LECTURE 26
MATH 229

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In the last week we will treat the stable case.

1. Rank

Let M be an ω-categorical structure.

Definition 1. For a definable set D and ordinal α, we define $\operatorname{rank}(D) \geq \alpha$ inductively:

(i) $\operatorname{rank}(D) \geq 0$ if $D \neq \emptyset$,
(ii) $\operatorname{rank}(D) \geq \alpha + 1$ if in M^{eq} there is a definable family $(X_{\overline{a}})$, where $\overline{a} \models \varphi(x)$, of subsets of D which is k-inconsistent[1] for some k for $k < \omega$ and $\operatorname{rank}(X_i) \geq \alpha$ for all i.
(iii) $\operatorname{rank}(D) \geq \lambda$, λ limit if $\operatorname{rank}(D) \geq \alpha$ for all $\alpha < \lambda$.

We say $\operatorname{rank}(D) = \emptyset$ if $\operatorname{rank}(D) \geq \alpha$ for all α.

Example 1.
- If $(M, \leq)$ is DLO then $\operatorname{rank}(M) = 1$.
- The random graph has rank 1.
- If M is ω-stable then the rank is the Morley rank.
- The generic tree has infinite rank.

Note that (ii) is equivalent to the following: there is a definable $\begin{array}{c} D' \\ \downarrow^{\pi} \\ D \end{array}$ with finite fibers and a definable equivalence relation on D' with infinitely many classes of rank $\geq \alpha$.

1.1. Properties of rank.

First note that:

$$\operatorname{rank}(a/A) = \operatorname{rank}(\operatorname{tp}(a/A)) = \min\{\operatorname{rank}(D) \mid DA\text{-definable}, a \in D\} \ .$$

We have the following properties:

1. $\operatorname{rank}(a/A) = 0$ iff $a \in \operatorname{acl}(A)$
2. $\operatorname{rank}(D_1 \cup D_2) = \max\{\operatorname{rank}(D_1), \operatorname{rank}(D_2)\}$
3. If D is definable over S there is $a \in D$ with $\operatorname{rank}(a/A) = \operatorname{rank}(D)$.
4. If $\operatorname{rank}(a/bc)$, $\operatorname{rank}(b/c)$ are finite, then $\operatorname{rank}(ab/c) = \operatorname{rank}(a/bc) + \operatorname{rank}(b/c)$.

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[1] I.e. the intersection of any k of them is empty.

If $\operatorname{rank}(M) < \omega$ we write $a \perp_c b$ if

$$\begin{aligned}\operatorname{rank}(ab/c) = \operatorname{rank}(a/c) + \operatorname{rank}(b/c) &\iff \operatorname{rank}(a/bc) = \operatorname{rank}(a/c) \\ &\iff \operatorname{rank}(b/ac) = \operatorname{rank}(b/c) \\ &\iff b \underset{c}{\perp} a\ .\end{aligned}$$

Equivalently $a \perp_E bc$ iff $a \perp_{Ec} b$ and $a \perp_E c$.

Note that if $\operatorname{rank}(D) = 1$, then acl has the exchange property on D. This means for $A \subseteq D$, $a, b \in D$ we have that $a \in \operatorname{acl}(Ab) \setminus \operatorname{acl}(A)$ iff $b \in \operatorname{acl}(Aa) \setminus \operatorname{acl}(A)$ which is equivalent to

$$\operatorname{rank}(Aab) = \operatorname{rank}(A) + 1 = \operatorname{rank}(Ab) = \operatorname{rank}(Aa)\ .$$

What this means is that acl defines a *pregeometry* on D.

A pregeometry is a closure operation that satisfies certain properties:

- $A \subseteq B \implies \operatorname{acl}(A) \subseteq \operatorname{acl}(B)$,
- $\operatorname{acl}(\operatorname{acl}(A)) = \operatorname{acl}(A)$:
- exchange property
- maybe another axiom...

The associated dimension is the rank.

Example 2. $\mathbb{F}_p$ vector space is rank one and has a nontrivial acl.

If $\operatorname{rank}(D) = 1$ and D is primitive then acl defines a *geometry* on D. This means that we additionally have

- $\operatorname{acl}(\emptyset) = \emptyset$ and
- $\operatorname{acl}(\{a\}) = \{a\}$ for $a \in D$.

1.2. Main theorem.

Theorem 1. *If M is finite homogeneous, $\operatorname{rank}(M) = 1$, there is a unique n-type of an independent tuple for all n, then it has trivial geometry.*

Proof. Assume M has rank $= 1$ and is primitive for $A \subseteq M$ finite. Let

$$(A)_+ = \operatorname{cl}(A) - \bigcup_{B \subsetneq A} \operatorname{cl}(B)\ .$$

Assume there is $\{a, b\}$ such that $(\{a, b\})_+ \neq \emptyset$.

Lemma 1. *If $A \cup \{a\}$ is independent, $b \in (A)_+$, $c \in (\{a, b\})_+$, then $c \in (A \cup \{a\})_+$.*

Proof. Assume there is $B \subsetneq A \cup \{a\}$, $c \in \operatorname{cl}(B)$. If $B \subseteq A$, $c \in \operatorname{cl}(A)$, by exchange, $a \in \operatorname{cl}(\{b, c\}) \subseteq \operatorname{cl}(A)$ which is a contradiction.

Otherwise, $B = B' \cup \{a\}$ for $B' \trianglelefteq A$. Let $d \in A \setminus B$. By exchange $b \in \operatorname{cl}(\{a, c\}) \subseteq \operatorname{cl}(B)$,

$$b \in \operatorname{cl}(A \setminus \{d\} \cup \{d\}) \setminus \operatorname{cl}(A \setminus \{d\})\ .$$

Again by exchange,

$$d \in \operatorname{cl}(A \setminus \{d\} \cup \{b\}) \subseteq \operatorname{cl}\left(\underbrace{A \setminus \{d\} \cup \{a\}}_{\supseteq B}\right)$$

so certainly B is in this closure. This contradicts the independence of $A \cup \{a\}$. ☐

Lemma 2. *For any independent A, $A_0, A_1 \subseteq A$ ($A_0 \neq A_1$) then*

$$(A_0)_+ \cap (A_1)_+ = \emptyset \ .$$

Proof. True iff $A_0 \subseteq A_1$ or $A_1 \subseteq A_0$. Otherwise we can find $c_0 \in A_0 \setminus A_1$ or $c_1 \in A_1 \setminus A_0$ and contradict independence.

Let $d \in (A_0)_+ \cap (A_1)_+$. Then by exchange

$$c_0 \in \mathrm{cl}\,(A_0 \setminus \{c_0\} \cup \{d\}) \qquad c_1 \in \mathrm{cl}\,(A_1 \setminus \{c_1\} \cup \{d\})$$

so $A \subseteq \mathrm{cl}\,(A \setminus \{c_0, c_1\} \cup \{d\})$ which is a contradiction to the fact that $\mathrm{rank}\,(A) = |A|$. □

Now let A be an independent set of size n by induction (by first lemma), for all nonempty $B \subseteq A$, $(B)_+ \neq \emptyset$. By the second lemma we get 2^n many one-types over A. This contradicts finite homogeneity.

In general assume $n \geq 2$ is minimal such that

$$(\{a_1, \dots, a_n\})_+ \neq \emptyset \ .$$

Then add $a_1, \dots, a_{n-2}$ as constraints to the language and apply the previous case. ■

Definition 2. A definable set D is *strongly-minimal* if any parameter-definable subset of D is finite or co-finite.

If D is strongly-minimal then $\mathrm{rank}\,(D) = 1$ and any two independent n-tuples have the same tuple.

The conclusion is that a strongly minimal primitive set definable in a finitely homogeneous structure is an indiscernible set, i.e. isomorphic to $(A, =)$.

2. Binary structure

Proposition 1. *Let M be a binary structure. Then* $\mathrm{rank}\,(M) < \omega$.

Proof. Assume $\mathrm{rank}\,(M) \geq \omega$. Fix $N < \omega$ sufficiently large. We can find:

- An increasing family $(c\,(n) \mid n < N)$ of finite tuples,
- a $c\,(n)$-definable set D_n transitive over $c\,(n)$,
- a $c\,(n)$ definable family $(X^n_t \mid t \in E_n)$, $k\,(n)$-inconsistent, where there exists some t such that $D_{n+1} \subseteq X^n_t$.

Claim 1. We can assume $X^n_t \cap X^n_{t'}$ is finite for $t \neq t'$.

To see this replace X^n_t by the max infinite intersections $X_{t_1} \cap \ldots \cap X^n_{t_k}$.

Claim 2. For each n there are $x, y \in D_n$ such that for no $t \in R_n$ do we have both $x_n \in X_t$, $y_n \in X_t$.

We assume this for now.

For each n let $\varphi_n\,(x, y)$ say that for some t we have $x \in X_t$, $y \in X_t$. By binary, on D_n we have that $\varphi_n\,(x, y)$ is equivalent to a $\emptyset$-definable formula $\psi_n\,(x, y)$. We have $\not\psi_n\,(x_n, y_n)$, and for $m < n$ we have $\psi_m\,(x_n, y_n)$ so the ψ_ns are pairwise distinct which is impossible. □