LECTURE 23
MATH 229

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1. Few substructure

Definition 1. Let $\mathcal{M}$ be an ω-categorical structure. We say it has *few-substructures* if for no polynomial $p(x)$ do we have

$$f_n(\mathcal{M}) \geq \frac{2^n}{p(n)} .$$

The theorem we will prove over the next two classes is the following.

Theorem 1. *If primitive $\mathcal{M}$ has few substructures, then either*

(i) $\mathcal{M}$ is stable, not ω-stable, or
(ii) $\mathcal{M}$ is one of the 5 reducts of DLA.

Remark 1. In (i) we cannot hope for $\mathcal{M}$ to be finitely homogeneous. Conjecturally, it never happens.

Lemma 1. *If $\mathcal{M}$ has few substructures, then any expansion of $\mathcal{M}$ by adding finitely many constants also has few substructures.*

Exercise 1. Prove lemma 1.

We have seen the following.

Fact 1. *If $\mathcal{M}$ has few substructure then it is NIP.*

Fact 2. *If $\mathcal{M}$ is primitive with few substructures, and ω-stable, then $\mathcal{M} \simeq (M, =)$.*

Fact 3 (Shelah)**.** *If $\mathcal{M}$ is NIP unstable, then there is a formula $\varphi(x, y)$ where $|x| = |y| = 1$ which defines a partial order with infinite chains.*

Fact 4. *If $\mathcal{M}$ is ω-categorical NIP, unstable, then there is a definable equivalence relation E on $\mathcal{M}$ and a definable infinite linear order on $\mathcal{M}/E$*

Proof for few substructure case. The idea is to take this partial order and show that if it isn't close to linear then we have lots of substructures.

Let $\leq$ be a definable (over some A) partial order on M. First we can assume $A = \emptyset$. Let $D \subseteq M$ be a transitive set (i.e. complete type over $\emptyset$) where $\leq$ has infinite chains. Now we have two cases.

Case 1: Any definable $X \subseteq D$ with an finite chain has an infinite anti-chain.
Case 2: D has no infinite anti-chain.

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Case 1: Assume $C_0 \subseteq D$ is an infinite anti-chain and let $a_0 \in X_0$. By transitivity a_0 belongs to some infinite chain so

$$D_0 = \{x \geq a_0, x \in D\}$$

has an infinite anti-chain, say X_1. Now pick $a_1 \in C_1$ and iterate. Let $n < \omega$, and consider an ordered partition

$$n = m_1 + \ldots + m_k$$

with $m_i > 0$. Now we will associate to this a structure of size n. Consider $A_{\overline{m}} \subseteq M$ obtained by taking m_i elements from C_i (including a_i) and nothing else. Then we claim that if $\overline{m} \neq \overline{m'}$ then

$$A_{\overline{m}} \ncong A_{\overline{m'}} \ .$$

The reason is as follows. We know $A_{\overline{m}}$ has exactly m_0 minimal elements. Then removing them we have exactly m_1 and so on. Now there are 2^{n-1} ordered partitions, hence

$$f_n(M) \geq 2^{n-1} \ .$$

Case 2: We will construct a linear order by induction on the size of a max antichain. Assume D has no antichain of size $n+1$. Now we again have two cases:

Case A: There is $a \in D$ such that the set $V(a)$ of elements comparable to a is infinite.

Case B: Otherwise.

Case A: We are done by induction as $V(a)$ has no antichain of size n.

Case B: Say $|V(a)| \leq k$. Then we induct on k. For $a, b \in D$ define $a \to b$ if $b \in V(a)$ is a maximal element of $V(a)$. Then define $a \trianglelefteq b$ if either $a \leq b$ or $a \to b$.

Claim 1. If $a \trianglelefteq b$ and $b \trianglelefteq c$ then $a \trianglelefteq c$.

Assume not, i.e. $a \ntrianglelefteq c$. Then either $c < a$ or $c \in V(a)$ not maximal. Now we again have cases.

Case α: $c < a$.

Case β: $c \in V(a)$ not maximal.

Case α: If $a \leq b$ then $c < b$ which is a contradiction. If $a \to b$ and $b < c$ then since $c < a$ we have $b < a$ so $b \to c$ but now both $a, c \in V(b)$ which contradicts maximality of c.

Case β: Let $d > c \in V(a)$. If $b \leq x$, $b \leq d$, and $b \to c$ then $d \notin V(b)$ In either case $d \notin V(b)$. If $d \leq b$ then $c < b$. So $b < d$ and $a \nleq b$ which means $a \to b$ which is a contradiction since $d > b$.

So $\trianglelefteq$ is a quasi-order. If $a \trianglelefteq b$ and $b \trianglelefteq a$ for $a \neq b$ then it must be that $a \to b$ and $b \to a$ so the equivalence classes of the quasi-order are finite in the quotient, $\trianglelefteq$ induces an infinite order with $|V(a)| \leq k-1$ for all a. □

2. Interpretable orders in $\mathcal{M}$ with few substructures

From now on assume $\mathcal{M}$ has few substructures.

Lemma 2. *Let $D \subseteq M$ and $\pi : D \to V$ an interpretable map with V linearly ordered. If V is infinite and transitive then π has finite fibers.*

Proof. Take the partial order $\pi(a) \leq \pi(b)$ on D. This is the same argument as before. □

Lemma 3. *Let $D \subseteq M$ and $\pi : D \to V$ an interpretable map with V linearly ordered. Then any parameter-defined subset of V is a finite union of convex sets.*

Proof. Let X be a definable set which is not a finite union of convex sets. For any $n < \omega$ and any $\sigma : n \to 2$ we can find

$$D_\sigma = \{d_0, \dots, d_{n-1}\} \subseteq D$$

with $\pi(d_i) < \pi(d_{i+1})$ and $\pi(d_i) \in X$ iff $\sigma(i) = 1$. Then we get 2^n many. □

For V linearly ordered, let $\bar{V}$ denote the completion of V. A function $f : X \to \bar{V}$ (for $X \subseteq M^k$ definable) is definable if the set

$$\{(x,t) \in X \times V \mid t < f(x)\} \subseteq X \times V$$

is definable.

Lemma 4. *Let $X \subseteq M$ be definable and transitive.Let $D \subseteq M$ and $\pi : D \to V$ as above, V transitive. If $f : X \to \bar{V}$ is definable then $X = D$ and $f = \pi$.*

Proof. Assume $D \setminus X$ is not empty. Then by transitivity of V we have that $\pi : D \setminus X \to V$ is onto. So now fix $n < \omega$. For $\sigma \in \mathrm{Fun}(n,2)$ take

$$(a_i^\sigma \mid i < n)$$

in M such that

- $a_i^\sigma \in D \setminus X$ if $\sigma(i) = 0$ and $a_i^\sigma \in X$ if $\sigma(i) = 1$
- $g_i(a_i^\sigma) < g_{i+1}\left(a_{i+1}^\sigma\right)$ where $g_0 \in \{f, \pi\}$.

This gives 2^n substructures of size n. It follows that $D \subseteq X$ and therefore $X = D$ by transitivity of X.

So now we have $D = X$ and potentially two different maps to V which we want to show are the same. By transitivity $f(a) > \pi(a)$. Now we construct 2^n substructures. Take any point a_0 then act f on it. This lands somewhere else, and now when we choose our next point we can either choose it ahead of $f(a_0)$ or behind it. Now just iterate it. □