LECTURE 25
MATH 229

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1. More lemmas

Today we will finish the proof we have been doing. The last thing we proved was that if we have an intersection of D_a and $D_{a'}$ then there is a neighborhood of the point in common such that the orders are either the same or opposite. Now we have the following.

Lemma 1. *Let D_1, D_2 be almost linear, transitive (and definable over some parameters $\overline{a}$ and $\overline{b}$ respectively). Assume they have an interval I in common. Take the maximal such I. Then I is cofinal either in D_1 or D_2.*

Proof. Assume I is not cofinal in either. Let $c \in D_1$ be the sup of I in D_1, and $d \in D_2$ be the sup of I in D_2. Then $c \in \operatorname{dcl}\left(\overline{a} \wedge \overline{b}\right)$. Therefore c is interalgebraic with some element of $\overline{b}$. Let $d = d_0 > d_1 > d_2 > \ldots$ be a sequence of elements of D_2. By transitivity, for each i there is $\sigma_i \in \operatorname{Aut}\left(M/\overline{b}\right)$ such that $\sigma_i(d) = d_i$. Set $D_{1,i} = \sigma_i(D_1)$, and $c_i = \sigma_i(c)$.

Claim 1. The elements c_i are pairwise distinct.

This must be the case because otherwise $D_{1,i}$ and $D_{1,j}$ would have a neighborhood of $c_i = c_j$ in common, but this is impossible by construction.

Since each c_i is n $\operatorname{acl}\left(\overline{b}\right)$, this is a contradiction. □

Corollary 1. *Let D_0, D_1 be as above and $I = D_0 \cap D_1$. Then we have one of the following:*

- *(a) $I = \emptyset$,*
- *(b) I is an initial segment of D_i and a final segment of D_j and the two orders disagree,*
- *(c) I is an initial segment of D_i and a final segment of D_j and the two orders agree, and*
- *(d) $I = I_1 \cup I_2$ where each of I_1 and I_2 are as in (b), or*
- *(e) $I = I_1 \cup I_2$ where each of I_1 and I_2 are as in (c).*

Let $\mathcal{L}$ be the set of definable subsets of M which are almost linear and transitive over some $\overline{a}$. If $D_0, D_1 \in \mathcal{L}$, write $D_0 \trianglelefteq D_1$ if $D_0 \cap D_1$ is as in case (b) or (d).

Say that $c \in M$ is of *order type* if there is $D \in \mathcal{L}$ such that $c \in D$. Let Ω be the set of $c \in M$ of order type. Define an equivalence relation $\mathcal{E}$ on Ω by $c\mathcal{E}d$ if there are $D_0 \trianglelefteq D_1 \trianglelefteq \ldots \trianglelefteq D_k$ with $c \in D_0$ and $d \in D_k$ (or $d \in D_0$ and $c \in D_k$).

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On each $\mathcal{E}$-equivalence class we have a definable separation relation $S(a,b,c,d)$ defined as follows. Every path as above from a to b contains either c or d. Note that D is really on the quotient of the class by a finite equivalence relation.[1]

2. The main theorem

Theorem 1. *Assume that M has few substructures. Then there is a $\emptyset$-definable equivalence relation F with finite classes and a $\emptyset$-definable equivalence relation E on the quotient $M_0 = M/F$ such that the following is true. Let $N := M_0/E$, and M_* denote the reduct of M_0 to pullbacks of definable subsets of N. Then M_* is stable, and*

$$f_n(M_*) = f_n(M_0)$$

for all n.

Proof. F is interalgebraicity for points of order type and equality elsewhere. Then for $M_0 = M/F$, $f_n(M_0) \leq f_n(M)$, so M_0 also has few substructures.

Let E be the equivalence relation $\mathcal{E}$ on M_0 defined above (extended by $=$ on points which are not of order type). Let $N = M_0/E$.

Claim 2. N is stable.

Otherwise, as N is NIP, there would be a definable map $f : N \to V$ (V linearly ordered) which would lift to an almost linear subset of M_0 contradicting the definition of E.

Define M_* as in the statement. Then M_* is stable. It just remains to see that $f_n(M_*) = f_n(M_0)$. I.e. a finite substructure $A \subseteq M_*$ has a unique expansion (up to isomorphism) to a substructure of M_0.

Proceed by induction on $|A|$. Write

$$A = A_0 \amalg A_1 \amalg \cdots \amalg A_k$$

where A_0 consists of the points not of order type, and the remaining A_i consist of points of order type grouped by E-equivalence classes. We expand A to an M_0 substructure one A_i at a time. At stage i the E-class of A_i has a structure over $A_0 \ldots A_{i-1}$ which is isomorphic to one of the four unstable reducts of DLO. Therefore there is a unique embedding of A_i into M_0 up to isomorphisms over the previous $A_{<i}$. □

Theorem 2. *Assume M has few substructures and is primitive. Then M is either stable or M is one of the four unstable reducts of DLO.*

In fact we recover the classification of the unstable reducts of DLO.

Proof. As M is primitive F is equality, and E is either equality or empty and this gives us the two cases. □

Corollary 2. *Assume that for no polynomial p we have*

$$f_n(M) \geq \frac{\Phi^n}{p(n)}$$

for $\Phi \approx 1.618$ is the golden ratio. Then M has a stable reduct M^* such that $f_n(M) = f_n(M^*)$.

[1] This is just interalgebraicity as we saw.

Proof. We only need to show that F is equality. If it wasn't, then we get at least F_n substructures, where F_n is the nth Fibonacci number, and we know $F_n \sim \Phi^n/\sqrt{5}$.

Note that this is optimal because if we do take a trivial 2-cover of DLO, then $f_n(M) = F_n$. For any stable reduct M_*, $f_n(M_*) \leq n/2$. □

3. The stable case

We will now deal with the stable case and then consider more general NIP. What remains:

- Understand the stable case:
  - ω stable: well understood,
  - strictly stable: not well understood at all.
- generalize to $f_n(M) \sim c^n$ or even $f_n(M) \sim e^{2n \log n}$.

Question 1. Are there uncountable many M such that $f_n(M) \sim e^{cn \log n}$?

The idea to approach this is the following. Reduce to finite homogeneous case and induct on the (pseudo-)arity.

Definition 1. The pseudo-arity of M is the minimal arity of a finitely homogeneous structure N in which M can be interpreted.

Example 1. The arity of the circular order is 3, however the pseudo-arity is 2.

Note that a structure of arity 2 cannot interpret a random 3-hypergraph. Similarly a structure of arity 2 cannot interpret a tree $(T, \leq)$ with > 1 branch above every node.

There is another parameter called the "dimension" which is the number of independent linear orders.

Example 2. The dimension of DLO is 1, and DLO^2 has dimension 1. We also want something with two independent linear orders to have $\dim(M, \leq_1, \leq_1) = 2$.

The point is that somehow arity 2 is well understood, and as arity increases thins get more complicated. Trees are dimension 1 and arity 3 which is the first case which is somehow not well understood.