left(|A|^k\right)$$

where $S_\varphi(A)$ consists of the φ-types over A, i.e. the maximal consistent set of formulas of the form $\varphi\left(x, \overline{b}\right)$ where $\overline{b} \in A$.

Note if φ has IP

$$|S_\varphi(A)| = 2^{|A|}$$

for some A of arbitrarily large size.

This has the following consequence. Assume M is homogeneous in a finite relational language and every relation in L is NIP (for every partition of the variables) then

$$f_n(M) = \mathcal{O}\left(e^{cn \ln n}\right)$$

for some $c > 0$ so its growth rate is at most factorial.

For simplicity assume we have 1-formula. Consider some n-elements $a_1, \ldots, a_n$. We might as well count $F_n(M)$ since the factorial won't change whether or not we satisfy this bound. Then the number of n-types over $\emptyset$ is at most

$$S_n(\emptyset) \leq S_1(\emptyset) \cdot S_1(\{a_0\}) \cdot S_1(\{a_0, a_1\}) \cdot \ldots$$

and by NIP this is a product which looks like:

$$S_n(\emptyset) \leq 1^k \cdot 2^k \cdot 3^k \cdot \ldots \cdot n^k \leq (n!)^c$$

for some c.

Fact 4 (Macpherson)**.** *If some $\varphi(\overline{x}, \overline{y})$ has IP then*

$$f_n(M) \geq 2^{p(n)}$$

for some polynomial p of degree at least 2.

For finitely homogeneous structures we get a gap that either

$$f_n(M) = \mathcal{O}\left(e^{cn \ln n}\right)$$

where M is NIP, and

$$f_n(M) \geq e^{p(n)}$$

(for $\deg p \geq 2$) in the IP case. So the picture is:

$$\underbrace{1 \quad |\text{gap}| \quad \underbrace{e^{cn}}_{\text{trees}} \quad \cdots \quad \underbrace{e^{cn \ln n}}_{\text{2 lin ord}}}_{\text{NIP}} \quad |\text{gap}| \quad e^{cn^2} \ldots$$

New goal: understand NIP for finite homogeneous (or ω-categorical) structures.

1.3. **Stability.**

Definition 3. $\varphi(\overline{x}, \overline{y})$ is unstable (in M) if for every n there are $(\overline{a}_i \mid i < n)$ $\left(\overline{b}_j \mid j < n\right)$ such that

$$M \models \varphi\left(\overline{a}_i, \overline{b}_j\right) \iff i \leq j\ .$$

Definition 4. M is stable if all formulas are stable.

Note that stable implies NIP.

Example 7. $L = \{E\}$ where E is an equivalence relation then M is stable. DLO is NIP but not stable.

Example 8. $\mathbb{F}_p$ vector spaces are stable ω-categorical but not finitely homogeneous.

(1) If M is finitely homogeneous and stable then it is ω-stable.
(2) ω-categorical and ω-stable structures are very understood. They are all build out of $(M, =)$, $\mathbb{F}_q$ vector spaces by families of covers. More precisely:

Fact 5.

- *A primitive ω-categorical ω-stable is a Grassmannian over one of:*
  - $(M, =)$
  - *Affine or projective space over a finite field.*
- *A finite homogeneous stable primitive structure is interdefinable with unordered k-tuples of distinct elements of $(M, =)$.*

Once we know that these are the primitive ones, one can just check what happens with f_n. For $(M, =)$, $f_n = 1$. The 2-Grassmannian of $(M, =)$, then f_n is roughly the set of graphs with n edges. This is hard to compute, but certainly it grows faster than c^n but the actual asymptotic behavior is not known.

In the case of the $\mathbb{F}_p$ vector space f_n looks like the following. This has at least the number of partitions, so f_n also grows faster than c^n.

The conclusion is that if $\mathcal{M}$ is ω-stable, primitive, and $f_n = \mathcal{O}(c^n)$, then $\mathcal{M} \cong (M, =)$. So then the unstable case is left which we will deal with next time.

Stable not ω-stable things are not understood at all. Pseudo-planes can get some lower bounds of f_n by powers of $\sqrt{n}$ but even powers of n hasn't been done.