LECTURE 24
MATH 229

LECTURE: PROFESSOR PIERRE SIMON
NOTES: JACKSON VAN DYKE

Recall we're still in this situation where M has few substructures. Then we're studying tameness of definable linear orders. The last lemma we had was:

Lemma 1. *Let $\overline{V}$ be transitive and linearly ordered. Then for $f : X \to \overline{V}$, $\pi : D \to \overline{V}$ we have $X = D$ and $f = \pi$.*

It follows that the induced structure on B is o-minimal.[1] We now have a stronger statement which will effectively tell us that V has no structure.

Proposition 1. *Let $\pi : D \to V$ be as above. In particular V is transitive and linear ordered. Then π has finite fibers and any formula $\varphi(x_1, \ldots, x_n)$ (defining a subset of D^n) definable over some $\overline{b}$ is equivalent to a Boolean combination of formulas of the form*

(a) $\pi(x_i) \square \pi(x_j)$, where $\square \in \{=, \leq\}$,
(b) $\pi(x_i) < \pi(a)$, where $a \in \overline{b} \cap D$, and
(c) $x_i = a$ for some $a \in \pi^{-1}\left(\pi\left(\overline{b} \cap D\right)\right)$.

Proof. Let $\overline{c} = (c_0, \ldots, c_{l-1}) \in M^l$, and let $v \in V$ be definable over $\overline{c}$. Then there is some index $i < l$ such that $v = \pi(c_i)$. Take l minimal such that this is false. Then let $\overline{c_*} = (c_0, \ldots, c_{l-2})$, $\overline{cc_*} \wedge c_{l-1}$. First notice c_{l-1} is not algebraic over $\overline{c_*}$ because otherwise v would be algebraic over $\overline{c_*}$, and

$$v \in \operatorname{acl}(\overline{c_*}) = \operatorname{dcl}(\overline{c_*}) \cap V \ .$$

So over $\overline{c_*}$ we have a definable function $f_{\overline{c_*}}$ such that

$$f_{\overline{c_*}}(c_{l-1}) = v \ .$$

Let

$$W = \{w \in V \mid \operatorname{tp}(w/\overline{c_*}) = \operatorname{tp}(v/\overline{c_*})\}$$

and apply the previous lemma to this (replacing D with $\pi^{-1}(W)$). So $f_{\overline{c_*}} = \pi$ and thus $v = \pi(c_{l-1})$. We also have that $\operatorname{acl}(\overline{c}) \cap D = \pi^{-1}(\pi(\overline{c} \cap D))$.

We prove the proposition by induction on n. For $n = 1$ $\varphi(x_1)$ is a Boolean combination of convex subsets of V. The end cuts of those convex sets are in

$$\operatorname{dcl}\left(\overline{b}\right) \cap \overline{V} = \pi(\overline{v} \cap D)$$

so $\varphi(x_1)$ is as desired.

Now we complete the inductive step. Fix values for $x_1, \ldots, x_{n-1}$ then apply the case $n = 1$. There are finitely many possibilities for the set $\varphi(c_1, \ldots, x_{n-1}, x_n)$. Now for each of these possibilities we can define the set of $c_1, \ldots, c_{n-1}$ for which each possibility is realized. Hence by induction the formula has the desired form. $\square$

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[1] This means a definable set is a finite union of intervals.

As a consequence of this, the induced structure on V is just DLO.

Proposition 2. *A similar statement holds for circular orders, i.e.* $\pi : D \to V$, V *transitive, has a definable circular order.*

The point is that instead of inequalities we have things like $C(\pi(x_i), \pi(x_j), \pi(c))$ for $c \in \pi\left(D \cap \overline{b}\right)$.

Proof. Take $\varphi(x_1, \dots, x_n)$ over $\overline{b}$ as in the previous proposition. If $D \cap \overline{b} \neq \emptyset$, then over $\overline{b}$, V splits into linear orders and we can apply proposition 1.

In general, pick any $a \in D$, then working over a we can apply proposition 1. Since we can change a, a cannot appear in the decomposition of φ. □

The conclusion is the following.

Corollary 1. *Let* D *be definable,* $\pi : D \to V$ *an interpretable map, where* V *is transitive, infinite, and has a definable separation relation. Take* $\overline{b} \in M \setminus D$*. Then* V *is transitive over* $\overline{b}$ *and its structure over* $\overline{b}$ *is precisely one of the four unstable reducts of DLO.*

1. Gluing orders

Say that a definable set $D_{\overline{a}} \subseteq M$ (over $\overline{a}$) is *almost linear (over* $\overline{a}$*)* if there is $\pi : D_{\overline{a}} \to V_{\overline{a}}$ interpretable over $\overline{a}$ and $V_{\overline{a}}$ has an $\overline{a}$-definable linear order. The point is that π might have finite fibers so it might not actually be linear. We can see $\leq_{\overline{a}}$ as a quasi-order over $D_{\overline{a}}$.

Now we want to see what happens when we vary $\overline{a}$.

Lemma 2. *Let* $D_{\overline{a}}$ *be almost linear, transitive over* $\overline{a}$, $\pi : D_{\overline{a}} \to V_{\overline{a}}$*. Define the equivalence relation* $E_{\overline{a}}$ *as:*

$$xE_{\overline{a}}y \iff \pi(x) = \pi(y) \ .$$

Let $c \in D_{\overline{a}}$*. Then for* $c' \in M$ *the following are equivalent:*

1. $c' \in D_{\overline{a}}$, $E_{\overline{a}}$*-equivalent to* c
2. $c \in \operatorname{acl}(c')$.

Proof. (2) $\implies$ (1): This follows from proposition 1. The point is that if c is algebraic over c' so is its projection, but the only way a point in M can know a point in V is if it lies above it.

(1) $\implies$ (2): Enumerate the $E_{\overline{a}}$ class of c as $\overline{c}_0 \wedge \overline{c_1}$ where $\overline{c_0} \in \operatorname{acl}(c)$, $\overline{c_1} \notin \operatorname{acl}(c)$. There is $\sigma \in \operatorname{Aut}(M)$ fixing c and such that

$$\sigma(\overline{c_1}) \cap \operatorname{acl}(\overline{a} \wedge c) = \emptyset \ .$$

Let $\overline{a'} = \sigma(\overline{a})$. Then we claim that $c \in \operatorname{acl}\left(\overline{a} \wedge \overline{a'}\right)$, as the set of $E_{\overline{a}}$-classes which $D_{\overline{a}'}$ intersects non-trivially is finite. Now $\overline{a}$ must contain a point in the $E_{\overline{a}'}$-class of c. Now this point is algebraic over $\overline{a} \wedge c$ since it is in $\overline{a}$. But this means it is in $\overline{c_0}$. But this is not possible since $D_{\overline{a}}$ is transitive. So we are done. □

Note that $E_{\overline{a}}$ is just the relation of inter-algebraicity over $\emptyset$.

Lemma 3. *Let* $D_{\overline{a}}$ *and* $D'_{\overline{b}}$ *be transitive and almost-linear over* $\overline{a}$ *and* $\overline{b}$ *respectively. Assume there is* $c \in D_{\overline{a}} \cap D'_{\overline{b}}$*. Then* $D_{\overline{a}}$ *and* $D'_{\overline{b}}$ *agree up to reversal on an open neighborhood of* c*.*

Proof. First we know that the $E_{\overline{a}}$-class of c and the $E'_{\overline{b}}$-class of c coincide. We claim that $c \notin \operatorname{acl}\left(\overline{a} \wedge \overline{b}\right)$. Working over $\overline{a}$, if $c \in \operatorname{acl}\left(\overline{a} \wedge \overline{b}\right)$, then $\overline{b}$ has a point in the $E_{\overline{a}}$-class of c. This is the $E_{\overline{b}}$-class of c, so this would contradict transitivity of $D'_{\overline{b}}$.

It follows that c is not an endpoint of the intersection (in $D_{\overline{a}}$). Therefore there is an open neighborhood of c in $D_{\overline{a}}$ inside the intersection. Then the order induced on this intersection by $D'_{\overline{b}}$ must either be the original order, or the reversed order. More specifically we can work in a small neighborhood of c which doesn't intersect $\operatorname{acl}\left(\overline{a} \wedge \overline{b}\right)$ so $\overline{b}$ cannot do anything besides reverse the order in this neighborhood.

Now repeat the same argument for $\overline{b}$ and it is easy to see we are done. □